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Numerical Solution of Fractional Integro-differential Equation by Using Cubic B-spline Wavelets

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Abstract—**A numerical scheme, based on the cubic B-spline wavelets for solving fractional integro-differential equations is presented. The fractional derivative of these wavelets are utilized to reduce the fractional integro-differential equation to system of algebraic equations. Numerical examples are provided to demonstrate the accuracy and efficiency and simplicity of the method.**

Index Terms—**fractional integro-differential equation, Caputo fractional differential equation, cubic B-spline wavelets, collocation method.**

I. Introduction

THE objective of this paper is to introduce a comparative study to examine the performance of the Galerkin method via cubic B-spline wavelets in solving fractional integro-differential equations of the type:

$$D^{\alpha}y(t) = p(t)y(t) + f(t) + \int_0^t K(t,s)y(s)ds, \qquad (1)$$
$$0 \le t \le 1,$$

with initial condition

$$y(0) = y_0, \qquad (2)$$

where the functions $f, p : [0,1] \to \mathbf{R}$ and $K : [0,1]\times[0,1] \to \mathbf{R}$ are given and supposed to be sufficiently smooth and $0 < \alpha \le 1$.

The B-spline wavelets used in this work have compact support, vanishing moments and also they are semi orthogonal. These properties causes many of the operational matrix entries be very small compared with the largest ones. Consequently, these elements can be set to zero with an opportune threshold technique without significantly affecting the solution.

The article is organized as follows: We begin by introducing some necessary definitions and mathematical preliminaries of the fractional calculus theory. Then cubic B-spline wavelets and function approximation by them are purposed which are required for establishing our results. Section 4 is devoted to applying the fractional differential of cubic B-spline scaling functions and wavelets for solving fractional integro-differential equation. In Section 5 the proposed method is applied to several examples. Also a conclusion is given in Section 6.

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II. Some preliminaries in fractional calculus

In this section we briefly present some definitions and results in fractional calculus for our subsequent discussion [1]. The fractional calculus is the name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration [1]-[2]. There are various definitions of fractional integration and differentiation, such as Grunwald-Letnikov and Riemann-Liouville's definitions.

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. The reason for adopting the Caputo definition, as pointed by [3], is as follows: to solve differential equations (both classical and fractional), we need to specify additional conditions in order to produce a unique solution. For the case of the Caputo fractional differential equations, these additional conditions are just the traditional conditions, which are akin to those of classical differential equations and are therefore familiar to us. In contrast, for the Riemann-Liouville fractional differential equations, these additional conditions constitute certain fractional derivatives (and/or integrals) of the unknown solution at the initial point $x = 0$, which are functions of x. These initial conditions are not physical; furthermore, it is not clear how such quantities are to be measured from experiment, say, so that they can be appropriately assigned in an analysis. For more details see [4]. Therefore, we shall introduce a modified fractional differential operator D^{α} proposed by Caputo in his work on the theory of viscoelasticity [5].

Definition: The Caputo definition of the fractional-order derivative of function $f : [a,b] \to R$ is defined as:

$$D^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}dt, \qquad (3)$$
$$n-1 < \alpha \le n, \quad n \in N$$

where α is the order of the derivative and n is the smallest integer greater than α. For the Caputo derivative we have [6]:

$$D^{\alpha}C = 0, \quad (C \textit{ is a constant}),$$

$$D^{\alpha}x^{\beta} = 0, \quad \beta \in N, \quad \beta < \lceil\alpha\rceil,$$

$$D^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha},$$

$$\beta \in N, \ \beta \ge \lceil\alpha\rceil \ or \ \beta \in R - N_0, \ \beta > \lfloor\alpha\rfloor,$$

where $\lceil\alpha\rceil$ denotes the smallest integer greater than or equal to α and $\lfloor\alpha\rfloor$ denotes the largest integer less than or equal

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to α and $N_0 = 0, 1, 2, ...$
For the Laplace transformation of $f(x)$ we have:

$$L(D^\alpha f(x)) = s^\alpha Lf(x) - \sum_{j=0}^{n-1} s^{\alpha-1-j} f^{(j)}(0). \tag{4}$$

It is clear that for $\alpha \in N$ the Caputo differential operator coincides with the usual differential operator of an integer order. Similar to the integer-order differentiation, the Caputo fractional differentiation is a linear operator:

$$D^\alpha(\lambda f(x) + \mu g(x)) = \lambda D^\alpha f(x) + \mu D^\alpha g(x),$$

where λ and μ are constants. In the present work, the fractional derivatives are considered in the Caputo sense.

