The Application of the Hybrid Method to Solving the Volterra Integro-differential Equation

G. Mehdiyeva, M. Imanova, and V. Ibrahimov

***Abstract*— There are several works dedicated to the investigation of Volterra integro-differential equations. In addition, there are theoretical and practical representations of stable methods that have a high order of accuracy and extended stability regions; these representations were constructed using the minimum values of arithmetic operations. Here, hybrid methods are proposed for the construction of numerical methods with these properties; one of these hybrid methods is well known. We constructed concrete methods with orders of accuracy of $p = 6$ and $p = 8$ using information pertaining to the solution of the considered problem with one and two mesh points, respectively.**

***Index Terms*— Volterra integro-differential equations, hybrid method, degree and stability, necessary and sufficient conditions, initial-value problem.**

I. Introduction

It is well known that in the early XXth century, to solve some problems in the field of mechanics Vito Volterra had to solve integro-differential equations with variable boundaries. In the 1930s, Volterra showed that mathematical models for some seasonal diseases, e.g., influenza, are formulated as integral and differential equations (see [1, pp. 22-34]); this work gave impetus to the development of approximate methods for solving integro-differential equations. One popular method for solving what are now known as Volterra integro-differential equations is the method of quadratures. Note that the quadrature method was first used by Volterra to solve integro-differential equations with variable boundaries.

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Consider the following initial-value problem in Volterra integro-differential equations:

$$y' = F(x, y, z(x)),\ y(x_0) = y_0,\ x_0 \le x \le X, \qquad (1)$$

where $y(x)$ is a solution of the problem. The function $z(x)$ is defined as follows:

$$z(x) = \int_{x_0}^{x} K(x, s, y(s))ds,\ x_0 \le s \le x \le X. \qquad (2)$$

Obviously, if the function $z(x)$ is known, then problem (1) can be rewritten as:

$$y' = f(x, y(x)),\ y(x_0) = y_0. \qquad (3)$$

Therefore, using the known quantities $y_1, y_2, ..., y_{k-1}$ and $z_1, z_2, ..., z_{k-1}$ to solve problem (1) can be done by applying the k-step method with constant coefficients. Then, we have:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i F(x_{n+i}, y_{n+i}, z_{n+i}), \qquad (4)$$

Here, y_m, z_m $(m = 0,1,2,...)$ are the approximate values of the function $y(x)$ and $z(x)$ for the points $x_m = x_0 + mh$, $(m = 0,1,2,...)$, where the parameter $h > 0$ is the integration step, which is divided by the segment $[x_0, X]$ into N equal parts. It is easy to see that if there is a way to determine z_{n+k} $(n \ge 0)$, then it should be used in formula (4). Furthermore, we can calculate the values of the function $y(x)$ of the mesh points x_{n+k} $(n = 0,1,2,..., N-k)$. In this case, solving problem (1) is equivalent to solving an initial-value problem with ordinary differential and integral equations (see, e.g., [2] - [6]). Thus, by means of multistep methods with constant coefficients we can find the solution to problem (1). Note that solving integral equations can be accomplished with several different approximation methods (see, e.g., [7] - [9]).

In the class of problems (1), the most basic research is on the following problem:

$$y' = f(x, y) + \int_{x_0}^{x} K(x, s, y(s))ds,\ \ y(x_0) = y_0, \qquad (5)$$
$$x_0 \le s \le x \le X.$$

To solve problem (5) one can use the following multistep method (see, e.g., [10] or [11]):

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$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i} + h\sum_{i=0}^{k}\gamma_i z_{n+i} \,, \qquad (6)$$

$$\sum_{i=0}^{k}\hat{\alpha}_i z_{n+i} = h\sum_{j=0}^{k}\sum_{i=0}^{k}\gamma_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}). \qquad (7)$$

