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Compressible Pipe Flow and Water Flow over a Hump

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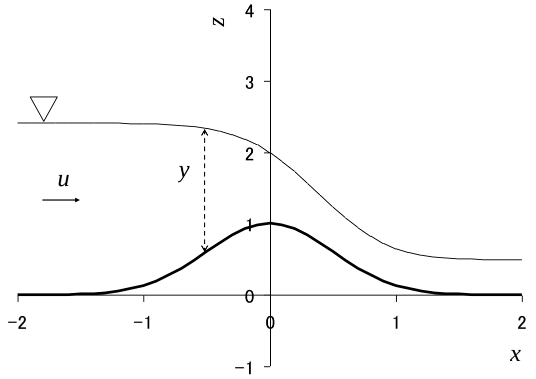
***Abstract*—There is a marked similarity between the open channel hydraulics and the high-speed gas dynamics. This paper introduces another new analogy between the two types of flows and shows that the Mach number varies by the gravity effect as in the Froude number. A transition from the subcritical to the supercritical condition is observed in an open channel flow over a hump. An analogous gravity effect is expected for a one-dimensional isentropic compressible flow of constant cross section with up and down. The Mach number varies with the flow elevation as is observed in an open channel flow over a hump. The equation of continuity and the energy equation lead to the analogy and the compressible flow change with elevation is clarified. The analytical solutions and the jump conditions are derived for the compressible flow equivalent to those of hydraulics. The sonic condition is reached at the maximum flow elevation and the subsonic-supersonic transition is possible even for a flow of constant cross-sectional area.**

***Index Terms*—compressible flow, flow analogy , free surface , hydraulics,**

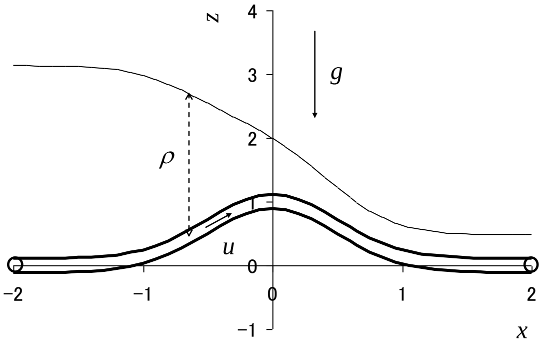
I. Introduction

Analogy with compressible flow is well known for open channel flow [1]. The open channel flows are equivalent to the gas flows of specific heat ratio $\gamma = 2$. An ideal horizontal open channel flow of varying width behaves as a Laval nozzle. Froude number Fr is obtained in an analytical form as a function of the width ratio to the critical condition B/B_* [2] (see also Appendix) as the Mach number M is calculated from the cross-sectional area ratio to the sonic condition A/A_* in a Laval nozzle. A horizontal open channel flow with bottom friction has an integrated analytical solution [3]. This type of flow is called the Fanno flow in gas dynamics, a one-dimensional compressible flow of constant cross-sectional area with friction. A hydraulic jump corresponds to a normal shock wave. Oblique shock waves and Prandtl-Mayer expansion waves are also observed in open channel flows. The shallow water flow pattern resembles a two-dimensional compressible flow.

It is customary to neglect the gravity effect in aerodynamics and most of the compressible flow solutions are derived without the gravitational acceleration. But there is a spherically symmetric astrophysical flow in gas



(a) Open channel flow over a hump



(b) One-dimensional compressible flow under gravity

Fig.1 Flow Analogy

dynamics [4] where the gravity plays an important role. In this paper, another analogy between open channel hydraulics and gas dynamics is introduced. For a one-dimensional isentropic compressible flow of constant cross sectional area with up and down under gravity, the solutions are identical with those of an open channel flow over a hump. Although the similarity appears unnoticed so far to the author's knowledge, the analogy is a direct consequence of the governing equations. A possible experimental validation is also proposed.