III. Cubic B-spline scaling and wavelet functions

The general theory and basic concepts of the wavelet theory and MRA are given in [7]-[12].
Wavelets and scaling functions are defined on the entire real line so that they could be outside of the integration domain. This behavior may require an explicit enforcement of the boundary conditions. In order to avid this occurrence, semiorthogonal compactly supported spline wavelets, constructed for the bounded interval $[0, 1]$, have been taken into account in this paper. These wavelets satisfy all the properties verified by the usual wavelets on the real line.
Definition: Let m and n be two positive integers and

$$a = x_{-m+1} = \ldots = x_0 < \ldots < x_n = \ldots = x_{n+m-1} = b,$$

be an equally spaced knots sequence. The functions

$$B_{m,j,X}(x) = \frac{x - x_j}{x_{j+m-1} - x_j} B_{m-1,j,X}(x) + \frac{x_{j+m} - x}{x_{j+m} - x_{j+1}} B_{m-1,j,X}(x), \quad j = -m+1, \ldots, n-1,$$

and

$$B_{1,j,X}(x) = \begin{cases} 1 & x \in [x_j, x_{j+1}), \\ 0 & otherwise, \end{cases}$$

are called cardinal B-spline functions of order $m \geq 2$ for the knot sequence $X = \{x_i\}_{i=-m+1}^{n+m-1}$, and $Supp\ [B_{m,j,X}(x)] = [x_j, x_{j+m}] \bigcap [a, b]$.
For the sake of simplicity , suppose

$$[a, b] = [0, n], \quad x_k = k, \quad k = 0, ..., n.$$

The $B_{m,j,X} = B_m(x - j)$, $j = 0, ..., n - m$, are interior B-spline functions, while the remaining $B_{m,j,X}, j = -m+1, ..., -1$ and $j = n-m+1, ..., n-1$ are boundary B-spline functions, for the bounded interval $[0, n]$. Since the boundary B-spline functions at 0 are symmetric reflections of those at n, it is sufficient to construct only the first half functions by simply replacing x with $n - x$.
By considering the interval $[a, b] = [0, 1]$, at any level $j \in Z^+$, the discretization step is 2^{-j} and this generates $n = 2^j$ number of segments in $[0, 1]$ with knot sequence

$$X^{(j)} = \begin{cases} x_{-m+1}^{(j)} = ... = x_0^{(j)} = 0, & \\ x_k^{(j)} = \frac{k}{2^j} & k = 1, ..., n-1, \\ x_n^{(j)} = ... = x_{n+m-1}^{(j)} = 1. & \end{cases}$$

Let j_0 be the level for which

$$2^{j_0} \geq 2m - 1,$$

for each level $j \geq j_0$ the scaling functions of order m can be defined as follows:

$$\varphi_{m,k}^{(j)}(x) = \begin{cases} B_{m,j_0,k}(2^{j-j_0}x) \\ k = -m+1, ..., -1 \\ \\ B_{m,j_0,2^j-m-k}(1 - 2^{j-j_0}x) \\ k = 2^j - m + 1, ..., 2^j - 1 \\ \\ B_{m,j_0,0}(2^{j-j_0}x - 2^{-j_0}k) \\ k = 0, ..., 2^j - m. \end{cases}$$