This method is obtained by using a multistep method to solve both integral equation (2) and initial-value problem (5). Therefore, solving problem (1) can be accomplished with one of the approximate methods of ordinary differential equations by employing some combinations of the methods proposed for solving integral equations with variable boundaries. The order of accuracy of the stable methods, which is constructed by the scheme (6) - (7), does not exceed $k+2$; this result was established by Dahlquiste (see [12]). Therefore, scientists have proposed various ways to construct stable methods with an order of accuracy greater than $k+2$. To this end, in work [13] a hybrid method that was first investigated by Gear and Butcher was applied to solve the problem (1) (see [14], [15]). However, in [11] the existence of stable forward jumping methods with a higher order of accuracy than $k+2$ was proven and a method to solve Volterra integral equations was proposed. We remark that in [16], stable hybrid methods with a higher order of accuracy than $2k$ were constructed, but in [17], a hybrid method was applied to extend Makroglou's ideas for solving equation (2). Thus, we find that the numerical methods of ordinary differential equations can be applied to solve both integral equations of type (2) and initial-value problems with the form of (1). Note that if one wishes to solve Volterra integral equations using quadrature or other methods that are different from method (7), then one cannot exclusively use the methods of ordinary differential equations to solve problem (1). However, if the kernel of the integral is degenerate, i.e., if

$$K(x,z,y) = \sum_{\nu=1}^{m} a_\nu(x) b_\nu(z,y), \qquad (8)$$

then problem (1) can be reduced to a system of ordinary differential equations. Obviously, in this case problem (1) can be solved using the methods of ordinary differential equations.

In this work, we constructed stable hybrid methods with a high order of accuracy that used information about the solution of problem (1) only minimally. The proposed work is a continuation of the investigations conducted in [16].

Consider the application of the following method for solving problem (1):

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\left(\beta_i y'_{n+i} + \gamma_i y'_{n+i+\nu_i}\right), \qquad (9)$$
$$\left(|\nu_i| < 1;\ i = 0,1,...,k\right)$$

from which we may obtain many well-known hybrid methods. In [18], method (9) is applied to solve problem (3), and it was proved that there exist stable methods of type (9) with degree $p = 3k+1$.

II. The Application of the Generalised Hybrid Method to Solve Problem (1)

Among the numerical methods of both theoretical and practical interest are converging methods. It is known that stability is a necessary and sufficient condition for the convergence of multistep methods. Thus, we investigate the stable hybrid methods that are applied to solve problem (1). Usually, the study of multistep methods imposes certain restrictions on the coefficients (see, e.g., [12]). These constraints on the coefficients of method (9) can be written in the following form:

A. The values of the variables $\alpha_i, \beta_i, \gamma_i, \nu_i \ (i = 0,1,...,k)$ are real numbers and $\alpha_k \neq 0$.

B. The characteristic polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k}\alpha_i\lambda^i;\ \ \vartheta(\lambda) \equiv \sum_{i=0}^{k}\beta_i\lambda^i;\ \ \gamma(\lambda) \equiv \sum_{i=0}^{k}\gamma_i\lambda^{i+\nu_i}$$

of method (9) have no common multiplier that is not a constant.

C. $\vartheta(1) + \gamma(1) \neq 0$ and $p \geq 1$.

Here, p is the order of accuracy of method (9), which is defined in the following form:

Definition 1. For a sufficiently smooth function $y(x)$, method (9) has the degree $p > 0$ if the following holds:

$$\sum_{i=0}^{k}\alpha_i y(x+ih) - h\sum_{i=0}^{k}\left(\beta_i y'(x+ih) + \gamma_i y'(x+(i+\nu_i)h)\right) =$$
$$= O(h^{p+1}),\ h \to 0. \qquad (10)$$

