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Stochastic Optimal Control Problem for Switching Systems with Controlled Diffusion Coefficients

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Abstract—**This paper provides necessary conditions of optimality, in the form of a maximum principle, for optimal control problems of switching systems. Dynamics of the constituent processes take the form of stochastic differential equations with control terms in the drift and diffusion coefficients. The restrictions on the transitions or switches between operating modes, are described by collections of functional equality constraints**

Keywords: stochastic differential equations, stochastic control systems, optimal control problem, maximum principle, switching system, switching law.

1 Introduction

In general, a lot of real systems have abrupt changes in their dynamics that result from causes such as connections or disconnections of some components and success or failures in outcomes. These systems have stochastic behaviour and have been modeled by the class of stochastic differential equations [8, 24].

Change of the structure of the system means that at some moment it may go over from one law of movement to another. After changing the structure, the characteristics of the initial condition of the system depends on its previous state. This situation joins them into a single system with variable structure [6, 15].

A switching systems have the benefit of modeling dynamic phenomena with the continuous law of movement. Recently, optimization problems for switching systems have attracted a lot of theoretical and practical interest [6, 9, 16, 17, 18, 21, 28, 29, 30].

Stochastic control problems have a variety of practical applications in fields such as physics, biology, economics, management sciences, etc. [1, 22]. The modern stochastic optimal control theory has been developed along the lines of Pontryagin's maximum principle and Bellman's dynamic programming [20, 31]. The stochastic maximum principle has been first considered by Kushner [25]. Earliest results on the extension of Pontryagin's maximum principle to stochastic control problems are obtained in [7, 10, 12, 23]. A general theory of stochastic maximum principle based on random convex analysis was given by Bismut [13]. Modern presentations of stochastic maximum principle with backward stochastic differential equations are considered in [14, 26, 27].

In this paper, backward stochastic differential equations have been used to establish a maximum principle for stochastic optimal control problems of switching systems. Such kind of problems have been considered by the authors in [2, 3, 5], where the optimal control problem of switching systems for stochastic systems with uncontrolled diffusion coefficients are studied. The problems with controlled diffusion coefficients without endpoint constraints are considered in [4] .

In this paper, the optimal control problem of stochastic switching systems with control terms in the drift and diffusion coefficients and with endpoint constraints is considered. We obtain necessary condition of optimality in the form of a maximum principle for such systems, where the restrictions on transitions are described by equality constraints.

The rest of the paper is organized as follows. The notations, some basic definitions and the description of the main problem are given in section 2. Section 3 is devoted to stochastic optimal control problem of switching system with endpoint constraints. In this section we give some important facts for our goal and establish necessary condition of optimality for the case of controlled diffusion coefficient.

2 Preliminaries and Statement of problem.

Throughout this paper, we use the following notations. Let $\mathbf{N}$ be some positive constant, R^n denotes the n dimensional real vector space, $|.|$ denotes the Euclidean norm in R^n and E represents the mathematical expectation. Assume that $w_t^1, w_t^2, ..., w_t^r$ are independent Wiener processes, which generate filtration

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$F_t^l = \bar{\sigma}(w_t^l, t_{l-1}, t_l), l = \overline{1,r}$, $0 = t_0 < t_1 < ... < t_r = T$. Let (Ω, F, P), $l = \overline{1,r}$ be a probability space with filtration$\{ F_t, t \in [0,T] \}$, where $F_t = \bigcup_{l=1}^{r} F_t^l$. $L_F^2(a,b;R^n)$ denotes the space of all predictable processes $x_t(\omega)$ such that: $E\int_a^b |x_t(\omega)|^2 dt < +\infty$. $R^{m\times n}$is the space of linear transformations from R^m to R^n. Let $O_l \subset R^{n_l}$, $Q_l \subset R^{m_l}$, $l = \overline{1,r}$, be open sets.

Consider the following stochastic control system:

$$dx_t^l = g^l\left(x_t^l, u_t^l, t\right)dt + f^l\left(x_t^l, u_t^l, t\right)dw_t,$$

$$t \in (t_{l-1}, t_l]\ ,\ l = \overline{1,r}; \tag{1}$$

$$x_{t_{l-1}}^l = \Phi^{l-1}\left(x_{t_{l-1}}^{l-1}, t_{l-1}\right)\ l = \overline{2,r}\ ; x_{t_0}^1 = x_0, \tag{2}$$

$$u_t^l \in U_\partial^l \equiv \left\{u^l(\cdot,\cdot) \in L_{F^l}^2(t_{l-1}, t_l; R^{m_l})\right\}.(3)$$

Elements of U_∂^l, $l = \overline{1,r}$ are called admissible controls. The problem is to find optimal inputs $\left(x^1, x^2, ..., x^r, u^1, u^2, ..., u^r\right)$ and switching sequence $t_1, t_2, ..., t_r$, such that the cost functional :

$$J(u) = \sum_{l=1}^{r} E\left[\varphi^l\left(x_{t_l}^l\right) + \int_{t_{l-1}}^{t_l} p^l\left(x_t^l, u_t^l, t\right)dt\right] \tag{4}$$

is minimized on the decisions of the system (1)-(3), which are generated by all admissible controls $U = U^1 \times U^2 \times ... \times U^r$ at conditions:

$$Eq^l\left(x_{t_r}^l\right) = 0, l = \overline{1,r} \tag{5}$$

Assume that the following requirements are satisfied:

