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Threshold of Isoperimetric Ratio based on Compressibility via Invertible Affine Transformations

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Abstract—**The isoperimetric ratio of a given planar closed curve is the ratio of the square of the perimeter to the area. We show the following theorem: (1) Any convex quadrilateral is affine-equivalent to a quadrilateral whose isoperimetric ratio is less than 20.784. (2) The above result is optimal. And, we propose to use 20.784 as a threshold when we judge that a given polygon is not a succinct quadrilateral. In particular, we illustrate which shapes have larger isoperimetric ratio than the threshold 20.784, with real street patterns. Finally, we discuss possible application to complexity measure of street patterns.**

Index Terms—**affine transformation; isoperimetric ratio; elementary geometry; threshold; street pattern.**

I. Introduction

The isoperimetric problem asks, given a length of perimeter, the maximum area that such a perimeter surrounds. The problem begins in ancient Greek mathematics, and rigorous theory is developed in 19th century. Fore more precise, see [3], [6]. Given a planar closed curve, we consider the isoperimetric ratio; It is defiend to be the ratio of the square of the perimeter to the area that surrounds. The isoperimetric ratio is invariant under similarity transformation. In psychology and engineering, the isoperimetric ratio is used as a complexity measure of a figure. For example, see [1], [2] and [7].

It is well-known that the minimum of the isoperimetric ratio is achieved by a circle. Note that, in our setting, the numerator of the ratio is the square of the perimeter. Thus, unlike the case where the numerator is the area, the isoperimetric ratio of a circle is not the maximum but the minimum.

Now, we investigate the opposite direction; We investigate an upper bound of the isoperimetric ratio, under the setting that we can compress a given shape by invertible affine transformations.

As a main result, we prove the following theorem in section III:

(1) Any convex quadrilateral is affine equivalent to a quadrilateral whose isoperimetric ratio is less than $12\sqrt{3} \simeq 20.784$.

(2) The above result is optimal; We cannot replace $12\sqrt{3}$ by a smaller number.

Here, $12\sqrt{3}$ is the isoperimetric ratio of a regular triangle.

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In section IV, with real street patterns, we illustrate which shapes have larger isoperimetric ratio than the threshold $12\sqrt{3}$.

II. Notation

A. Isoperimetric ratio

Definition 1. Suppose that γ is a convex polygon in the plane. *The isoperimetric ratio of* γ is the ratio of the squared length of γ to the area γ surrounds. We denote the isoperimetric ratio by $I(\gamma)$. That is, $I(\gamma) = (\text{perimeter})^2/(\text{area})$.

We remark that the definitions of the terminology "isoperimetric ratio" or "isoperimetric quotient" slightly differ depending on literatures. In particular, they may be defined as to be the reciprocal of ours. For example, in some literature, $4\pi \times (\text{area})/(\text{perimeter})^2$ is called isoperimetric quotient.

In the remainder of this section, we review concepts of an affine transformation and uniform convergence.

B. Affine transformation

A transformation on xy plane is called an *affine transformation on xy plane* if it is sum of a linear transformation and a translation. In other words, a transformation f is an affine transformation if it is of the following form, where $a_{i,j}$ and b_i are real numbers.

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad (1)$$

We say f is *invertible* (also called *regular*) if so is the linear mapping part, that is, $a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \neq 0$.

Two polygons γ and γ' are *affine-equivalent* if there exists an invertible affine transformation f such that γ' is the image of γ by f. More precisely, regarding γ and γ' as point sets, it holds that $\gamma' = \{f(x) : x \in \gamma\}$.

We observe some examples. Any triangle is affine-equivalent to a regular triangle. It achieves isoperimetric ratio $12\sqrt{3}$, which is the minimum among all triangles. Any parallelogram is affine-equivalent to a square. It achieves isoperimetric ratio 16, the minimum among all quadrilaterals. The supremum of isoperimetric ratio among all quadrilaterals is infinity; To verify this, consider a rectangle such that the ratio of length of shorter edge to that of longer edge is 1 to n, and take limit of $n \to \infty$.