Condition A is obvious. Therefore, we consider condition B and assume the converse. Then, the polynomials $\rho(\lambda)$, $\vartheta(\lambda)$, and $\gamma(\lambda)$ have a common multiplier, which we denote by $\varphi(\lambda)$. After taking into account the shift operator E $(E^\nu y(x) = y(x+\nu h))$, the finite-difference equation (9) can be rewritten as follows:

$$\rho(E)y_n - h\vartheta(E)y'_n - h\gamma(E)y'_n = 0. \qquad (11)$$

Let us use the given assumptions to rewrite equation (11) in the following form:

$$\varphi(E)(\rho_1(E)y_n - h\vartheta_1(E)y'_n - h\gamma_1(E)y'_n) = 0.$$

Here,

$$\rho_1(\lambda) = \rho(\lambda)/\varphi(\lambda);\ \ \vartheta_1(\lambda) = \vartheta(\lambda)/\varphi(\lambda);$$
$$\gamma_1(\lambda) = \gamma(\lambda)/\varphi(\lambda).$$

Hence, we find that

$$\rho_1(E)y_n - h\vartheta_1(E)y'_n - h\gamma_1(E)y'_n = 0, \qquad (12)$$

because $\varphi(\lambda) \neq const$. Obviously, to have a unique solution of finite-difference equation (12), there should be no more than $k-1$ initial data. However, from the theory of finite-difference equations it is known that for a finite-difference equation of order k to have a unique solution, k initial data are required. However, the difference equations (12) and (9) are equivalent. Hence, difference equation (9) has a unique solution despite having no more than $k-1$ initial data, which contradicts the above-mentioned theory. Consequently, the assumption that there is a common factor of the polynomials $\rho(\lambda)$, $\vartheta(\lambda)$ and $\gamma(\lambda)$ is incorrect. Now, consider the validity of condition C. Assume that method (9) converges. Then, as (9) approaches the limit and as $h \to 0$ we have:

$$\sum_{i=0}^{k}\alpha_i y(x) = 0,\ \left(x = x_0 + nh\right). \qquad (13)$$

Because $y(x) \neq 0$, from equation (13) we have:

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$$\rho(1)=0. \quad (14)$$

Equation (14) is a necessary condition for the convergence of the method defined by formula (9), and by using it we can write

$$\rho(\lambda)=(\lambda-1)\rho_1(\lambda).$$

Furthermore, by using (11) we obtain:

$$\rho_1(E)(y_{j+1}-y_j)-h\vartheta(E)y'_n-h\gamma(E)y'_n=0. \quad (15)$$

Here, by changing the value of variable j from 0 to n and summing the resulting equations, we obtain:

$$\rho_1(E)(y_{n+1}-y_0)-h\vartheta(E)\sum_{j=0}^{n}y'_j-h\gamma(E)\sum_{j=0}^{n}y'_j=0.$$

Then, as $h\to 0$, we have:

$$\rho_1(1)(y(x)-y_0)=\left(\vartheta(1)+\gamma(1)\right)\int_{x_0}^{x}y'(\xi)d\xi. \quad (16)$$

However, from problem (1) we can write:

$$y(x)=y(x_0)+\int_{x_0}^{x}f(\xi,y(\xi))d\xi. \quad (17)$$

By comparing (16) and (17), it is clear that

$$\rho_1(1)=\rho'(1)=\vartheta(1)+\gamma(1).$$

It is easy to prove that due to the conditions

$$\rho(1)=0;\ \ \rho'(1)=\vartheta(1)+\gamma(1),$$

$p\ge 1$.

Now we must prove that $\vartheta(1)+\gamma(1)\neq 0$. Assume otherwise. Then, from the conditions $\rho(1)=0$ and $\rho'(1)=0$ we obtain that $\lambda=1$ is a double root of the polynomial $\rho(\lambda)$.

Consider the homogeneous finite-difference equation

$$\alpha_k y_{n+k}+\alpha_{k-1}y_{n+k-1}+...+\alpha_1 y_{n+1}+\alpha_0 y_n=0,$$

whose general solution can be written in the following form:

$$y_m=c_1\lambda_1^m+c_1 m\lambda_1^m+c_3\lambda_3^m+...+c_k\lambda_k^m,$$

where $\lambda_i\ (i=1,2,...,k)$ are the roots of the polynomial $\rho(\lambda)$. Hence, as $h\to 0$ we know that $y_m\to\infty$ because $m\to\infty$. Thus, if $\vartheta(1)+\gamma(1)=0$, then the method does not converge. It follows that $\vartheta(1)+\gamma(1)\neq 0$. If we use the conditions