I. Functions g^l, f^l, p^l, $l = \overline{1,r}$ are twice continuously differentiable with respect to x.

II. Functions g^l, f^l, p^l, $l = \overline{1,r}$ and all their derivatives are continuous in (x,u). $g_x^l, g_{xx}^l, f_x^l, f_{xx}^l, p_{xx}^l$are bounded and hold the condition: $(1+|x|)^{-1}\left(\left|g^l(x,u,t)\right| + \left|g_x^l(x,u,t)\right| + \left|f^l(x,u,t)\right| +\right.$

$$\left. + \left|f_x^l(x,u,t)\right| + \left|p^l(x,u,t)\right| + \left|p_x^l(x,u,t)\right|\right) \le N.$$

III.Functions $\varphi^l(x) : R^{n_l} \to R$, $l = \overline{1,r}$ are twice continuously differentiable and satisfy the condition:

$$\left|\varphi^l(x)\right| + \left|\varphi_x^l(x)\right| \le N(1+|x|), \quad \left|\varphi_{xx}^l(x)\right| \le N.$$

IV Functions $\Phi^l(x,t) : R^{n_l} \times \mathrm{T} \to R^1$, $l = \overline{1,r-1}$ are continuously differentiable with respect to (x,t) and hold the condition:

$$\left|\Phi^l(x,t)\right| + \left|\Phi_x^l(x,t)\right| \le N(1+|x|).$$

V Functions $q^l(x) : R^{n_l} \times R^1 \to R^1$, $l = \overline{1,r}$ are twice continuously differentiable and meet the condition:

$$\left|q^l(x,t)\right| + \left|q_x^l(x,t)\right| \le N(1+|x|).$$

Consider the sets:

$$A_i = \mathrm{T}^{i+1} \times \prod_{j=1}^{i} O_j \times \prod_{j=1}^{i} \Lambda_j \times \prod_{j=1}^{i} Q_j,\ i = \overline{1,r},$$

with the elements

$$\pi^i = (t_0, t_1, t_i, x_t^1, x_t^2, ..., x_t^i, u^1, u^2, ..., u^i).$$

Definition 1 The set of functions $\left\{x_t^l = x^l\left(t, \pi^l\right)\right\}$, $t \in [t_{l-1}, t_l]$, $l = \overline{1,r}$ is said to be a solution of equations (1)-(2) with variable structure which corresponding to an element $\pi^r \in \mathrm{A}_r$, if function $x_t^l \in O_l$ on the interval $[t_{l-1}, t_l]$ satisfies the condition (2) on point t_l, while it is absolutely continuous on the interval $[t_{l-1}, t_l]$ with probability 1 and satisfies the equation (1) almost everywhere.

Definition 2 The element $\pi^r \in A_r$ is said to be admissible if the pairs $\left(x_t^l,\ u_t^l\right)$, $t \in [t_{l-1}, t_l]$, $\quad l = \overline{1,r}$ are the solutions of system (1)-(3) which satisfy the conditions (5).

Definition 3 Let A_r^0 be the set of admissible elements. The element $\tilde{\pi}^r \in A_r^0$, is said to be an optimal solution of problem (1)-(5) if there exist admissible controls $\tilde{u}_t^l$, $t \in [t_{l-1}, t_l]$, $l = \overline{1,r}$ and solutions of system (1)-(2) such that pairs$\left(\tilde{x}_t^l, \tilde{u}_t^l\right)$, $l = \overline{1,r}$ minimize the functional (4).

3 Main Result

To state the main result of this paper, we need to introduce the following theorem is proved in [4].

Theorem 1 Suppose that, conditions I-IV hold and $\pi^r = (t_0, t_1, ..., t_r, x_{t_1}, x_{t_2}, , ..., x_{t_r}, u^1, u^2, ..., u^r)$ is an optimal solution of problem (1)-(4). Then,

a) there exist random processes $(\psi_t^l, \beta_t^l) \in L_{F^l}^2(t_{l-1}, t_l; R^{n_l}) \times L_{F^l}^2(t_{l-1}, t_l; R^{n_l x n_l})$ and $(\Psi_t^l, \mathrm{K}_t^l) \in L_{F^l}^2(t_{l-1}, t_l; R^{n_l}) \times L_{F^l}^2(t_{l-1}, t_l; R^{n_l x n_l})$ which are the solutions of the following conjugate equations:

$$\begin{cases} d\psi_t^l = -H_x^l(\psi_t^l, x_t^l, u_t^l, t)dt + \beta_t^l dw_t,\ \ t_{l-1}t < t_l, \quad l = \overline{1,r}, \\ \psi_{t_l}^l = -\varphi_x^l(x_{t_l}^l) + \psi_{t_{l+1}}^l \Phi_x^l(x_{t_l}^l, t_l),\ l = \overline{1,r-1}, \\ \psi_{t_r}^l = -\varphi_x^l(x_{t_r}^l),\ l = \overline{1,r}; \end{cases} \tag{6}$$