C. Uniform convergence

A statement "If x approaches to a then $g(x, y, z)$ *converges to b uniformly with respect to y and z*" denotes the following: "For any $\varepsilon > 0$, there exists a $\delta > 0$, where the choice of δ does not depend on y and z, such that for every x, y and z, if $0 < |x - a| < \delta$ then $|g(x, y, z) - b| < \varepsilon$."

III. Theory

In this section, we show our main theorem.

Theorem 1. 1) *For any convex quadrilateral γ, there exists a quadrilateral γ' such that γ' is affine equivalent to γ and such that $I(\gamma') < 12\sqrt{3}$.*

2) *The upper bound $12\sqrt{3}$ in the above statement is optimal. More precisely, for any positive real number ε, there exists a convex quadrilateral γ such that for any quadrilateral γ' that is affine-equivalent to γ, it holds that $I(\gamma') > 12\sqrt{3} - \varepsilon$.*

Proof: 1. In xy plane, consider four points $O(0,0)$, $A(1,0)$, $B(1/2, \sqrt{3}/2)$ and $C(3/2, \sqrt{3}/2)$ (Fig. 1). Without loss of generality, we may assume that a given convex quadrilateral is $OAXB$, where the point X is on the triangle ABC and X is not on the segment AB.

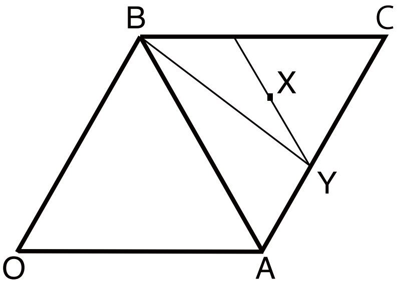

Fig. 1. $O(0,0), A(1,0), B(1/2, \sqrt{3}/2)$ and $C(3/2, \sqrt{3}/2)$

Consider the segment pararell to AB on which X is. Suppose that the segment intersects AC at a point Y.

Then, the isoperimetric ratio of the quadrilateral $OAXB$ is not greater than that of the quadrilateral $OAYB$. Let $t (0 < t \leq 1)$ be the length of AY. Then, the isoperimetric ratio of $OAYB$ is not greater than the followoing.

$$\frac{(3+t)^2}{(1+t)\sqrt{3}/4} = \frac{4\sqrt{3}}{3}\{(t+1) + 4 + 4/(t+1)\} \qquad (2)$$

Now, $(t+1) + 4/(t+1)$ is decreasing in the interval $0 < t < 1$, and therefore it is less than $(0+1)+4/(0+1) = 5$. Thus, (2) is less than $12\sqrt{3}$. Hence, the first assertion of the theorem holds.

2. Consider a triangle OBC. Take points A on OB and D on OC so that the ratios of lengths OA/OB and OD/OC have the same value; Let $\varrho > 0$ be the value. (Fig. 2).

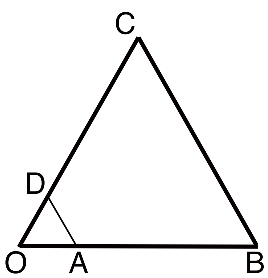

Fig. 2. $OA/OB = OD/OC = \varrho \to 0+$

If $\varrho \to 0+$ then the isoperimetric ratio of $ABCD$ converges to the isoperimetric ratio of OBC. And, the convergence is uniform with respect to the angle BOC and the ratio OC/OB.

In addition, the minimum of the isoperimetric ratio of the triangle OBC is achieved when it is a regular triangle.

Hence, the second assertion of the theorem holds. ■

IV. Practice

We illustrate, with real street patterns, which shapes have larger isoperimetric ratio than the threshold $12\sqrt{3}$. The black pixels in Fig. 3, except for those consisting the canvas frame, are center lines of roads.

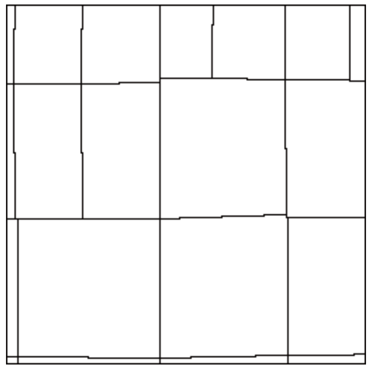

Fig. 3. Kyoto (1), before processing

We do the following procedure. Here, a block denotes a region surrounded by a closed curve.

