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An Improved Bound on Weak Independence Number of a Graph

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***Abstract*— A vertex v in a graph $G = (V, X)$ *is said to be weak* if $d(v) \le d(u)$ for every u adjacent to v in G. A set $S \subseteq V$ is said to be *weak* if every vertex in S is a weak vertex in G. A weak set which is independent is called a *weak independent set* (WIS). The *weak independence number* $w\beta_0(G)$ *is the maximum cardinality of a* WIS. We proved that $w\beta_0(G) \le p - \delta$. This bound is further refined in this paper and we characterize the graphs for which the new bound is attained.**

***Index Terms*— Weak Degree , Weak Independence Number, Weak Domination.**

I. INTRODUCTION

For standard terminologies we refer [2]. The domination parameters are well studied in [5]. The strong domination is introduced by E.Sampathkumar and L. Pushpalatha [12] and further studied in [1], [3], [4] and [11]. Varieties of strong domination are studied in [7], [8], [9] and [10]. The strong (weak) independence numbers and vertex coverings are discussed in [6]. A vertex v in a graph $G = (V, X)$ is said to be *strong* if $d(v) \ge d(u)$ (similarly *weak if* $d(v) \le d(u)$) for every u adjacent to v in G. A set $S \subseteq V$ is said to be *strong* (*weak*) if every vertex in S is a strong (weak) vertex in G. A strong (weak) set which is independent is called a *strong independent set* [SIS] (*weak independent set* [WIS]). The *strong* (*weak*) *independence number* $s\beta_0(G)$ ($w\beta_0(G)$) is the maximum cardinality of a SIS (WIS). For an edge $x = uv$, v *strongly* (*weakly*) *covers* the edge x if $d(v) \ge d(u)$ ($d(v) \le d(u)$) in G. A set $S \subseteq V$ is a *strong vertex cover* [SVC] (*weak vertex cover* [WVC]) if every edge in G is strongly (weakly) covered by some vertex in S. The *strong* (*weak*) *vertex covering number* $s\alpha_0(G)$ ($w\alpha_0(G)$) is the minimum cardinality of a SVC (WVC).

The following results appear in [6].

Theorem 1. *For any isolate free graph* $G = (V, X)$ *with* p *vertices,*

$$s\alpha_0 + w\beta_0 = p$$
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Theorem 2. *For any connected graph G with p vertices,*

$$w\beta_0(G) \le p - \delta$$

The following new degree concepts are defined in [7]. For any vertex $v \in V, N(V) = \{u \in V | u \textit{ is adjacent to } v\}$. $N_s(v) = \{u \in N(v) | d(v) \ge d(u)\}$ and $N_w(v)\{u \in N(v) | d(v) \le d(u)\}$. Then degree of v denoted as $d(v) = |N(v)|$, *strong degree* of v is $d_s(v) = |N_s(v)|$ and *weak degree* of v is $d_w(v) = |N_w(v)|$. We then have the following new graph parameters – maximum strong degree $\Delta_s(G)$, minimum strong degree $\delta_s(G)$, maximum weak degree $\Delta_w(G)$ and minimum weak degree $\delta_w(G)$. It is proved in [7] that if v is a weak vertex then $d(v) = d_w(v)$ and if v is a strong vertex then $d(v) = d_s(v)$.

Theorem 3 [7]. For any graph G,

$$\delta_s(G), \delta_w(G) \le \delta(G) \le \Delta_w(G) \le \Delta_s(G) = \Delta(G)$$

II. AN IMPROVED BOUND ON WEAK INDEPENDENCE NUMBER

We improve the upper bound obtained in Theorem 2 using another graph parameter, maximum weak degree Δ_w of a graph defined above.

Proposition 4. *For any connected graph G with p vertices and maximum weak degree* Δ_w,

$$w\beta_0(G) \le p - \Delta_w \qquad (1)$$

Proof. Let D be any maximum WIS and V_{Δ_w} be the set of all maximum weak degree vertices in G. Then there are two possibilities.