$$\begin{cases} d\Psi_t^l = -[\boldsymbol{H}_x^l\left(\Psi_t^l, x_t^l, u_t^l, t\right) + H_{xx}^l(\psi_t^l, x_t^l, u_t^l, t) \\ +f_x^{l*}(x_t^l, u_t^l, t)\Psi_t^l f_x^l(x_t^l, u_t^l, t)]\ dt + \mathrm{K}_t^l dw_t^l, t \in [t_{l-1}, t_l) \\ \Psi_{t_l}^l = -\varphi_{xx}^l(x_{t_l}^l) + \psi_{t_{l+1}}^l \Phi_{xx}^l(x_{t_l}^l, t_l),\ l = \overline{1,r-1}, \\ \Psi_{t_r}^l = -\varphi_{xx}^l(x_{t_r}^l) \end{cases} \tag{7}$$

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b) a.e. $\theta \in [t_{l-1}, t_l]$ and $\forall\ \tilde{u}^l \in U^l$, $l = \overline{1, r}$, a.c. fulfills the maximum principle:

$$H^l(\psi^l_\theta, x^l_\theta, u^l, \theta) - H^l(\psi^l_\theta, x^l_\theta, u^l_\theta, \theta) + \\ +0.5\Delta_{u^l} f^{l*}(x^l_\theta, u^l_\theta, \theta)\Psi^l_\theta \Delta_{u^l} f^l(x^l_\theta, u^l_\theta, \theta) \le 0 \tag{8}$$

c) following transversality conditions hold.:

$$\psi^{l+1}_{t_l} \Phi^l_t \left(x^l_{t_l}, t_l\right) = 0\,,\ l = \overline{1, r-1}\,,\ a.c. \tag{9}$$

Here $H^l\left(\psi_t, x_t, u_t, t\right) = \psi_t g^l\left(x_t, u_t, t\right) +$
$+\, \beta_t f^l\left(x_t, u_t, t\right) - p^l\left(x_t, u_t, t\right);$
$\boldsymbol{H}^l\left(\Psi_t, x_t, u_t, t\right) = \Psi_t g^l\left(x_t, u_t, t\right) + g^{l*}\left(x_t, u_t, t\right)\Psi_t +$
$+\, \mathrm{K}_t f^l\left(x_t, u_t, t\right) + f^{l*}\left(x_t, u_t, t\right)\mathrm{K}_t.$

Then using the obtained result of the Theorem 1 and Ekeland's variational principle [19] the following theorem for stochastic optimal control problem of switching systems with constraints (5) is proved.

Theorem 2. Suppose that, conditions I-V hold and $\pi^r = (t_0, t_1, ..., t_r, x_{t_1}, x_{t_2}, ..., x_{t_r}, u^1, u^2, ..., u^r)$ is an optimal solution of problem (1)- (5). Then,
a) there exist non-zero vectors $(\lambda_0, \lambda_1, ..., \lambda_r) \in R^{r+1}$ and random processes $(\psi^l_t, \beta^l_t) \in L^2_{F^l}(t_{l-1}, t_l; R^{n_l}) \times L^2_{F^l}(t_{l-1}, t_l; R^{n_l x n_l})$ and $(\Psi^l_t, \mathrm{K}^l_t) \in L^2_{F^l}(t_{l-1}, t_l; R^{n_l}) \times L^2_{F^l}(t_{l-1}, t_l; R^{n_l x n_l})$ which are the solutions of the conjugate equations:

$$\begin{cases} d\psi^l_t = -H^l_x(\psi^l_t, x^l_t, u^l_t, t)dt + \beta^l_t dw^l_t, \ \ t_{l-1}t_l, l = \overline{1, r} \\ \psi^l_{t_l} = -\varphi^l_x(x^l_{t_l}) + \psi^l_{t_{l+1}} \Phi^l_x(x^l_{t_l}, t_l),\ l = \overline{1, r-1}, \\ \psi^l_{t_r} = -\lambda_0 \varphi^l_x(x^l_{t_r}) - \sum\limits_{l=1}^{r} \lambda_l q^l_x(x^l_{t_r}); \end{cases} \tag{10}$$

$$\begin{cases} d\Psi^l_t = -[\boldsymbol{H}^l_x\left(\Psi^l_t, x^l_t, u^l_t, t\right) + H^l_{xx}(\psi^l_t, x^l_t, u^l_t, t) \\ +f^{l*}_x(x^l_t, u^l_t, t)\Psi^l_t f^l_x(x^l_t, u^l_t, t)]\ dt + \mathrm{K}^l_t dw^l_t, t \in [t_{l-1}, t_l) \\ \Psi^l_{t_l} = -\varphi^l_{xx}(x^l_{t_l}) + \psi^l_{t_{l+1}} \Phi^l_{xx}(x^l_{t_l}, t_l),\ l = \overline{1, r-1}, \\ \Psi^l_{t_r} = -\lambda_0 \varphi^l_{xx}(x^l_{t_r}) - \sum\limits_{l=1}^{r} \lambda_l q^l_{xx}(x^l_{t_r}). \end{cases} \tag{11}$$

b) a.e. $\theta \in [t_{l-1}, t_l]$ and $\forall\ \tilde{u}^l \in U^l$, $l = \overline{1, r}$, a.c. fulfills the maximum principle:

$$H^l(\psi^l_\theta, x^l_\theta, u^l, \theta) - H^l(\psi^l_\theta, x^l_\theta, u^l_\theta, \theta) + \\ +0.5\Delta_{u^l} f^{l*}(x^l_\theta, u^l_\theta, \theta)\Psi^l_\theta \Delta_{u^l} f^l(x^l_\theta, u^l_\theta, \theta) \le 0 \tag{12}$$

c) following transversality conditions holds:

$$\psi^{l+1}_{t_l} \Phi^l_{t_i}\left(x^l_{t_l}, t_l\right) = 0, a.c.,\ l = \overline{1, r-1} \tag{13}$$