And the two-scale relation for the m-order semi orthogonal compactly supported B-wavelet functions are defined as follows:

$$\psi_{m,j,i-m} = \sum_{k=i}^{2i+2m-2} q_{i,k} B_{m,j,k-m}\ , \ i = 1, ..., m-1, \tag{5}$$

$$\psi_{m,j,i-m} = \sum_{k=2i-m}^{2i+2m-2} q_{i,k} B_{m,j,k-m}\ , \ i = m, ..., n-m+1, \tag{6}$$

$$\psi_{m,j,i-m} = \sum_{k=2i-m}^{n+i+m-1} q_{i,k} B_{m,j,k-m}\ , \ i = n-m+2, ..., n, \tag{7}$$

where $q_{i,k} = q_{k-2i}$.
Hence, there are $2(m-1)$ boundary wavelets and $(n-2m+2)$ inner wavelets in the boundary interval $[a, b]$. Finally by considering the level j with $j \geq j_0$, the B-wavelet functions in $[0, 1]$ can be expressed as follows:

$$\psi_{m,j,i}(x) = \begin{cases} \psi_{m,j_0,i}(2^{j-j_0}x) \\ i = -m+1, ..., -1 \\ \\ \psi_{m,2^j-2m+1-i,i}(1 - 2^{j-j_0}x) \\ i = 2^j - 2m + 2, ..., 2^j - m \\ \\ \psi_{m,j_0,0}(2^{j-j_0}x - 2^{-j_0}i) \\ i = 0, ..., 2^j - 2m + 1. \end{cases} \tag{8}$$

The scaling functions $\varphi_{m,k}^{(j)}(x)$, occupy m segments and the wavelet functions $\psi_{m,i}^{(j)}(x)$ occupy $2m - 1$ segments.
Therefore the condition $2^j \geq 2m - 1$, must be satisfied in order to have at least one inner wavelet.
Cubic B-spline scaling function $\varphi_4(x)$ is given by:

$$\varphi_4(x) = \frac{1}{6} \sum_{k=0}^{4} \begin{pmatrix} 4 \\ k \end{pmatrix} (-1)^k (x-k)_+^3 =$$

$$\begin{cases} \frac{1}{6}x^3 & x \in [0, 1) \\ \frac{1}{6}(-3x^3 + 12x^2 - 12x + 4) & x \in [1, 2) \\ \frac{1}{6}(3x^3 - 24x^2 + 60x - 44) & x \in [2, 3) \\ \frac{1}{6}(4 - x)^3 & x \in [3, 4) \\ 0 & otherwise, \end{cases} \tag{9}$$

where

$$x_+^n = \begin{cases} x^n & , x > 0 \\ 0 & , x \leq 0. \end{cases}$$

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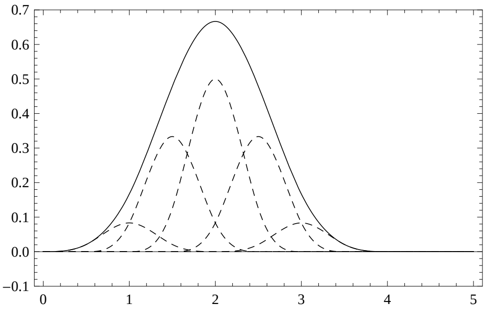

Fig. 1. Two scale relation of $\varphi_4(x)$

And its two-scale dilation equation defined as follows:

$$\varphi_4(x) = \sum_{k=0}^{4} \frac{1}{8} \left(\begin{array}{c} 4 \\ k \end{array} \right) \varphi_4(2x-k). \tag{10}$$

Fig 1 shows the two scale relation of cubic B-spline scaling functions. In this section, the scaling functions used in this work, for $j_0 = j = 3$ and $m = 4$, are reported :

Boundary scalings
Three left boundary cubic B-spline scaling functions are constructed by the following formula:

$$\varphi_{4,k}^{(3)}(x) = \varphi_4(8x-k).\chi_{[0,1]}(x), \tag{11}$$
$$k = -3, -2, -1,$$

and for other levels of j, we have:

$$\varphi_{4,k}^{(j)}(x) = \varphi_{4,k}^{(3)}(2^{j-3}x), \tag{12}$$
$$k = -3, -2, -1, \quad j = 4, 5, \ldots\cdot$$

left and right boundary scaling functions are symmetric with respect to 0, so right boundary scalings are constructed by:

$$\varphi_{4,5}^{(3)}(x) = \varphi_{4,-1}^{(3)}(1-x), \tag{13}$$

$$\varphi_{4,6}^{(3)}(x) = \varphi_{4,-2}^{(3)}(1-x), \tag{14}$$

$$\varphi_{4,7}^{(3)} = \varphi_{4,-3}^{(3)}(1-x), \tag{15}$$

and for other levels of j, we have:

$$\varphi_{4,2^j-k-3}^{(j)}(x) = \varphi_{4,k}^{(3)}(2^{j-3}x), \tag{16}$$
$$k = -3, -2, -1, \quad j = 4, 5, \ldots\cdot$$