$$\rho(1)=0 \text{ and } \vartheta(1)+\gamma(1)=\rho'(1)$$

in asymptotic relation (10), then we obtain that

$$p\ge 1.$$

Now, consider the application of method (9) to solve problem (1). To this end, we investigate the numerical solution of problem (1) by using the following methods:

$$\sum_{i=0}^{k}\alpha_i y_{n+i}=h\sum_{i=0}^{k}\beta_i F_{n+i}+h\sum_{i=0}^{k}\gamma_i F_{n+i+\nu_i}, \quad (18)$$

$$\sum_{i=0}^{k}\alpha_i z_{n+i}=h\sum_{j=0}^{k}\sum_{i=0}^{k}\beta_i^{(j)}K(x_{n+j},x_{n+i},y_{n+i})+$$
$$+h\sum_{j=0}^{k}\sum_{i=0}^{k}\gamma_i^{(j)}K(x_{n+j+m_j},x_{n+i+\nu_i},y_{n+i+\nu_i}), \quad (19)$$
$$(|m_j|<1,\ j=0,1,...,k).$$

To study method (19), we suggest that the kernel of the integral $K(x,s,y)$ is a continuous function that is defined in the region $G=\{x_0\le s\le x\le X,|y|\le b\}$ and that has continuous derivatives up to and including some order p. If in method (19) we take into account the properties of the function $K(x,s,y)$, then we have following (see, e.g., [17]):

$$\beta_i^{(j)}=0;\ \gamma_i^{(j)}=0\ \ (i>j).$$

Method (19) as a numerical method for solving Volterra integral equations is studied in [17]. We remark that method (18) is a generalisation of hybrid methods. In the past few years, scientists have thoroughly studied the application of hybrid methods to solving initial-value problems with ordinary differential equations and Volterra integro-differential equations (see, e.g., [13] - [21]). Let us consider finding the coefficients in methods (9) and (19).

It can be shown that by using the Taylor expansions

$$y(x+ih)=y(x)+ihy'(x)+\frac{(ih)^2}{2!}y''(x)+...$$
$$+\frac{(ih)^p}{p!}y^{(p)}(x)+O(h^{p+1}), \quad (20)$$

$$y'(x+l_ih)=y'(x)+l_ihy''(x)+\frac{(l_ih)^2}{2!}y'''(x)+...$$
$$+\frac{(l_ih)^{p-1}}{(p-1)!}y^{(p)}(x)+O(h^p), \quad (21)$$

in asymptotic equation (10), we can obtain the necessary and sufficient conditions for equation (10), where $x=x_0+nh$ is the fixed point and $l_i=i+\nu_i\ \ (i=0,1,2,...,k)$. These conditions can be written in the form of systems of equations that consist of the following nonlinear equations:

$$\sum_{i=0}^{k}\alpha_i=0,\ \sum_{i=0}^{k}(i\alpha_i-\beta_i-\gamma_i)=0,$$

$$\sum_{i=0}^{k}\left(\frac{i^p}{p!}\alpha_i-\frac{i^{p-1}}{(p-1)!}\beta_i-\frac{(i+\nu_i)^{p-1}}{(p-1)!}\gamma_i\right)=0, \quad (22)$$
$$(l=0,1,.,...,p).$$

It is easy to determine that system (22) for the values $\nu_i=0\ (i=0,1,...,k)$ is linear and coincides with known systems that are used to determine the coefficients of the multistep method with constant coefficients. Furthermore, for the conditions $|\nu_0|+|\nu_1|+...+|\nu_k|\neq 0$, system (22) is nonlinear; by solving it, we determine the coefficients of method (9). In this system, the number of unknowns is equal to $4k+4$ and the number of equations is equal to $p+1$. Because system (22) is homogeneous, it always has a trivial solution, but to ensure that system (22) will have a solution that is different from zero, the condition $4k+4>p+1$ must hold. Thus, one can be write the following:

$$p\le 4k+2.$$

Note that if we take $\beta_i=0\ (i=0,1,2,...,k)$, then the relationship between the degree and the order of method (9) will be as follows:

$$p \le 3k+1 .$$

It is known that if we consider the case $\gamma_i = 0\ (i = 0,1,2,...,k)$, then the degree of the stable method received from formula (9) satisfies the condition $p \le 2[k/2]+2$ (see [12]).