STEP 1. Remove all blind alleys.
STEP 2. For each block
  if (the isoperimetric ratio $\geq 12\sqrt{3}$)
  { Paint out it black. }

STEP 1 is just for making the issue simpler. In the case of Fig. 3, we have no blind alleys. The result of the above procedure is Figure 4.

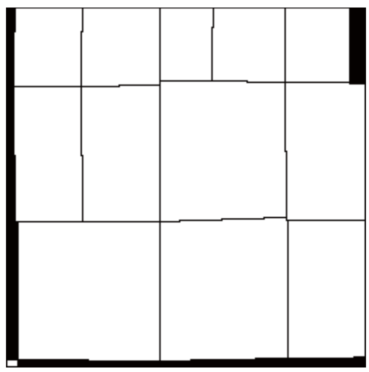

Fig. 4. Kyoto (1), after processing (8.6%)

The percentage in the caption (8.6%) denotes the ratio of black pixels in the canvas. In the same way, we apply the procedure to some figures. All the figures of street patterns in this section are drawn on the same scale. Fig. 3 and Fig. 5 are clipped out of Kyoto [5]. Other figures of odd numbers $\geq$ 7 are clipped out of Tokyo [4]. Among them, Fig. 7, Fig. 9 and Fig. 11 are clipped out of areas which had land readjustments.

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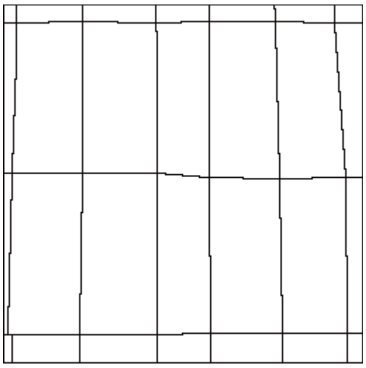

Fig. 5. Kyoto (2), before processing

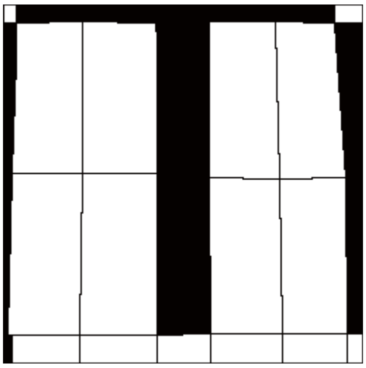

Fig. 6. Kyoto (2), after processing (27%)

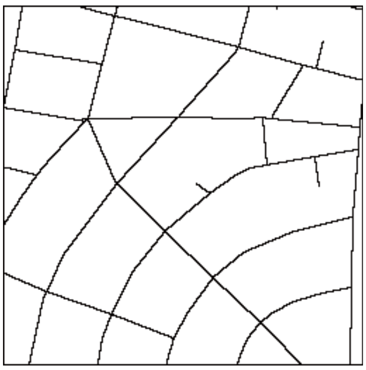

Fig. 7. Tokyo (1), before processing

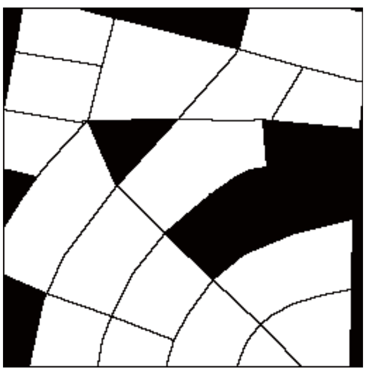

Fig. 8. Tokyo (1), after processing (27%)