Inner scalings
Five inner cubic B-spline scaling functions are constructed by the following formula:

$$\varphi_{4,k}^{(3)}(x) = \varphi_4(8x-k).\chi_{[0,1]}(x), \tag{17}$$
$$k = 0, 1, 2, 3, 4, 5,$$

and for other levels of j, we get:

$$\varphi_{4,k}^{(j)}(x) = \varphi_{4,k}^{(3)}(2^{j-3}x-k),$$
$$k = 0, 1, ..., 2^j - 4, \; j = 4, 5, \ldots\cdot \tag{18}$$

Two scale delation equation for cubic B-spline wavelet is given by:

$$\psi_4(x) = \sum_{k=0}^{10} \frac{(-1)^k}{8} \sum_{l=0}^{4} \left(\begin{array}{c} 4 \\ l \end{array} \right) \varphi_8(k-l+1)\varphi_4(2x-k).$$

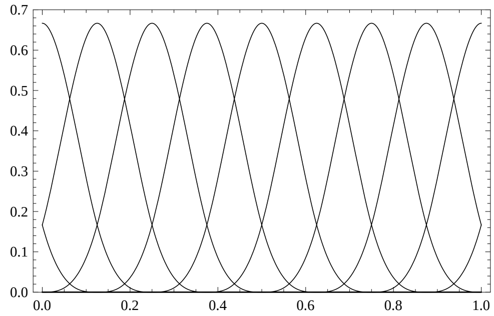

Fig. 2. Boundary and inner scaling functions

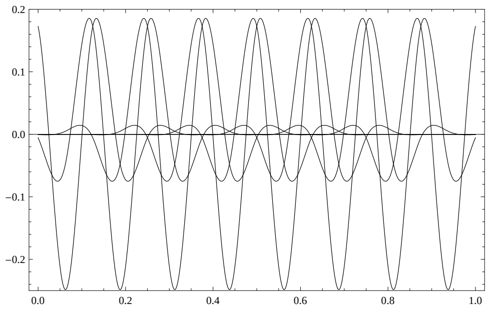

Fig. 3. Boundary and inner wavelets

Other inner and boundary wavelets are made similarly by equations 5-8 [13].

Figures 2 and 3 show the boundary and inner scaling and wavelet functions for cubic B-spline wavelet.

A. Function approximation

A function $f(x)$ defined over $[0, 1]$ may be approximated by cubic B-spline wavelets as:

$$f(x) = \sum_{i=-3}^{2^{j_0}-1} c_{j_0,i}\varphi_{j_0,i}(x) + \sum_{k=j_0}^{\infty} \sum_{j=-3}^{2^k-4} d_{k,j}\psi_{k,j}(x), \tag{19}$$

where $\varphi_{j_0,i}$ and $\psi_{k,j}$ are scaling and wavelets functions, respectively. If the infinite series in equation 19 is truncated, then it can be written as:

$$f(x) \simeq \sum_{i=-3}^{i=2^{j_0}-1} c_{j_0,i}\varphi_{j_0,i}(x) + \sum_{k=j_0}^{j_u} \sum_{j=-3}^{2^k-4} d_{k,j}\psi_{k,j}(x),$$

or

$$f(x) \simeq C^T \Upsilon(x), \tag{20}$$

where C and Υ are $(2^{(j_u+1)} + 3)$ column vectors given by

$$C = \left(c_{j_0,-3}, ..., c_{j_0,2^{j_0}-1}, d_{j_0,-3}, ..., d_{j_u,2^{j_u}-4}\right)^T, \tag{21}$$

$$\Upsilon = \left(\varphi_{j_0,-3}, ..., \varphi_{j_0,2^{j_0}-1}, \psi_{j_0,-3}, ..., \psi_{j_u,2^{j_u}-4}\right)^T, \tag{22}$$