Case (i). $D \cap V_{\Delta_w} \ne \emptyset$. Let $v \in D \cap V_{\Delta_w}$. Since D is independent $D \cap N_w(v) \ne \emptyset$. Therefore we have $D \subseteq V - N_w(v)$. Hence the result follows.

Case (ii). $D \cap V_{\Delta_w} = \emptyset$. Then there exists a vertex $v \in V_{\Delta_w}$ such that $v \notin D$. Let $u \in N_w(v)$ then $d(u) \ge d(v)$. Suppose $u \in D$. Since D is a WIS, every vertex in D is a weak vertex. Thus u is also a weak vertex and hence $d(u) \le d(v)$. Therefore $d(v) = d(u)$. Now, since u is a weak vertex, we have $d(u) = d_w(u)$. But then $d_w(u) = d(u) = d(v) \ge d_w(v)$. If $d_w(u) > d_w(v)$ we get a contradiction to the statement that v is a maximum weak degree vertex. On the other hand if $d_w(u) = d_w(v)$, then

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we get a contradiction to the statement that $D \cap V_{\Delta_w} = \emptyset$. Hence we conclude that $u \notin D$. Since u is arbitrary we have $u \notin D$ for every $u \in N_w(v)$. This implies that $D \subseteq V - N_w[v]$. Hence $w\beta_0 \leq p - (\Delta_w + 1) < p - \Delta_w$. ■

Since $\Delta_w \geq \delta$ the above bound is a better bound than the bound obtained in Theorem 2. For the graph, in Fig. 1, $w\beta_0 = 6 = 10 - 4 = p - \Delta_w$. Also for any complete bipartite graph $K_{m,n}$ the above bound is attained.

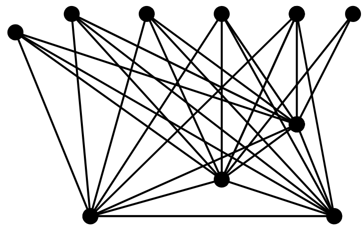

Fig. 1. A graph for which $w\beta_0 = p - \Delta_w$

From the case (ii) of Proposition 4, the above upper bound is further reduced by one.

Corollary 4.1 *Let G be a connected graph with p vertices, maximum weak degree* Δ_w. *Let D be the maximum weak independent set and* Δ_w be the set of all maximum weak degree vertices in *G. If* $D \cap V_{\Delta_w} = \emptyset$ then

$$w\beta_0 \leq p - (\Delta_w + 1).$$

For the graph shown in the Fig. 2, $\Delta_w = 3$ and *v* is the vertex with maximum weak degree and attains the bound $w\beta_0 = 5 = 9 - (3 + 1) = p - (\Delta_w + 1)$. Observe that in this case $\cap\ V_{\Delta_w} = \emptyset$.

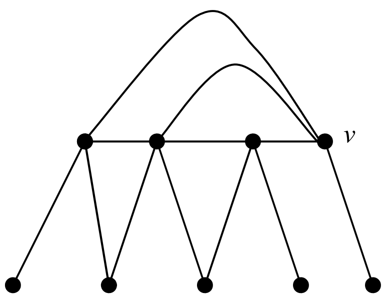

Fig. 2. A graph for which $w\beta_0 = p - (\Delta_w + 1)$

Proposition 5. *Let G be any graph.* V_{Δ_w} *and S be the set of all maximum weak degree vertices in G. Further, W be any maximum independent set of vertices in* $\langle V_{\Delta_w} \rangle$ *and let* $w\beta_0 = p - \Delta_w$. Then there exists a $w\beta_0$ *set D such that* $D \cap V_{\Delta_w} \neq \emptyset$.