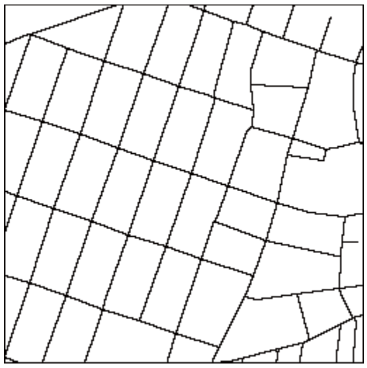

Fig. 9. Tokyo (2), before processing

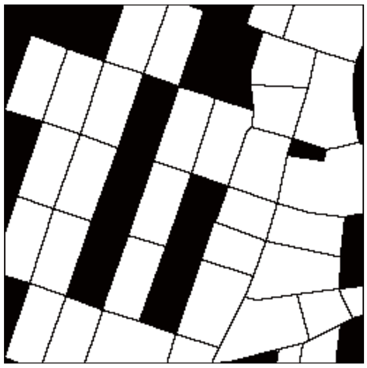

Fig. 10. Tokyo (2), after processing (32%)

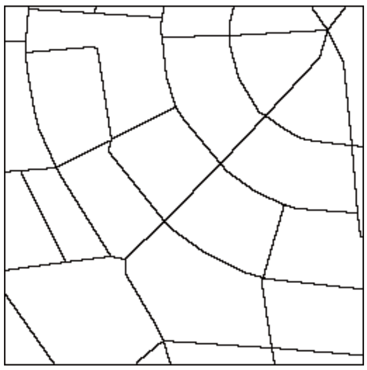

Fig. 11. Tokyo (3), before processing

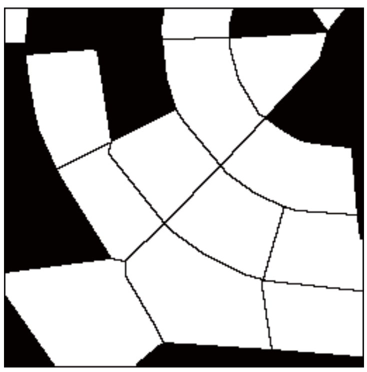

Fig. 12. Tokyo (3), after processing (34%)

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Fig. 13. Tokyo (4), before processing

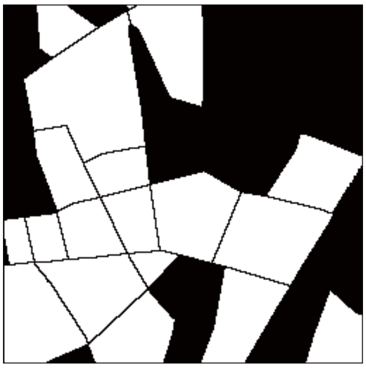

Fig. 14. Tokyo (4), after processing (45%)

Fig. 15. Tokyo (5), before processing

Fig. 16. Tokyo (5), after processing (51%)

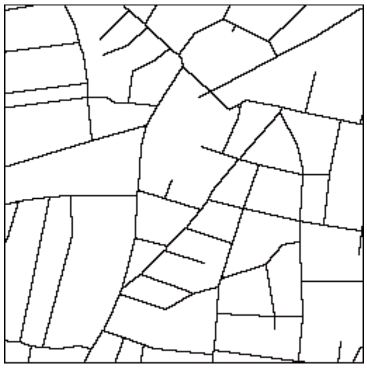

Fig. 17. Tokyo (6), before processing

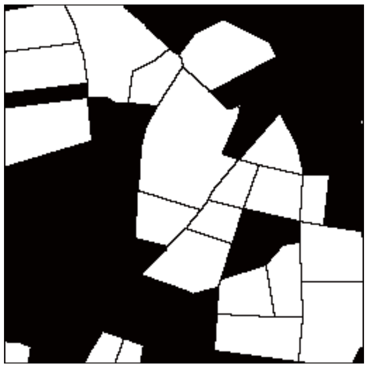

Fig. 18. Tokyo (6), after processing (53%)

Fig. 19. Tokyo (7), before processing

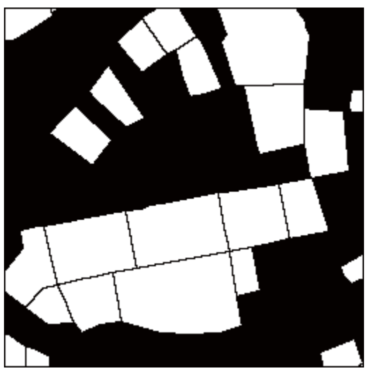

Fig. 20. Tokyo (7), after processing (60%)