with

$$c_{j_0,i} = \int_0^1 f(x)\tilde{\varphi}_{j_0,i}(x)dx \quad , \quad i = -3, ..., 2^{j_0} - 1,$$

$$d_{k,j} = \int_0^1 f(x)\tilde{\psi}_{k,j}(x)dx$$

$$k = j_0, ..., j_u \; , \; j = -3, ..., 2^{j_u} - 4,$$

and $\tilde{\varphi}_{j_0,i}$ and $\tilde{\psi}_{k,j}$ are dual functions of $\varphi_{j_0,i}, i = -3, ..., 2^{j_0} - 1$ and $\psi_{k,j}, j = j_0, ..., j_u, \; k = -3, ..., 2^j - 4,$

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respectively. These can be obtained by linear combinations of $\varphi_{j_0,i}$ and $\psi_{k,j}$.
Let

$$\varphi(x) = \left(\varphi^{(3)}_{4,-3}(x), \varphi^{(3)}_{4,-2}(x), ..., \varphi^{(3)}_{4,7}(x)\right)^T, \tag{23}$$

$$\psi(x) = \left(\psi^{(3)}_{4,-3}(x), ..., \psi^{(3)}_{4,4}(x), ..., \psi^{(j_u)}_{4,2^{j_u}-4}(x)\right)^T, \tag{24}$$

and

$$\int_0^1 \varphi(x)\varphi^T(x)dx = P_1, \tag{25}$$

$$\int_0^1 \psi(x)\psi^T(x)dx = P_2, \tag{26}$$

where P_1 and P_2 are 11×11 and $(2^{j_u+1}-8) \times (2^{j_u+1}-8)$ matrices, respectively. Suppose $\tilde{\varphi}(x)$ and $\tilde{\psi}(x)$ are the dual functions of $\varphi(x)$ and $\psi(x)$, given by

$$\tilde{\varphi}(x) = \left(\tilde{\varphi}^{(3)}_{4,-3}(x), \tilde{\varphi}^{(3)}_{4,-2}(x), ..., \tilde{\varphi}^{(3)}_{4,7}(x)\right)^T, \tag{27}$$

$$\tilde{\psi}(x) = \left(\tilde{\psi}^{(3)}_{4,-3}(x), ..., \tilde{\psi}^{(3)}_{4,4}(x), ..., \tilde{\psi}^{(j_u)}_{4,2^{j_u}-4}(x)\right)^T. \tag{28}$$

Using equations 23-24, 27 and equation 28 we have

$$\int_0^1 \tilde{\varphi}(x)\varphi^T(x)dx = I_{11}, \quad \int_0^1 \tilde{\psi}(x)\psi^T(x)dx = I_{2^{j_u+1}-8}.$$

where I_{11} and $I_{2^{j_u+1}-8}$ are 11×11 and $(2^{j_u+1}-8)\times(2^{j_u+1}-8)$ identity matrices, respectively. So we get

$$\tilde{\varphi} = P_1^{-1}\varphi, \qquad \tilde{\psi} = P_2^{-1}\psi.$$

Thus, the dual function of Υ can be constructed as:

$$\tilde{\Upsilon}(x) = P^{-1}\Upsilon(x),$$

where

$$P = \begin{pmatrix} P_1 & \\ & P_2 \end{pmatrix}.$$

Now, we found a bound for wavelet coefficients.
Theorem 1: [13] We assume that $f \in C^4[0,1]$ is represented by cubic B-spline wavelets as 20, where ψ has 4 vanishing moments, then

$$|d_{j,k}| \le \alpha\beta\frac{2^{-5j}}{4!}, \tag{29}$$

where

$$\alpha = \max|f^{(4)}(t)|_{t\in[0,1]}, \qquad \beta = \int_0^1 |x^4\tilde{\psi}_4(x)|dx.$$

Theorem 2: [13] Consider the previous theorem assume that $e_j(x)$ be error of approximation in V_j, then

$$|e_j(x)| = O(2^{-4j}). \tag{30}$$

As is shown in equation 30, the order of the error depends on the level j. Obviously, for larger level of j, the error of approximation will be smaller.