Proof. Let $w\beta_0 = p - \Delta_w$. Then there exists a $w\beta_o$ *set D* such that $D \cap V_{\Delta_w} \neq \emptyset$ for otherwise as in the proof of case (ii) of Proposition 4, we get $w\beta_0 \leq p - (\Delta_w + 1) < p - \Delta_w$ - a contradiction. If W_1 is a maximum WIS in *G* such that $d(u) < \Delta_w$ for every $u \in W_1$ then $D = W \cup W_1$ is a $w\beta_0$ *set*. Since $W_1 \cap V_{\Delta_w} = \emptyset$ we have $D \cap V_{\Delta_w} = W$. ■

We now characterize the graphs for which $w\beta_0 = p - \Delta_w$.

Proposition 6. *For any connected graph G with p vertices* $w\beta_0 = p - \Delta_w$ *if and only if* $V - N_w(v)$ *is a WIS for every* $v \in W$.

Proof. Let $w\beta_0 = p - \Delta_w$. Then from the Proposition 5, there exists a $w\beta_0$ set D such that $D \cap V_{\Delta_w} = W \neq \emptyset$. Let $v \in W$. If $V - N_w(v)$ is not a weak independent set then there are at least two vertices which are adjacent in $V - N_w(v)$ and hence we can remove one of the two vertices which are adjacent. But then $w\beta_0 \leq p - (\Delta_w + 1) < p - \Delta_w$ - a contradiction.
Conversely, let $V - N_w(v)$ is a WIS and *D* be a maximum WIS of *G*. Then $w\beta_0 = |D| \geq |V - N_w(v)|$. Further since $v \in W$, as in Proposition 4, $D \subseteq V - N_w(v)$. Hence $w\beta_0 = |D| \leq |V - N_w(v)|$. Thus we have $w\beta_0 = p - \Delta_w$. ■

When $\Delta_w = \delta$ the above bound becomes $w\beta_0 = p - \delta$. We have already characterized the graphs for which $w\beta_0 = p - \delta$ and $s\beta_0 = p - \Delta$ in [6] and we quote those results for our reference.

Theorem 7 [6]. *Let G be a connected graph with* $p \geq 2$ *vertices. Then* $w\beta_0 = p - \delta$ *if and only if the vertex set of G can be partitioned into two sets* V_1 *and* V_2 *satisfying the following conditions.*
(*i*) V_1 *is a WIS.* (*ii*) *every vertex in* V_1 *is adjacent to every vertex in* V_1.

Theorem 8 [6]. *For any connected graph G with* $p \geq 2$ *vertices,* $s\beta_0 = p - \Delta$, *if and only if the vertex set of G can be partitioned in to two sets* V_1 *and* V_2 *satisfying the following conditions.*
(*i*) V_1 *is a SIS.* (*ii*) *there exists a vertex* $v \in V_1$ *such that* $N(v) = V_2$.

We now characterize the graphs for which $w\beta_0 = p - \Delta_w$ when $\Delta_w > \delta$. Since the proof is similar to the proof of Theorem 8, we state the theorem without proof.

Theorem 9. *Let G be a connected graph with* $p \geq 2$ *vertices. Then* $w\beta_0 = p - \Delta_w$ *if and only if the vertex set of G can be partitioned into two sets* V_1 *and* V_2 *satisfying the following conditions.*
(*i*) V_1 *is a WIS.* (*ii*) *there exists a vertex* $v \in V_1$ *such that* $N(v) = V_2$.

In the next result we get a bound for the number of edges when the weak independence number is known.

Theorem 10. *Let G(p, q) be a simple connected graph with weak independence number* $w\beta_0 = k$. *Let* $\Delta_w > \delta$ *so that* $\Delta_w - \delta = r$ *where r is a positive integer. Then ,*
$q \leq \frac{(p+k-1)(p-k)-2r}{2}$. *Further this bound is sharp.*