To determine the coefficients in method (19), consider a special case and let $K(x,s,y) = F(x,y)$. Then, from (2) we have

$$\vartheta' = F(x,y), \quad \vartheta(x_0) = 0. \tag{23}$$

If we apply method (19) to solve problem (23), then we obtain:

$$\sum_{i=0}^{k} \hat{\alpha}_i \vartheta_{n+i} = h\sum_{i=0}^{k} \hat{\beta}_i F_{n+i} + h\sum_{i=0}^{k} \hat{\gamma}_i F_{n+i+\nu_i} , \tag{24}$$

where

$$\sum_{j=0}^{k} \beta_i^{(j)} = \hat{\beta}_i; \ \sum_{j=0}^{k} \gamma_i^{(j)} = \hat{\gamma}_i \quad (i = 0,1,...,k). \tag{25}$$

First, from system (22) we determine the values of $\hat{\alpha}_i, \hat{\beta}_i, \hat{\gamma}_i, \nu_i\ (i = 0,1,2,...,k)$, and then, by solving system (25) we find the coefficients of method (19). Note that in system (25), the number of equations is equal to $k+1$ and the number of unknowns is greater than $k+1$. Consequently, the solution of system (25) is not unique. Therefore, although the method of type (9) may be unique, the corresponding method of type (19) is not unique. This fact allows us to select some of the coefficients to construct the method with an extended region of stability.

Consider special cases and let $k=1$. Then, by solving system (22) and using the solution in system (25), we obtain a few methods of degree $p=6$. One of them is the following:

$$y_{n+1} = y_n + h(F_{n+1}+F_n)/12 + 5h(F_{n+1/2-\alpha}+F_{n+1/2+\alpha})/12, \tag{26}$$

where the corresponding method of type (19) in one variable can be written as:

$$\begin{aligned} z_{n+1} &= z_n + h(2K(x_{n+1},x_{n+1},y_{n+1}) + \\ &K(x_{n+1},x_n,y_n) + K(x_n,x_n,y_n))/24 + \\ &+5h(K(x_{n+1},x_{n+1/2+\alpha},y_{n+1/2+\alpha}) + \\ &K(x_{n+1/2+\alpha},x_{n+1/2+\alpha},y_{n+1/2+\alpha}) + \\ &+K(x_{n+1},x_{n+1/2-\alpha},y_{n+1/2-\alpha}) + \\ &+K(x_{n+1/2-\alpha},x_{n+1/2-\alpha},y_{n+1/2-\alpha}))/24, \ (\alpha = \sqrt{5}/10). \end{aligned} \tag{27}$$

Note that the method with degree $p=8$ for $k=2$ is as follows:

$$\begin{aligned} y_{n+2} &= y_n + h(64y'_{n+2} + 98y'_{n+1} + 18y'_n)/180 + \\ &+h(y'_{n+1+\beta} + 98y'_n + 64y'_{n+1-\beta})/180, \end{aligned} \tag{28}$$

where the value $\beta = \sqrt{21}/14$.