IV. Numerical Implementation

Since all the boundary and inner B-spline scaling functions and wavelets are composed by cardinal B-spline function of order $m = 4$, if the analytical expressions of $D^\alpha\varphi_4(x)$, is obtained, those of the boundary and inner B-spline scaling functions and wavelets can be naturally achieved.
Theorem 3: For $m \in N$ and $m < \alpha \le m+1$, $n > 0$, $x > 0$, $a > 0$, $b \ge 0$, if $\alpha \le n$ or $\alpha \in N$, then:

$$D^\alpha(ax-b)^n_+ = a^\alpha\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}(ax-b)^{n-\alpha}_+. \tag{31}$$

Proof: Let

$$f(x) = x^n_+, \quad g(x) = f(ax-b) = (ax-b)^n_+,$$

then the Laplace transform of $f(x)$ is:

$$F(s) = \int_0^\infty e^{-sx}f(x)dx = \frac{\Gamma(n+1)}{s^{n+1}},$$

by the property of the Laplace transform, Laplace transform of $g(x)$ is:

$$G(s) = \int_0^\infty e^{-sx}g(x)dx = \frac{1}{a}e^{-\frac{b}{a}s}F(\frac{s}{a}),$$

From the property of fractional derivative equation 6, we can obtain:

$$\begin{gathered} L\left(D^\alpha(ax-b)^n_+\right) = L\left(D^\alpha g(x)\right) = \\ s^\alpha G(s) - \sum_{j=0}^{m} s^{\alpha-1-j}g^{(j)}(0) = \\ a^n\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}L\left[\left(x-\frac{b}{a}\right)^{n-\alpha}_+\right] = \\ a^\alpha\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}L\left[(ax-b)^{n-\alpha}_+\right]. \end{gathered} \tag{32}$$

From the uniqueness of Laplace transform, we get:

$$D^\alpha(ax-b)^n_+ = a^\alpha\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}(ax-b)^{n-\alpha}_+.$$

Now, we derive the analytical expression of $D^\alpha\varphi_4(x)$.
Theorem 4:

$$D^\alpha\varphi_4(x) = \frac{1}{\Gamma(4-\alpha)}\sum_{k=0}^{4}\begin{pmatrix} 4 \\ k \end{pmatrix}(-1)^k(x-k)^{3-\alpha}_+. \tag{33}$$

Proof: By substituting 9 in 31, proof is completed.
So by the fractional derivative of $\varphi_4(x)$ we can obtain the fractional derivative of boundary scaling functions as follows:

$$\begin{gathered} D^\alpha\varphi^{(3)}_{4,k}(x) = D^\alpha\varphi_4(8x-k).\chi_{[0,1]}(x) = \\ \left(\frac{8^\alpha}{\Gamma(4-\alpha)}\sum_{i=0}^{4}\begin{pmatrix} 4 \\ i \end{pmatrix}(-1)^i(8x-k-i)^{3-\alpha}\right).\chi_{[0,1]}(x), \\ K = -3, -2, -1. \end{gathered} \tag{34}$$

And for other levels of j, we get:

$$\begin{gathered} D^\alpha\varphi^{(j)}_{4,k}(x) = D^\alpha\varphi^{(3)}_{4,k}(2^{j-3}x) = \\ \left(\frac{2^{j\alpha}}{\Gamma(4-\alpha)}\sum_{i=0}^{4}\begin{pmatrix} 4 \\ i \end{pmatrix}(-1)^i(2^jx-k-i)^{3-\alpha}\right).\chi_{[0,1]}(x), \end{gathered} \tag{35}$$

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$$K = -3, -2, -1, \qquad j = 4, 5, \ldots.$$

By the symmetry property of B-spline scaling functions, the fractional derivative of right boundary scaling functions are constructed similarly:

$$D^{\alpha}\varphi_{4,k}^{(3)}(x) = D^{\alpha}\varphi_{4,4-k}^{(3)}(x). \quad k = 5, 6, 7. \tag{36}$$