Proof. Fist we discuss the existence of uniquely solutions of adjoint equations (10) and (11). In fact from [10, 11, 26, 27], the first-order adjoint processes (ψ^l_t, β^l_t) and second order adjoint processes $(\Psi^l_t, \mathrm{K}^l_t)$ described in a unique way by (10) and (11) respectively. Finally, we obtain maximum principle in the case when and endpoint constraints are imposed.

For any natural j let's introduce the following approximating functional for each $l = \overline{1, r}$:

$$I^l_j\left(u^l\right) = S^l_j(E\varphi^l\left(x^l_{t_l}\right) + E\int\limits_{t_{l-1}}^{t_l} p^l\left(x^l_t, u^l_t, t\right)dt, Eq^l(x^r_{t_r})) =$$

$$\min_{c^l_j \in \varepsilon} \sqrt{\left|c^l_j - 1/j - EM^l\left(x^l_{t_l}, u^l_{t_l}, t_l\right) + \left|Eq^l(x^l_{t_r})\right|^2\right.}$$

where $M^l\left(x^l_{t_l}, u^l_{t_l}, t_l\right) = \left[\varphi^l(x^l_{t_l}) + \int\limits_{t_{l-1}}^{t_l} p(x^l_t, u^l_t, t)dt\right]^2$
and $\varepsilon = \{\ c : c \le J^0\}$, J^0minimal value of the functional in the problem (1)-5).

Let $V^l \equiv (U^l_\partial, d)$ be space of controls obtained by means of the following metric: $d(u^l, v^l) = (l \otimes P)\ \left\{(t, \omega) \in [t_{l-1}, t_l] \times \Omega : \nu^l_t \ne u^l_t\right\}$. For each $l = \overline{1, r}$, the V^l is a complete metric space [19].

It is easy to prove the following fact:

Lemma 1. Assume that conditions I-IV hold, $u^{l,n}_t$, $l = \overline{1, r}$ be the sequence of admissible controls from V^l, and $x^{l,n}_t$be the sequence of corresponding trajectories of the system (1)-(3). If the following condition is met: $d(u^{l,n}_t, u^l_t) \to 0$, then, $\lim\limits_{n \to \infty}\left\{\sup\limits_{t_{l-1}tt_l} E\left|x^{l,n}_t - x^l_t\right|^2\right\} = 0,$ where x^l_t is a trajectory corresponding to an admissible controls u^l_t , $l = \overline{1, r}.$

Due to continuity of the functionals $I^l_j : V^l \to R^{n_l}$, according to Ekeland's variational principle, there are controls such as; $u^{l,j}_t : d(u^{l,j}_t, u^l_t) \le \sqrt{\varepsilon^l_j}$ and for $\forall u^l_t \in V^l$ the following is achieved: $I^l_j(u^{l,j}) \le I^l_j(u^l) + \sqrt{\varepsilon^l_j} d(u^{l,j}, u^l), \varepsilon^l_j = \frac{1}{j}$.

This inequality means that for each $t \in (t_{l-1}, t_l]$ $(t_1, ..., t_r, x^{1,j}_t, ..., x^{r,j}_t, u^{1,j}_t, ..., u^{r,j}_t)$ is a solution of the following problem:

$$\begin{cases} J^(_j u) = \sum\limits_{l=1}^{r}\left(I^l_j(u^l) + \sqrt{\varepsilon^l_j} E\int\limits_{t_{l-1}}^{t_l} \delta(u^l_t, u^{l,j}_t)dt\right) \to \min \\ dx^l_t = g^l(x^l_t, u^l_t, t)dt + f^l(x^l_t, u^l_t t)dw_t,\ l = \overline{1, r} \\ x^{l+1}_{t_l} = \Phi^l\left(x^l_{t_l}, t_l\right),\ l = \overline{1, r-1}\,; x^1_{t_0} = x_0, \\ u^l_t \in U^l_\partial \end{cases} \tag{14}$$

Function $\delta(u, v)$ is determined in the following way: $\delta(u, v) = \begin{cases} 0, u = v \\ 1, u \ne v. \end{cases}$

