On the Mechanical and Elastic Properties of Anisotropic Engineering Materials Based upon Harmonic Representations

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***Abstract*—This paper presents a new aspect of harmonic decomposition method for elastic constant tensor of various anisotropic materials. Some misprints found in literature are corrected. This procedure derived here, is applied to anisotropic engineering materials possessing different elastic symmetries. In order to gain insight about these applications, numerical illustrations are presented for anisotropic engineering materials. A new description of norm in terms of harmonic tensors is introduced instead of well-known form of norm. This case is a significant innovation for specifying the anisotropy degree of any engineering materials to have opinion about the mechanical and elastic properties of these materials.**

***Index Terms*—elasticconstanttensor,harmonic decomposition method, anisotropic engineering materials, norm concept, anisotropy degree.**

I. Introduction

A MATERIAL is said to be isotropic if its mechanical and elastic properties are the same in all directions. Anisotropic materials such as composites become the material of choice in a variety of engineering applications in the last century. Many materials are anisotropic and inhomogeneous due to the varying composition of their constituents. Polycrystalline materials generally show an elastic anisotropy due to texture and the anisotropy of single crystallites. Those materials are used in many applications in industry. Everyday passed, the number of anisotropic engineering materials is increasing by the addition of man-made anisotropic single crystals and technologically developed materials. In order to understand the physical properties of those materials, use of tensors by decomposing them is significant. Tensors are the most important mathematical entities to describe direction dependent physical properties of solids and the tensor components characterizing physical properties which must be specified without reference to any coordinate system.

The anisotropic form of Hooke's law in linear elasticity is often written in indicial notation as

$$\sigma_{ij=}C_{ijkl}\varepsilon_{kl} \qquad (1)$$

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Where σ_{ij} are components of stress tensor, ε_{kl} are components of infinitesimal strain tensor and C_{ijkl} are the components of elasticity tensor and satisfy three important symmetry restrictions. These are

$$C_{ijkl}=C_{jikl} \quad C_{ijkl}=C_{ijlk} \quad C_{ijkl}=C_{klij} \qquad (2)$$

which follow from the symmetry of the stress tensor, the symmetry of the strain tensor and the elastic strain energy. These restrictions reduce the number of independent elastic constants C_{ijkl} from 81 to 21 (see for instance;[1]). The indices are abbreviated according to the replacement rule given in the following table:

TABLE I
UNITS FOR MAGNETIC PROPERTIES

Four index notation	11	22	33	23,32	13,31	12,21
Double index notation	1	2	3	4	5	6

There is a close relationship between irreducible and harmonic decomposition methods. Like irreducible decomposition, two scalars, two deviators and the nonor part are obtained in harmonic decomposition method in this work. In literature, irreducible decomposition method had been studied extensively. To name some; [2], [3] and quoted by [4]- [9] realized decomposition in which elastic constant tensor was decomposed into its irreducible parts. Reference [10] derived certain results for the irreducible tensors in their natural form. Reference [11] followed the technique of [10] and gave the reduction of a fourth rank cartesian tensor into irreducible parts under the three-dimensional rotation group.

There are also other works for harmonic decomposition of tensors and harmonic representations. For instance, reference [12] proposed a representation of elastic constant tensor in terms of harmonic tensors. These are based on an isomorphism between the space of homogeneous harmonic polynomials of degree q and the space of totally symmetric tensors of order q. Furthermore according to [13], elastic constant tensor was decomposed with respect to general linear group and then orthogonal group O(3). Reference [14] followed [12] and developed the method.

One of the aims of this work is to represent the elastic constant tensor in terms of harmonic tensors by developing the existing theory given in literature. This is done by decomposing the elastic constant tensor explicitly. For the first time in the literature, this decomposition process is applied to elastic constant of materials exhibiting anisotropic

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elastic symmetries such as isotropy, transversely isotropy and orthotropy. Since critical and important engineering materials possess these symmetries. Numerical examples for them are also illustrated.