Fractional derivative of inner scaling functions can be formulated as follows:

$$D^{\alpha}\varphi_{4,k}^{(3)}(x) = D^{\alpha}\varphi_4(8x-k).\chi_{[0,1]}(x) =$$

$$\left(\frac{8^{\alpha}}{\Gamma(4-\alpha)}\sum_{i=0}^{4}\begin{pmatrix}4\\ i\end{pmatrix}(-1)^i(8x-k-i)^{3-\alpha}\right).\chi_{[0,1]}(x), \tag{37}$$

$$K = 0, 1, ..., 5.$$

And for other levels of j, we get:

$$D^{\alpha}\varphi_{4,k}^{(j)}(x) = D^{\alpha}\varphi_{4,k}^{(3)}(2^{j-3}x-k) =$$

$$\left(\frac{2^{j\alpha}}{\Gamma(4-\alpha)}\sum_{i=0}^{4}\begin{pmatrix}4\\ i\end{pmatrix}(-1)^i(2^jx-k-i)^{3-\alpha}\right).\chi_{[0,1]}(x), \tag{38}$$

$$K = 0, 1, \ldots, 2^j - 4, \qquad j = 4, 5, \ldots.$$

The fractional derivative of B-spline wavelets are made similarly. Therefore the fractional derivative of $\Upsilon(t)$ is as follows:

$$D^{\alpha}\Upsilon(t) = \left(D^{\alpha}\varphi_{j_0,-3}(t), ..., D^{\alpha}\psi_{j_u,2^{j_u}-4}(t)\right)^T, \tag{39}$$

Now for solving the equation 1, the fractional derivative of unknown function is approximated by cubic B-spline wavelets as:

$$D^{\alpha}y(t) = D^{\alpha}(C^T\Upsilon(t)) = C^TD^{\alpha}\Upsilon(t), \tag{40}$$

substituting the current equation in equation 1, we have:

$$C^TD^{\alpha}\Upsilon(t) = C^Tp(t)\Upsilon(t) + f(t) + C^T\int_0^t K(t,s)\Upsilon(t)dt, \tag{41}$$

To find the solution $y(t)$, we collocate the equation 41 in

$$t_i = \frac{i}{2^{j_u+1}+2}, \; i = 0, 1, \ldots, 2^{j_u+1}+2,$$

so the fractional integro-differential equation transform to some algebraic linear equation that can be solved by some iteration method.

The limits of integrations in equation 41 range from zero to one; the actual integration limits are much smaller because of the finite supports of the semiorthogonal scaling functions and wavelets. Moreover, a lot of integrals in equation 41 become zero due to the semiorthogonality and vanishing moments properties of the wavelet functions. So using cubic B-spline wavelets, a sparse system of equations can be obtained from fractional integro-differential equation. Therefore, by using present method, we can economize in computational time and memory requirement.

TABLE I
EXACT AND NUMERICAL SOLUTION OF EXAMPLE 1

x_i	$j_u = 4$	$j_u = 5$	Method of [14]	Exact
0	0.00003	0.000000	0.00056	0
0.2	0.0080371	0.00800	0.008694	0.008
0.4	0.064024	0.064000	0.64098	0.064
0.6	0.216084	0.216000	0.216058	0.216
0.8	0.512043	0.512000	0.512583	0.512
1	1.000028	1.000000	1.000064	1

TABLE II
EXACT AND NUMERICAL SOLUTION OF EXAMPLE 2

x_i	$j_u = 4$	$j_u = 5$	Method of [14]	Exact
0	0.000286	0.000000	0.000357	0.0
0.2	0.240010	0.240000	0.240035	0.24
0.4	0.560087	0.560000	0.560093	0.56
0.6	0.960053	0.960000	0.960026	0.96
0.8	1.440081	1.440000	1.440073	1.44
1	2.000617	2.000000	2.001013	2

V. ILLUSTRATIVE EXAMPLES

In this section, for showing the accuracy and efficiency of the described method we present some examples.

Example 1: Consider the following fractional integro-differential equation:

$$y^{(\frac{3}{4})}(t) = \left(-\frac{t^2e^t}{5}\right)y(t) + \frac{6\sqrt[4]{t^9}}{\Gamma(3.25)} + \int_0^t e^tsy(s)ds,$$

with the initial condition $y(0) = 0$ and the exact solution $y(t) = t^3$.