Then according to the Theorem 1, it is obtained as follows:
1) there exist the random processes $(\psi^{l,j}_t, \beta^{l,j}_t) \in \mathrm{L}^2_{F^l}(t_{l-1}, t_l; R^{n_l}) \times L^2_{F^l}(t_{l-1}, t_l; R^{n_l \times n_l})$, which are solu-

tions of the following system:

$$\begin{cases} d\psi_t^{l,j} = -H_x^l\left(\psi_t^{l,j}, x_t^{l,j}, u_t^{l,j}, t\right)dt + \beta_t^{l,j}dw_t, \\ t\in[t_{l-1},t_l)\,,\ l=\overline{1,r}; \\ \psi_{t_l}^{l,j} = -\varphi_x^l\left(x_{t_l}^{l,j}\right) + \psi_{t_{l+1}}^l\Phi_x^l(x_{t_l}^{l,j},t_l), l=\overline{1,r-1} \\ \psi_{t_r}^l = -\lambda_0^j\varphi_x^l\left(x_{t_r}^{l,j}\right) - \sum\limits_{l=1}^{r}\lambda_l^j q_x^l\left(x_{t_r}^{l,j}\right). \end{cases} \tag{15}$$

and the random processes $\Psi_t^{l,j}\in L_{F^l}^2(t_{l-1},t_l;R^{n_l})$, $\mathrm{K}_t^{l,j}\in L_{F^l}^2(t_{l-1},t_l;R^{n_l\times n_l})$, which are solutions of the following system:

$$\begin{cases} d\Psi_t^{l,j} = -[\boldsymbol{H}_x^l\left(\Psi_t^{l,j},x_t^{l,j},u_t^{l,j},t\right) + H_{xx}^l(\psi_t^{l,j},x_t^{l,j},u_t^{l,j},t) \\ +f_x^{l*}(x_t^{l,j},u_t^{l,j},t)\Psi_t^{l,j}f_x^l(x_t^{l,j},u_t^{l,j},t)]\,dt + \mathrm{K}_t^{l,j}dw_t^l \\ \Psi_{t_l}^{l,j} = -\varphi_{xx}^l(x_{t_l}^{l,j}) + \psi_{t_{l+1}}^{l,j}\Phi_{xx}^l(x_{t_l}^{l,j},t_l),\ l=\overline{1,r-1}, \\ \Psi_{t_r}^{l,j} = -\lambda_0^{l,j}\varphi_{xx}^l(x_{t_r}^{l,j}) - \sum\limits_{l=1}^{r}\lambda_l^{l,j}q_{xx}^l(x_{t_r}^{l,j}). \end{cases} \tag{16}$$

where non-zero $(\lambda_0^j,\lambda_1^j,...,\lambda_r^j)\in R^{r+1}$ meet the following requirement:

$$(\lambda_0^j,\lambda_1^j,...,\lambda_r^j) = \left(\sum_{l=1}^{r} -c_l + 1/j + E\varphi^l\left(x_{t_l}^{l,j}\right) + \right.$$

$$\left.\mathrm{E}\int\limits_{t_{l-1}}^{t_l} p^l(x_t^{l,j},u_t^{l,j},t)dt,\ Eq^1\left(x_{t_r}^{1,j}\right),...,Eq^r\left(x_{t_r}^{r,j}\right)\right)/J_j^0 \tag{17}$$

$$J_j^0 = \left(\sum_{l=1}^{r}\left|Eq^l(x_{t_r}^l)\right|^2 + \right.$$

$$\left.\sum_{l=1}^{r}\left|c_l - 1/j - E\left[\varphi^l(x_{t_l}^l) + \int\limits_{t_{l-1}}^{t_l} p(x_t^l,u_t^l,t)dt\right]\right|^2\right)^{1/2}$$

2) a.e. $t\in[t_{l-1},t_l]$ and $\forall\ \tilde{u}^l\in V^l$, $l=\overline{1,r}$, a.c. is satisfied:

$$H^l\left(\psi_t^{l,j},x_t^{l,j},\tilde{u}_t^l,t\right) - H^l\left(\psi_t^{l,j},x_t^{l,j},u_t^{l,j},t\right) + \\ +0.5\Delta_{\tilde{u}^l}f^{l*}(x_t^{l,j},u_t^{l,j},t)\Psi_t^{l,j}\Delta_{\tilde{u}^l}f^l(x_t^{l,j},u_t^{l,j},t)\le 0 \tag{18}$$

3) the following transversality conditions hold:

$$\psi_{t_l}^{l+1,j}\Phi_{t_i}^l\left(x_{t_l}^{l,j},t_l\right) = 0,\ l=\overline{1,r-1},\ a.c \tag{19}$$

Since the following has existed $\left\|(\lambda_0^j,\lambda_1^j,...,\lambda_r^j)\right\| = 1$, then according to conditions I-IV it is implied that

$(\lambda_0^j,\lambda_1^j,...,\lambda_r^j)\to(\lambda_0,\lambda_1,...,\lambda_r)$if $j\to\infty$.

Let us introduce the following results which will be needed in the future.