Proof. Let $w\beta_0 = k$ and W be the $w\beta_0$- set. Since $w\beta_0 = p - \Delta_w$, we have $\Delta_w \leq p - k$. As $\Delta_w > \delta$, to have maximum edges W must contain only one vertex of minimum degree δ and the remaining vertices in W are of maximum weak degree Δ_w. Hence there are at most $k-1$ vertices of degree Δ_w and one vertex of degree δ. Since W is a WIS the vertices in $V - W$ can be of maximum degree. Hence there can be at most δ vertices of degree $p-1$ and the remaining $(\Delta_w - \delta)$ vertices can be at most of degree $p-1$. Hence $2q \leq (k-1)\Delta_w + \delta + \delta(p-1) + (\Delta_w - \delta)(p-2)$. Since $\Delta_w \leq p-k$ and $\delta \leq p-k-r$ we have $2q \leq (k-1)(p-k) + (p-k-r) + (p+k-r)(p-1) + r(p-2) = (p+k-1)(p-k) - 2r$. Then the result follows. ■

Let $G = K_{p-k} + \overline{K_k}$ with $V = V_1 \cup V_2$ where $|V_1| = k$ and $|V_2| = p-k$. Identify any one vertex v in V_1 and remove any r edges incident on v. The new graph G' so obtained attains the upper bound in the Theorem 10.
The graph G' obtained from $G = K_4 + \overline{K_6}$ shown in the

Fig.1, satisfies $q = \frac{(p+k-1)(p-k)-2r}{2} = \frac{(15\times4)-4}{2} = 28.$

The above theorem suggests a better upper bound for $w\beta_0$ in terms of order and size of the graph.

Corollary 10.1. *Let G(p, q) be a simple connected graph. Then,*

$$w\beta_0 \leq \frac{1}{2}\sqrt{p(p-1) - 2q - 2r + \frac{1}{4}}$$

Proof. From Theorem 10, $q \leq \frac{(p+k-1)(p-k)-2r}{2}$. On simplification we get a quadratic equation in *k*. Solving this equation for *k*, we get the desired bound.

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REFERENCES

[1] G.S. Domke, J.H. Hattingh, L.R. Marcus, Elna Ungener, “On Parameters Related to Strong and Weak Domination in Graphs”, *Discrete Mathematics*, vol. 258, pp.1-11, 2002.
[2] F. Harary, *Graph Theory*. Addison Wesley, 1969.
[3] J.H. Hattingh and M.A. Henning, “On Strong Domination in Graphs,” *Journal. Combin. Math. Combin.Comput.* vol. 26, pp. 33-42, 1998.
[4] J.H. Hattingh and R.C. Laskar, “On Weak Domination in Graphs,” *Ars Combinatoria*, vol. 49, pp. 205-216, 1998.
[5] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs.* Marcel Dekker, Inc., N.Y., 1999
[6] S.S. Kamath and R.S. Bhat, “On Strong / Weak Independent Sets and Vertex Coverings of a Graph,” *Discrete Mathematics*, vol. 307, pp.1136 – 1145,Dec. 2007.
[7] R.S.Bhat, S.S.Kamath and Surekha, “A Bound On Weak Domination Number Using Strong (Weak) Degree Concepts in Graphs,” *Journal of International Academy of Physical Sciences*, vol. 15, no.11, pp. 1-15, Sept. 2011.
[8] R.S. Bhat, S.S. Kamath and Surekha, “Strong / Weak Edge Vertex Domination Number of a Graph,” *International Journal of Mathematical Sciences*, vol.11 (3-4), pp 433- 444, July.2012.
[9] R.S. Bhat, S.S. Kamath and Surekha, “Strong / Weak Edge - Edge Domination Number of a Graph,” *Applied Mathematical Sciences*, vol. 6, no. 111, pp. 5525 – 5531, May.2012.
[10] R.S. Bhat, S.S. Kamath and Surekha, “Strong / Weak Matchings and Edge Coverings of a Graph,” *International Journal of Mathematics and Computer Applications Research*, vol.2, no. 3, 85-91, Sep. 2012.
[11] D. Routenbach, “Domination and Degree,” Ph.D thesis, Shaker Verlag, 1998.
[12] E. Sampathkumar and L.Pushpalatha, “Strong Weak Domination and Domination Balance in a Graph,” *Discrete Mathematics*, vol. 161, pp 235-242, 1996.