The solution for $y(t)$ is obtained by the method in Section 5 at the octave level $j_0 = 3$ and at the levels $j_u = 4$ and 5. In Table I, we present exact and approximate solutions of Example 1 in some arbitrary points. As proved perviously, the error at the level $j_u = 5$ is smaller than the error at $j_u = 4$.

Example 2: Consider the following fractional integro-differential equation:

$$y^{(\frac{1}{2})}(t) = (cos(t) - sin(t))\,y(t) + f(t) + \int_0^t tsin(s)y(s)ds,$$

with the initial condition $y(0) = 0$ and $f(t)$ is chosen such that the exact solution of equation is $y(t) = t^2 + t$.

The solution for $y(t)$ is obtained by the method in Section 5 at the octave level $j_0 = 3$ and at the levels $j_u = 4$ and 5. In Table II, we present exact and approximate solutions of Example 2 in some arbitrary points. As proved perviously, the error at the level $j_u = 5$ is smaller than the error at $j_u = 4$.

VI. CONCLUSION

In this work a new approach for solving fractional integro-differential equation is purposed. Collocation method via cubic B-spline wavelets are used to reduce the fractional integro differential equation to soma algebraic equation. Because of some properties of these wavelets such as semi orthogonality, having compact support and vanishing moments, system of equations are so spars. The approach can be extended to non-linear fractional integral and integro-differential equations

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with little additional work. Further research along these lines is under progress and will be reported in due time.

REFERENCES

[1] A. A. Kilbas, H. M. Srivastava and Juan. J. Trujillo, "Theory and Applications of Fractional Differential Equations," *North Holland Mathematics Studies*, vol. 204, Elsevier Science, B.V., Amsterdam, 2006.
[2] S. Das, *Functional Fractional Calculus for System Identification and Controls*, Springer, New York, 2008.
[3] S. Momani and M. A. Noor, "Numerical methods for fourth-order fractional integro-differential equations," *Appl. Math. Comput.*, vol. 182, pp. 754-760, 2006.
[4] I. Podlubny, "Geometric and physical interpretation of fractional integration and fractional differentiation," *Fract. Calculus Appl.*, Anal. vol. 5, pp. 367-386, 2002.
[5] M. Caputo, "Linear models of dissipation whose Q is almost frequency independent. Part II," *J. Roy Austral. Soc.*, vol. 13, pp. 529-539, 1967.
[6] K. Diethelm, N. J. Ford, A. D. Freed and Yu. Luchko, "Algorithms for the fractional calculus: A selection of numerical methods," *Comput. Methods Appl. Mech. Eng.*, vol. 194, pp. 743-773, 2005.
[7] C. Chui, *An introduction to wavelets*,New york: Academic press, 1992.
[8] I. Daubechies, *Ten lectures on wavelets*, Philadelpia, PA; SIAM, 1992.
[9] G. Strang and T. Nguyen, *Wavelets and filter banks*, Cambridge, MA: Wellesley-Cambridge, 1997.
[10] L. Telesca, G. Hloupis, I. Nikolintaga and F. Vallianatos, "Temporal patterns in southern Aegean seismicity revealed by the multiresolution wavelet analysis," *Commun. Non. Sci. Num. Simul.*, vol. 12, pp. 1418-1426, 2007.
[11] C. Chui, *Wavelets: a mathematical tool for signal analysis*, Philadelpia, PA: SIAM, 1997.
[12] S. G. Mallat, "A theory for multiresolution signal decomposition: The wavelet representation, IEEE Trans," *Pattern Anal. Mach. Intell.*, vol. 11, pp. 674-693, 1989.
[13] K. Maleknejad, K. Nouri and M. Nosrati Sahlan, "Convergence of approximate solution of nonlinear Fredholm-Hammerstein integral equations," *Commun. Nonlinear. Sci. Num. Simul.*, vol. 15, no. 6, pp. 1432-1443, 2010.
[14] E. A. Rawashdeh, "Numerical solution of fractional integro-differential equations by collocation method," *Applied Mathematics and Computation*, vol. 176, pp. 1-6, 2006.