Lemma 2. Let $\psi_{t_l}^l$ be a solution of system (10), $\psi_{t_l}^{l,j}$ be a solution of system (15). If $d(u_t^{l,j},u_t^l)\to 0,\ j\to\infty$, then

$$E\int\limits_{t_{l-1}}^{t_l}|\psi_t^{l,j}-\psi_t^l|^2dt + E\int\limits_{t_{l-1}}^{t_l}|\beta_t^{l,j}-\beta_t^l|^2dt\to 0\,, l=\overline{1,r}\,,$$

Proof: It is clear that$\forall t\in[t_{l-1},t_l]\,,\ l=\overline{1,r}$:

$$d\left(\psi_t^{l,j}-\psi_t^l\right) = -\left[H_x^l\left(\psi_t^{l,j},x_t^{l,j},u_t^{l,j},t\right) - H_x^l\left(\psi_t^l,x_t^l,u_t^l,t\right)\right]dt$$

$$+\left(\beta_t^{l,j}-\beta_t^l\right)dw_t$$

According Ito's formula, for $\forall s\in[t_{r-1},t_r]$ it is satisfied:

$$E|\psi_{t_r}^{l,j}-\psi_{t_r}^l|^2 - E|\psi_s^{l,j}-\psi_s^l|^2 =$$

$2\mathrm{E}\int\limits_s^{t_r}[\psi_t^{l,j}-\psi_t^l][(g_x^{l*}(x_t^{l,j},u_t^{l,j},t) - g_x^{l*}(x_t^l,u_t^l,t))\psi_t^{l,j} + g_x^{l*}(x_t^l,u_t^l,t)(\psi_t^{l,j}-\psi_t^l) + f_x^{l*}(x_t^l,u_t^{l,j},t)\ (\beta_t^{l,j}-\beta_t^l) + (f_x^{l*}(x_t^{l,j},u_t^l,t) - f_x^{l*}(x_t^l,u_t^{l,j},t))\ \beta_t^{l,j} - p^l(x_t^{l,j},u_t^{l,j},t) + p_x^l(x_t^l,u_t^l,t)]dt + E\int\limits_s^{t_r}|\beta_t^{l,j}-\beta_t^l|^2dt$

Due to assumptions I-IV and using simple transformations, the following is obtained:

$$E|\beta_t^{l,j}-\beta_t^l|^2dt + E|\psi_s^{l,j}-\psi_s^l|^2\le$$

$$\mathrm{EN}\int\limits_s^{t_r}|\psi_t^{l,j}-\psi_t^l|^2dt + EN\varepsilon\int\limits_s^{t_r}|\beta_t^{l,j}-\beta_t^l|^2dt + E\left|\psi_{t_r}^{l,j}-\psi_{t_r}^l\right|^2.$$

Hence, according to Gronwall inequality [20] it suggests that:

$$E|\psi_s^{l,j}-\psi_s^l|^2\le De^{N(t_r-s)}\ \ a.e.in\ \ [t_{r-1},t_r] \tag{20}$$

where constant D is determined in the way below:

$$D = E|\psi_{t_r}^{l,j}-\psi_{t_r}^l|^2$$

According to (10) and (15), it is obtained that: $\psi_{t_r}^{l,j}\to\psi_{t_r}^l$, which leads to $D\to 0$. Consequently, from (20) it follows: $\psi_s^{l,j}\to\psi_s^l$ in $L_{F^r}^2(t_{r-1},t_r;R^{n_l})$ and $\beta_s^{l,j}\to\beta_s^l$ in $L_{F^r}^2(t_{r-1},t_r;R^{n_l\times n_l})$. Then, $\forall t\in[t_{l-1},t_l]\,,\ l=\overline{1,r-1}$ from the expression:

$$E|\psi_{t_l}^{l,j}-\psi_{t_l}^l|^2 - E|\psi_s^{l,j}-\psi_s^l|^2 =$$

$$2\mathrm{E}\int\limits_s^{t_l}(\psi_t^{l,j}-\psi_t^l)[(g_x^{l*}(x_t^{l,j},u_t^{l,j},t) - g_x^{l*}(x_t^l,u_t^l,t))\psi_t^{l,j} +$$

$$\mathrm{g}_x^{l*}(x_t^l,u_t^l,t)(\psi_t^{l,j}-\psi_t^l) + f_x^{l*}(x_t^l,u_t^l,t)(\beta_t^{l,j}-\beta_t^{l,j}) +$$

$$+(f_x^{l*}(x_t^{l,j},u_t^{l,j},t) - f_x^{l*}(x_t^l,u_t^l,t))\beta_t^j +$$

$$+\mathrm{p}_x^l(x_t^l,u_t^l,t) - p_x^l(x_t^{l,j},u_t^{l,j},t)]dt + E\int\limits_s^{t_l}|\beta_t^{l,j}-\beta_t^l|^2dt,$$

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by using simple transformations , in view of assumptions I-IV the following is obtained:

$$E\int_{s}^{t_l}|\beta_t^{l,j}-\beta_t^l|^2dt+E|\psi_s^{l,j}-\psi_s^l|^2\leq$$

$$\mathrm{EN}\int_s^{t_l}\left|\psi_t^{l,j}-\psi_t^l\right|^2dt+EN\varepsilon\int_s^{t_l}|\beta_t^{l,j}-\beta_t^l|^2dt+E|\psi_{t_l}^{l,j}-\psi_{t_l}^l|^2.$$

Hence, according to Gronwall inequality, the following result is achieved:

$E|\psi_s^{l,j}-\psi_s^l|^2\leq De^{N(t_l-s)}$a.e. in $[t_{l-1},t_l]$, $l=\overline{1,r-1}$,

where constant D is determined as follows: $D=E|\psi_{t_l}^{l,j}-\psi_{t_l}^l|^2$, which leads to $D\to 0$.