II. Analogy

An open channel flow over a hump in Fig. 1(a) is found to be similar to a one-dimensional isentropic compressible flow of constant cross-sectional area with up and down shown in Fig.1(b) where g :constant gravitational acceleration, u :velocity, x :horizontal coordinate (arbitrary scale), y :flow depth, z :vertical coordinate(elevation), and ρ : gas

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density. The flow changes from subcritical to supercritical beyond the peak at $z=1$ both in Figs. 1(a) and (b).

For the ideal open channel flow over a hump in Fig. 1(a), the continuity and the energy equations become

$$yu = y_* u_* = \text{const} \tag{1}$$

$$\frac{u^2}{2} + gy + gz = \frac{u_*^2}{2} + gy_* + gz_* = \text{const} \tag{2}$$

where *: critical (sonic) condition.

The continuity and energy equations for the one-dimensional compressible flow of constant cross-sectional area in Fig.1 (b) are given by

$$\rho u = \rho_* u_* = \text{const} \tag{3}$$

$$\frac{u^2}{2} + h + gz = \frac{u_*^2}{2} + h_* + gz_* = \text{const} \tag{4}$$

where $h = a^2/(\gamma - 1)$: specific enthalpy with a: acoustic velocity. The similarity between Eqs. (1) -(4) suggests that a one-dimensional subsonic compressible flow of constant cross-sectional area can be accelerated to supersonic after the peak elevation as in the open channel flow.

For an open channel flow over a hump, Eqs. (1) and (2) are transformed as follow.

$$\left(1 - Fr^2\right)\frac{du}{u} = \frac{g}{c^2}dz \tag{5}$$

$$\frac{y}{y_*} = \frac{1}{\left(\frac{u}{u_*}\right)} = \frac{1}{Fr^{\frac{2}{3}}} \tag{6}$$

$$\frac{3}{2} - \frac{Fr^{\frac{4}{3}}}{2} - \frac{1}{Fr^{\frac{2}{3}}} = \frac{z - z_*}{y_*} \tag{7}$$

where c: shallow water wave velocity.

For a one-dimensional isentropic compressible flow of constant cross-sectional area under gravity, Eqs. (3) and (4) are transformed as follow.

$$\left(1 - M^2\right)\frac{du}{u} = \frac{g}{a^2}dz \tag{8}$$

$$\frac{\rho}{\rho_*} = \frac{1}{\left(\frac{u}{u_*}\right)} = \frac{1}{M^{\frac{2}{\gamma+1}}} \tag{9}$$

$$\frac{\gamma+1}{2(\gamma-1)} - \frac{M^{\frac{4}{\gamma+1}}}{2} - \frac{1}{(\gamma-1)M^{\frac{2(\gamma-1)}{\gamma+1}}} = \frac{z - z_*}{\left(\frac{a_*^2}{g}\right)} \tag{10}$$

An ideal gas is assumed and the pressure p and the temperature T are determined by the isentropic relations from the density ρ in Eq. (9). Equations (6) and (7) are obtained for $\gamma = 2$ in Eqs. (9) and (10) where a_*^2/g is equivalent to h_*.

Equations (7) for Fr and Eq. (10) for M of air $\gamma = 1.4$ is shown in Fig.2. Note that two solutions are almost identical in the subcritical region.

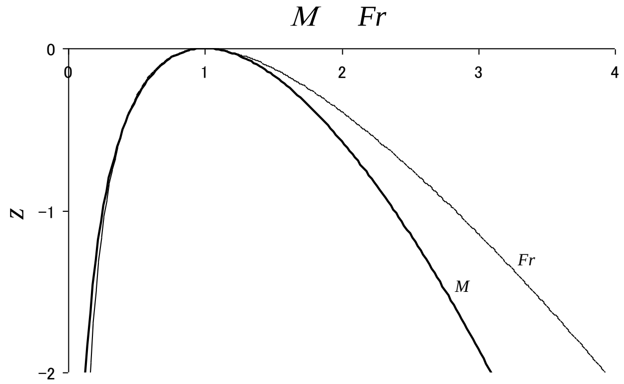

Fig. 2 Mach number and Froude number vs. z

(z in the figure: open channel flow $\frac{z - z_*}{y_*}$, compressible flow $\frac{z - z_*}{\left(\frac{a_*^2}{g}\right)}$)

III. Jump Conditions

For a hydraulic jump at Fr_1 the energy loss is equal to the decrease in the critical elevation z_* and expressed as follows [1].

$$\frac{z_{*1} - z_{*2}}{h_*} = \frac{\left(\sqrt{1 + 8Fr_1^2} - 3\right)^3}{16Fr_1^{\frac{2}{3}}\left(\sqrt{1 + 8Fr_1^2} - 1\right)} \tag{11}$$

where 1: before jump and 2:after jump.