For the sake of simplicity, let us consider the application of the following method:

$$y_{n+1} = y_n + h(y'_{n+1/2+\alpha} + y'_{n+1/2-\alpha})/2, \quad (\alpha = \sqrt{3}/6). \tag{29}$$

This method will be used to solve problem (1); to use it, one must define the values of $y_{n+1/2+\beta}$, $y_{n+1/2-\alpha}$, which can be done as follows:

$$\begin{aligned} y_{n+1/2+\beta} &= y_{n+1/2} + ((4\beta^3+6\beta^2)\hat{y}'_{n+1} - \\ &-(8\beta^3-24\beta)y'_{n+1/2} + (4\beta^3-6\beta^2)y'_n)/24. \end{aligned} \tag{30}$$

By the formula

$$\hat{y}_{n+1} = y_n + hy'_{n+1/2}, \tag{31}$$

we can find the variable $\hat{y}_{n+1}$. Then, we can determine $y_{n+3/2}$ by the following formula:

$$y_{n+3/2} = y_{n+1/2} + h(7y'_{n+1} - 2y'_{n+1/2} + y'_n)/6. \tag{32}$$

By using the next sequence of methods, one can solve initial-value problem (1).

Step 1. Calculate $\hat{y}_{n+1}$ by formula (31).

Step 2. Calculate $y_{n+1/2\pm\alpha}$ by formula (30).

Step 3. Calculate y_{n+1} by formula (29).

Step 4. Calculate $y_{n+3/2}$ by formula (32).

To illustrate these results, consider the following table.

One may also consider and compare the results obtained by methods of type (9) with other known methods using the following model problems:

$$1. y' = 1 + y - x\exp(-x^2) - 2\int_0^x xt\exp(-y^2(t))dt,$$
$$0 \le x \le 2, \ y(0) = 0$$

(the exact solution is $y(x) = x$).

Number of example	x	*By the work of Gragg & Stetter (see[13])*	*By the work of Kohfeld & Thomson (see[13])*	*By the method from [3]*	*For hybrid method (29)*
I $h = 0,05$	0.1 0.5 1.00	1.3E-09 5.2E-08 1.5E-06	3.5E-10 2.8E-08 1.4E-07		3.3E-10 1.0E-08 1.1E-06
II $h = 1/32$	1.031 1.50 2.0			Max error 1.8E-07	6.3E-09 2.7E-07 2.5E-08

$$2. y' = (4\exp(-y) - x^3)/3 + \frac{4}{3}\int_1^x \frac{1}{x}s^2\exp(y(s))ds,$$
$$1 \le x \le 2, \ y(1) = 0$$

(the exact solution is $y(x) = \ln x$).

Note that the received results are consistent with the theoretical results presented here.

Remark. It is known that scientists have investigated the numerical solutions of ordinary differential equations because they wished to solve integral and integro-differential equations by applying the methods of differential equations. For the sake of demonstration, suppose that the kernel is degenerate and has the following form:

$$K(x,s,y) = a(x)b(s,y) . \tag{33}$$

In this case, one can rewrite the problem as follows:

$$y' = f(x,y) + a(x)v(x), \quad y(x_0) = y_0, \tag{34}$$

$$v'(x) = b(x, y), v(x_0) = 0. \qquad (35)$$

Thus, one can replace solving problem (1) with solving problems (34) and (35), which are initial-value problems that can be solved with ordinary differential equations.

It is known that the problem encapsulated by (34) and (35) consists of two ODEs of the first order. Unfortunately, this simplification is not always correct. Indeed, by the derivative of (33) we obtain

$$y'' = \frac{df(x, y(x))}{dx} + a'x + v(x) + a(x)b(x, y).$$

The above equation is an integro-differential equation. In the considered examples, the kernels have the form (33).

Conclusion. In this paper, some information about solving integro-differential equations is given. We investigated solving an initial-value problem in the class of Volterra integro-differential equations using hybrid methods, and we began with the work of Makroglou (see [13]). Note that the constructed hybrid methods are symmetrical. However, asymmetric hybrid methods are usually more accurate than symmetric ones. We constructed an asymmetric stable hybrid method with degree $p = 9$ for the case $k = 2$. However, the application of this method to practical problems is more difficult than using symmetric methods. In a single article, it is not possible to investigate all aspects of this problem. We believe that the proposed method will have many applications in the future. Note that to apply (26) - (28) to solve certain problems, one can use block methods or methods of predictor-corrector.

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