It is inferred that $\psi_s^{l,j}\to\psi_s^l$ in$L^2_{F^l}(t_{l-1},t_l;R^n)$and $\beta_s^{l,j}\to\beta_s^l$ in $L^2_{F^l}(t_{l-1},t_l;R^{n\times n})$.

Lemma 2 is proved.

Lemma 3. Let $\Psi_{t_l}^{l,j}$be a solution of system (11), and $\Psi_{t_l}^l$ be a solution of system (16). Then

$$E\int_{t_{l-1}}^{t_l}|\Psi_t^{l,j}-\Psi_t^l|^2dt+E\int_{t_{l-1}}^{t_l}|\mathrm{K}_t^{l,j}-\mathrm{K}_t^l|^2dt\to 0$$
$l=\overline{1,r}$,if $j\to\infty$.

Proof:

Due to Ito's formula from expressions (11) and (16) for $\forall s\in[t_{l-1},t_l)$:

$$E|\Psi_{t_l}^{l,j}-\Psi_{t_l}^l|^2-E|\Psi_s^{l,j}-\Psi_s^l|^2\leq$$
$$2E\int_s^{t_l}[\Psi_t^{l,j}-\Psi_t^l][(g_x^{l*}(x_t^{l,j},u_t^{l,j},t)-g_x^{l,*}(x_t^j,u_t^j,t))\Psi_t^{l,j}+$$
$$g_x^{l,*}(x_t^j,u_t^j,t)(\Psi_t^{l,j}-\Psi_t^l)+H_{xx}^l\left(\psi_t^{l,j},x_t^{l,j},u_t^{l,j},t\right)-$$

$$H_{xx}^l\left(\psi_t^j,x_t^l,u_t^l,t\right)+H_{xx}^l\left(\psi_t^{l,j},x_t^l,u_t^l,t\right)-H_{xx}^l\left(\psi_t^l,x_t^l,u_t^l,t\right)$$

$$+\Big(f_x^{l*}(x_t^{l,j},u_t^{l,j},t)-f_x^{l*}(x_t^l,u_t^l,t)\Big)\ \mathrm{K}_t^{l,j}+$$

$$+\ \mathrm{f}_x^{l*}(x_t^l,u_t^l,t)(\mathrm{K}_t^{l,j}-\mathrm{K}_t^l)]dt+E\int_s^{t_l}|\mathrm{K}_t^{l,j}-\mathrm{K}_t^l|^2dt$$

Then with help simple transformations we obtain:

$$E\int_s^{t_l}|\mathrm{K}_t^{l,j}-\mathrm{K}_t^l|^2dt+E|\Psi_t^{l,j}-\Psi_t^l|^2\leq$$

$$\mathrm{EN}\int_s^{t_l}|\Psi_t^{l,j}-\Psi_t^l|^2dt+EN\varepsilon\int_s^{t_1}|\mathrm{K}_t^{l,j}-\mathrm{K}_t^l|^2dt+E|\Psi_{t_l}^{l,j}-\Psi_{t_l}^l|^2$$

According to Gronwall inequality a.e. in $[t_{l-1},t_l)$ we have:

$$E|\Psi_s^{l,j}-\Psi_s^l|^2\leq De^{-N(t_l-s)}$$

were the constant D defined as:

$$D=E|\Psi_{t_l}^{l,j}-\Psi_{t_l}^l|^2+EN\varepsilon\int_s^{t_l}|\mathrm{K}_t^{l,j}-\mathrm{K}_t^l|^2dt$$

So that $\Psi_{t_r}^{l,j}\to\Psi_{t_r}^l$, hence, according to assumptions I-IV and expressions (11), (15) we obtain : $\Psi_t^{l,j}\to\Psi_t^l$ in $L^2_{F^r}(t_{r-1},t_r;R^n)$ if $j\to\infty$.

Then according to sufficient smallness of ε follows, that $D\to 0$. Consequently: $\Psi_t^{l,j}\to\Psi_t^l$ in $L^2_{F^l}(t_{l-1},t_l;R^n)$ and $\mathrm{K}_t^{l,j}\to\mathrm{K}_t^l$ in $L^2_{F^l}(t_{l-1},t_l;R^{n\times n})$,$l=\overline{1,r-1}$. Lemma 3 is proved.

It follows from Lemma 2 and Lemma 3 that it can be proceeded to the limit in systems (15) , (16) and the fulfilments of (10),(11) are obtained. Following the similar scheme by taking limit in (18) and (19) it is proved that (12),(13) are true. Theorem 3 is proved.

References

[1] Afanasyev V.N., Kolmanovsky V.B., Nosov V.P. *Mathematical theory for synthesizing control systems*, Moscow, Higher school, 2003.

[2] Agayeva Ch., Abushov Q. *"Necessary condition of optimality for stochastic control systems with variable structure"*, Proceedings of the 20-th International Conference "Continuous, Optimization and Knowledge – based technologies" ,pp. 77–81, 2008.