The total enthalpy Eq. (4) is conserved across a shock wave and

$$\frac{z_{*1} - z_{*2}}{\left(\frac{a_{*1}^2}{g}\right)} = \frac{\gamma+1}{2(\gamma-1)}\left(\frac{a_{*2}^2}{a_{*1}^2} - 1\right) \tag{12}$$

From the continuity equation Eq. (3) at two sonic conditions across a shock wave,

$$\frac{a_{*2}^2}{a_{*1}^2} = \frac{1}{\left(\frac{p_{02}}{p_{01}}\right)^{2\frac{\gamma-1}{\gamma+1}}} \tag{13}$$

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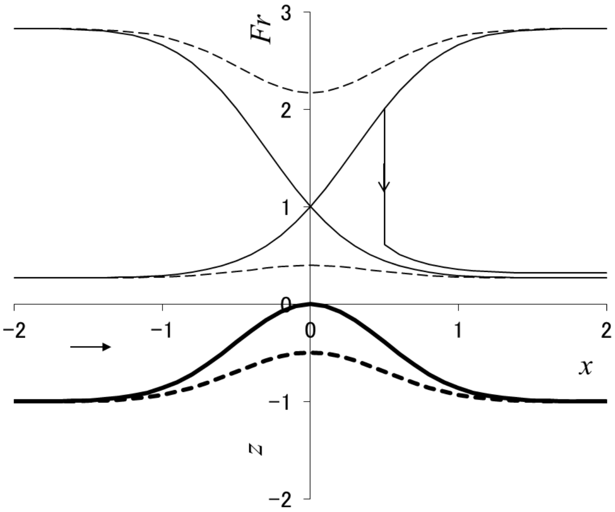
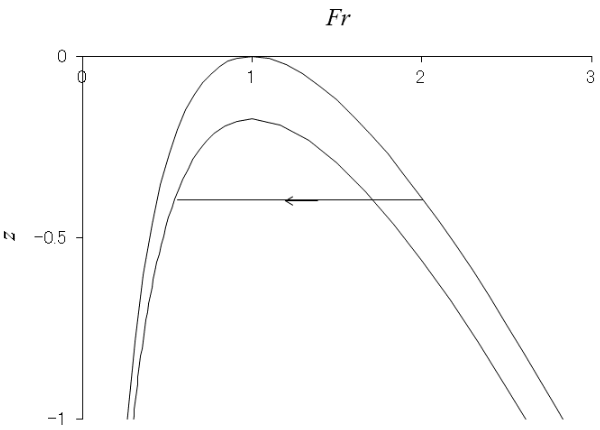


(a) Fr vs. x



(b) Fr vs. z

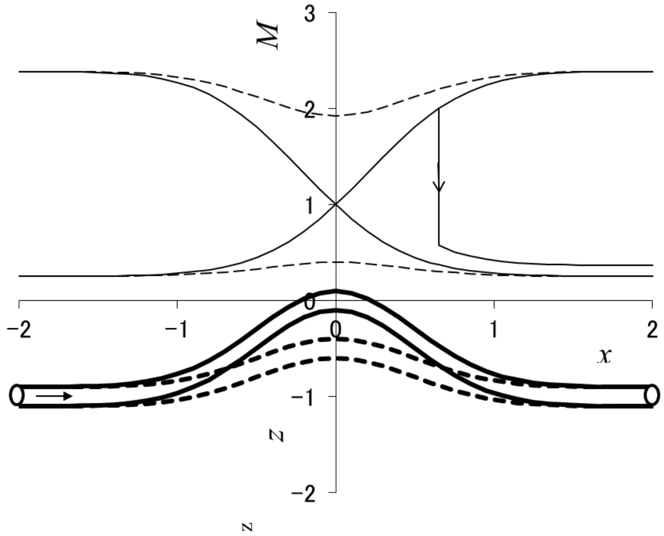
Fig.3 Open channel flow over a hump (z in the figure for $\frac{z-z_*}{y_*}$)