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Fig. 21. Tokyo (8), before processing

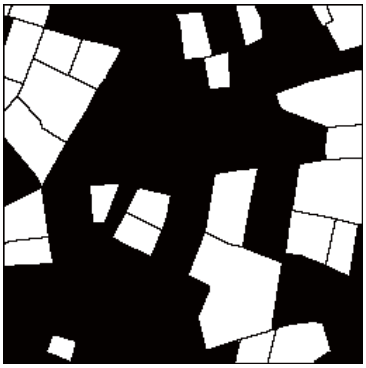

Fig. 22. Tokyo (8), after processing (65%)

V. Discussion

Suppose we have a closed curve γ approximately equal to a polygon. If γ has an isoperimetric ratio greater than or equal to $12\sqrt{3}$, we may conclude that either (Case 1) γ is not approximately equal to a convex quadrilateral, or (Case 2) γ is approximately equal to a convex quadrilateral such that we can reduce its isoperimetric ratio via an invertible affine transformation.

Thus, informally speaking, if a shape γ has an isoperimetric ratio greater than or equal to $12\sqrt{3}$, we may conclude that γ is not (a shape approximately equal to) a succinct quadrilateral.

On the other hand, if γ has an isoperimetric ratio less than $12\sqrt{3}$, we do not get a definite conclusion on γ by this method.

As examples of shapes approximately equal to polygons, we observed some real street patterns in the former section.

The percentage of black pixels in each processed square region is approximately equal to the probability of a randomly chosen point, from the square region, being included in a street block whose isoperimetric ratio is greater than or equal to $12\sqrt{3}$. In the examples of the former section, the range of percentages is 8.6%–65%.

A prospective application is to find the percentage of black pixels for a town or a ward. Although a town does not have a boundary of a square shape in general, in the same way as the examples of the former section, we can find the percentage for a town, and we can visualize the percentage in a monochrome picture. An administrative organ would utilize the percentage as an index (naturally, with other indices) showing priority of land readjustment.

VI. Conclusions and Summary

The isoperimetric ratio denotes the ratio of the square of the perimeter to the area. Note that, in our setting, the isoperimetric ratio achieved by a circle is not the maximum but the minimum.

In this paper, we observed an elementary but interesting theorem on an upper bound of isoperimetric ratio; Any convex quadrilateral can be compressed, via an invertible affine transformation, to a quadrilateral whose isoperimetric ratio is less than that of a regular triangle. And, this result is optimal.

In the discussion session, we proposed to use the isoperimetric ratio of a regular triangle as a threshold when we judge that a given polygon is not a succinct quadrilateral. Based on examples clipped out of real street patterns, we conclude that the percentage of black pixels has a prospective application to complexity measure of street patterns, where the percentage equals to the probability of a randomly chosen point being included in a block that exceeds the threshold. In addition, it is a merit of our method that the complexity measure is visualized by a monochrome picture.

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References

[1] Attneave, F. and Arnoult, M.D.: The quantitative study of shape and pattern perception. *Psychological Bulletin*, **53**, pp.452–471 (1956).

[2] Attneave, F.: Physical determinants of the judged complexity of shapes. *Journal of Experimental Psychology*, **53**, pp.221–227 (1957).

[3] Blåsjö, V.: The isoperimetric problem. *The Amer. Math. Monthly*, **112**, pp. 526-566 (2005).

[4] The Geospatial Information Authority of Japan: Digital Map 2500 (Spatial Data Framework) CD-ROM KANTO-3 (2003).

[5] The Geospatial Information Authority of Japan: Digital Map 2500 (Spatial Data Framework) CD-ROM KINKI-1 (2006).

[6] Osserman, R.: The Isoperimetric inquality. *Bull. Amer. Math. Soc.*, **86**, pp. 1182–1238 (1978).

[7] Sinclair, D. and Blake, A.: Isoperimetric normalization of planar curves. *IEEE Transactions on Pattern Analysis and Intelligence* , **16**, pp 769–777 (1994).