[3] Agayeva Ch., Abushov Q. *"The maximum principle for some stochastic control problem of switching systems"*, R. Kasmbeyli, C. Diner, S. zpeynirci and L. Sakalauskas (Eds.) Selected papers. 24th Mini EURO Conference on Continuous Optimization and Information-Based Technologies in the Financial Sector, June 23-26, 2010, Izmir University of Economics, Izmir, Turkey, pp.100–105,Vilnius Technika 2010.

[4] Aghayeva Ch. A., Abushov Q. U. *"The maximum principle for some nonlinear stochastic optimal control problem of swithching system"*, Transactions ANAS , mathematics and mechanics series of physical-technical mathematical science, XXXI, 3, pp. 74- 81 2011.

[5] Aghayeva Ch. A., Abushov Q. U. *"The maximum principle for the nonlinear stochastic optimal control problem of switching systems"* , Journal of Global Optimization, DOI information: 10.1007/s10898-011-9825-8.

[6] Antsaklis P.J., Nerode A. *Hybrid Control Systems: An Introductory Discussion to the Special Issue*, IEEE Transaction on Automatic Control **43** ,4, pp.457-460, 1998.

[7] Arkin V.I.,Saksonov M.T. *"The necessary conditions of optimality in control problems of stochastic differential equations"* , DAN SSSR**244** , 4, pp.11–16, 1979.

[8] Arnold L. *Stochastic Differential Equations: Theory and Applications* , Wiley, New York, 1974.

[9] Bengea S.C., Raymond A.C. *"Optimal Control of Switching systems"*, Automatica **41**, pp. 11–27, 2005.

[10] Bensoussan A. *Lectures on Stochastic Control in Nonlinear Filtering and Stochastic Control*, Springer Lectures Notes in Mathematics, Springer Verlag, 1982.

[11] Bensoussan A. *"Stochastic Maximum Principle for Distributed Parameter Systems"*, Journal of Franklin Institute, **315** , 5/6,pp.387–406, 1983.

[12] Bismut J.M. *"Linear quadratic optimal stochastic control with random coefficients"*, SIAM J. on Control **14** , pp.419–444,1976.

[13] Bismut J.M. *"An Introductory Approach to Duality in Optimal Stochastic Control"*, SIAM Reviewl **20** , pp.62–78, 1978.

[14] Borkar V. *"Controlled diffusion processes"*, Probability surveys **2** , 4, pp.213–244,2005.

[15] Boukas E. *Stochastic switching systems: analysis and design*, Birkhauser, Canada, 2006.

[16] Branicky M., Mitter S. *"Algorithms for optimal hybrid control"*, Proceedings of the 34th IEEE Conference on Decision and Control **3** ,pp.2661–2666,1995.

[17] Burachik, R.S., Gasimov, R. N., Ismayilova, N. A., Kaya, C. Y., *"On a Modified Subgradient Algorithm for Dual Problems via Sharp Augmented Lagrangian"*, Journal of Global Optimization **34** , pp. 55–78,2006.

[18] Capuzzo D.I., Evans L.C. *Optimal Switching for ordinary differential equations*, SIAM J. on Control and Optimization **22** , pp.143–161,1984.

[19] Ekeland I. *"On the variational principle"*, Journal Mathematical Analysis and Applications **47** , pp.324–353, 1974.

[20] Fleming W.H., Rishel R.W. *Deterministic and Stochastic Optimal Control*, Springer, 1975.

[21] Gao R., Liu X.,Yang J. *"On optimal control problems of a class of impulsive switching systems with terminal states constraints"*, Nonlinear Analysis: Theory, Methods and Applications **73**, pp.1940–1951, 2010.

[22] Gikhman I.I.,Skorokhod A.V. *Controlled stochastic processes*, Springer-Verlag. New York; Heidelberg. 1979.

[23] Haussman U.G. *"General necessary conditions for optimal control of stochastic systems", Stochastic Systems: Modeling, Identification and Optimization , II*, Mathematical Programming Studies **6** ,pp. 30–48,1976.

[24] Kolmanovsky V.B., Myshkis A.D. *Applied Theory of Functional Differential Equations*, Kluwer Academic Publishers, 1992.

[25] Kushner H.J. *"Necessary conditions for continuous parameter stochastic optimization problems"*, SIAM **10** ,pp. 550–565, 1976.

[26] Makhmudov, N. I. *"General necessary optimality conditions for stochastic systems with controllable diffusion"*, Statistics and control of random processes, Nauka, Moscow , pp. 135–138, 1989.

[27] Peng Shige *"A general stochastic maximum principle for Optimal control problem"*, SIAM. Journal on Control and optimization **28** , pp. 966–979, 1990.

[28] Picolli B. *"Hybrid systems and optimal control"*, Proceedings of the 37^{st} IEEE Conference on Decision and Control ,pp. 13–18,1989.

[29] Seidmann, T. I. *"Optimal control for switching systems"*, Proceedings of the 21^{st} annual conference on in formations science and systems , pp.485–489,1987.

[30] Xu, X., Antsaklis, P. J. *"Results and perspectives on computational methods for optimal control of switched systems"*, In O. Maler, & A. Pnueli (Eds.), Hybrid systems: computation and control. Lecture Notes in Computer Science 2623, pp.540–556, 2003.

[31] Yong J., Zhou X. Y. *Stochastic controls: Hamiltonian Systems and HJB equations*, Springer, New York,1999.

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