where p_{02}/p_{01} is the stagnation pressure ratio across a normal shock wave [5].

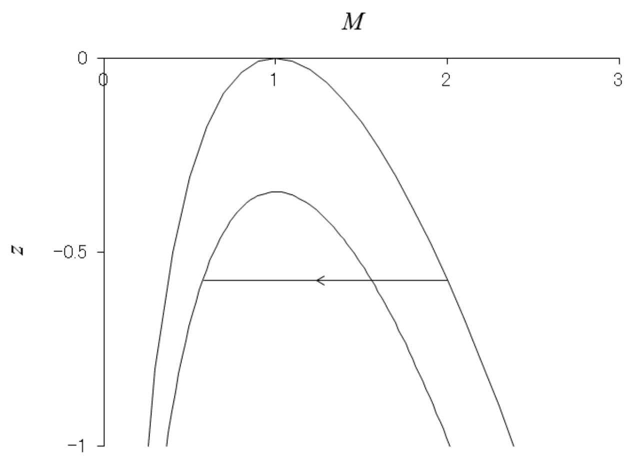
$$\frac{p_{02}}{p_{01}}=\left[\frac{\frac{\gamma+1}{2}M_1^2}{1+\frac{\gamma+1}{2}M_1^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{\frac{\gamma+1}{2}}{\gamma M_1^2-\frac{\gamma-1}{2}}\right]^{\frac{1}{\gamma-1}} \tag{14}$$

IV. Results and Discussions

Figure 3 shows a familiar result of an open channel flow over a hump. The solution of flow in Fig.3 can be expressed analytically [1], [6]-[8]. Although it appears almost the same as Fig.3, Fig.4 is the suggested novel analogical result of a one-dimensional isentropic compressible flow of constant cross-sectional area with up and down under gravity. The solid channel and pipe lines in Figs.3 (a) and 4(a)



(a) M vs. x



(b) M vs. z

Fig.4 One-dimensional isentropic compressible flow of constant cross-sectional area under gravity
($\gamma = 1.4$, z in the figure for $\frac{z-z_*}{\left(\frac{a_*^2}{g}\right)}$)

$(-1 \le z \le 0)$ represent the critical case at the maximum elevation. The broken channel and pipe lines in Figs.3 (a) and 4(a) $(-1 \le z \le -0.5)$ are entirely subcritical and supercritical. Possible jumps are also plotted in Figs. 3 and 4 at $Fr_1 = 2$ and $M_1 = 2$, respectively. The flow depth y/y_* and the density ρ/ρ_* are shown in Figs.1 (a) and (b) for the same flow passages in Figs. 3 and 4 respectively. Figures 3(b) and 4(b) show that the maximum elevation z_* at the critical and the sonic conditions decreases due to the loss generation across a hydraulic jump and a shock wave in Eqs. (11) and (12).

From the engineering point of view, the density and the Mach number of a compressible flow in Fig.1 (b) hardly change under the earth gravity field because obviously $a^2/g \approx 300^2/9.8 >> 1$ (see Eq. (10)) and friction dominates. However, the mathematical resemblance is astounding and

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the experimental verification of the analogy could be possible at low temperature where the acoustic velocity is small. Another possibility for the theory validation lies in the transonic region where the Mach number is sensitive to a small elevation change.

V. Conclusion

An open channel flow over a hump is analogous to a one-dimensional isentropic compressible flow of constant cross-sectional area with up and down under gravity. In theory, the one-dimensional isentropic compressible flow exceeds the sonic velocity even with a constant cross-sectional area under the gravity effect as in the open channel flow over a hump. The Mach number - elevation relation is derived and the shock condition is examined. The analogy is newly introduced to the author's knowledge and the experimental validation could be possible at low acoustic velocity and/or transonic conditions.

Appendix

The Laval nozzle relation for compressible flow is given as follows.

$$\frac{A}{A_*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} \qquad \text{(A1)}$$

where A : cross-sectional area . When the left-hand-side value of A/A_* is given, Mach number M is obtained numerically by iteration, i.e., no analytical form is available for M .

For an ideal horizontal open channel flow of varying width, a similar relationship to Eq. (A1) is obtained as follows.

$$\frac{B}{B_*} = \frac{1}{Fr}\left[\frac{2}{3}\left(1+\frac{1}{2}Fr^2\right)\right]^{\frac{3}{2}} \qquad \text{(A2)}$$

where B : channel width. So we may call this open channel with varying width as a hydraulic Laval nozzle where Fr is equivalent of M in Eq. (A1) and $\gamma = 2$.

Although Eq. (A1) has no analytical solution for M as a function of A/A_* , Eq. (A2) can be solved for Fr analytically as follows [2].

$$Fr^2 = -6\left(\frac{B}{B_*}\right)\cos\frac{\delta+\pi}{3} - 2 \quad (Fr \le 1) \quad \left[= \frac{\left(-1+\sqrt{3}\tan\frac{\delta}{3}\right)^2}{1+\sqrt{3}\tan\frac{\delta}{3}}\right] \qquad \text{(A3)}$$

$$Fr^2 = 6\left(\frac{B}{B_*}\right)\cos\frac{\delta}{3} - 2 \quad (Fr \ge 1) \quad \left[= \frac{8\cos^2\frac{\delta}{3}}{3-4\cos^2\frac{\delta}{3}}\right] \qquad \text{(A4)}$$

$$Fr^2 = -6\left(\frac{B}{B_*}\right)\cos\frac{\delta-\pi}{3} - 2 \quad (Fr^2 < 0) \quad \left[= \frac{\left(-1-\sqrt{3}\tan\frac{\delta}{3}\right)^2}{1-\sqrt{3}\tan\frac{\delta}{3}}\right] \qquad \text{(A5)}$$

$$\cos\delta = -\frac{1}{\left(\frac{B}{B_*}\right)} \quad \left(\frac{\pi}{2} \le \delta \le \pi = \delta_*\right) \qquad \text{(A6)}$$

Equation (A5) is not a physically meaningful solution. These solutions give Froude number Fr directly from the section width B/B_* . It is impossible in Eq. (A1) for gas flow. In many hydraulics text books, it is recommended to calculate Fr numerically from Eq. (2). But it is also possible to use the above Eqs. (A3) and (A4) to get Fr , although it is not popular so far.

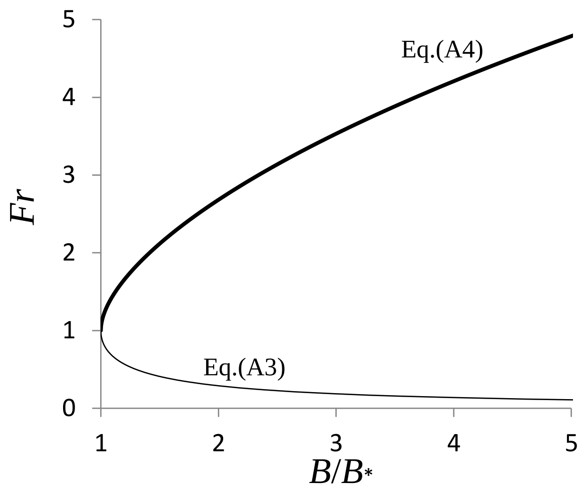

Fig. A1 Hydraulic Laval nozzle -
Horizontal open-channel flow with varying width

(Froude number Fr can be calculated analytically and directly from channel width B/B_* . Mach number M must be calculated numerically in Eq. (A1) of gas dynamics for a given A/A_* .)

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