Another aim is to establish a general norm concept and norm ratios for harmonic representation of anisotropic materials to investigate the elastic and mechanical properties of these materials by determining the anisotropy degree of each material.

II. Harmonic Decomposition of Elastic Constant Tensor

In harmonic decomposition, the action of $SO(3)$ on a vector space is said to be irreducible when there are no proper invariant subspaces. It is deduced that there is a decomposition of the space of elastic constant tensors

($\mathbf{E}$ la) into a direct sum of orthogonal subspaces on which the action of $SO(3)$ is irreducible. An important theorem of group representation theory can be summarized as: every space on which the group of rotations acts irreducibly is isomorphic through an $SO(3)$ -invariant map with an appropriate space of harmonic polynomials. In view of isomorphism, there is a decomposition of $\mathbf{E}$ la into a direct sum of spaces of harmonic tensors. (See, for instance; [5]) Besides, there is an $SO(3)$ -invariant isomorphism between $\mathbf{E}$ la and the direct sum R $\oplus$ R $\oplus$ Dev $\oplus$ Dev $\oplus$ $\mathbf{H}$ rm.

Elastic constant tensor with fourth rank in three dimensions, can be written in the following form:

$$C=S+A \tag{3}$$

Symmetric part represented by S, is expressed as

$$\boldsymbol{S}_{ijkl}=\tfrac{1}{3}\left(C_{ijkl}+C_{iklj}+C_{iljk}\right) \tag{4}$$

Asymmetric part represented by $\boldsymbol{A}$, is expressed as

$$\boldsymbol{A}_{ijkl}=C_{ijkl}-\boldsymbol{S}_{ijkl}=\tfrac{2}{3}C_{ijkl}-\tfrac{1}{3}C_{iklj}-\tfrac{1}{3}C_{iljk} \tag{5}$$

The total symmetric part can be rewritten in terms of H and H_{ij} .

$$\boldsymbol{S}_{ijkl}=H_{ijkl}+\left[\delta_{ij}H_{kl}+\delta_{kl}H_{ij}+\delta_{ik}H_{jl}+\delta_{jl}H_{ik}+\delta_{il}H_{jk}+\delta_{jk}H_{il}\right]+H\left(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right) \tag{6}$$

Similarly, the total asymmetric part can be obtained in terms of h and h_{ij} as

$$\boldsymbol{A}_{ijkl}=\delta_{ij}h_{kl}+\delta_{kl}h_{ij}-\tfrac{1}{2}\delta_{ik}h_{jl}-\tfrac{1}{2}\delta_{jl}h_{ik}-\tfrac{1}{2}\delta_{il}h_{jk}-\tfrac{1}{2}\delta_{jk}h_{il}+h(\delta_{ij}\delta_{kl}-\tfrac{1}{2}\delta_{ik}h_{jl}-\tfrac{1}{2}\delta_{il}h_{jk}) \tag{7}$$

By adding (6) and (7) gives the harmonic representation of elastic constant tensor for anisotropic materials possessing triclinic symmetry, which is

$$C_{ijkl}=H_{ijkl}+[\delta_{ij}H_{kl}+\delta_{kl}H_{ij}+\delta_{ik}H_{jl}+\delta_{jl}H_{ik}+\delta_{il}H_{jk}+\delta_{jk}H_{il}]+H[\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}]+\delta_{ij}h_{kl}+\delta_{kl}h_{ij}-\frac{1}{2}\delta_{ik}h_{jl}-\frac{1}{2}\delta_{jl}h_{ik}-\frac{1}{2}\delta_{il}h_{jk}-\frac{1}{2}\delta_{jk}h_{il}+h(\delta_{ij}\delta_{kl}-\frac{1}{2}\delta_{ik}\delta_{jl}-\frac{1}{2}\delta_{il}\delta_{jk}). \tag{8}$$

where $H=\frac{1}{45}(C_{ppqq}+2C_{pqpq}),\quad h=\frac{1}{9}(C_{ppqq}-C_{pqpq}),$

$$H_{ij}=\frac{1}{21}\left[C_{ijkk}-\frac{1}{3}C_{jjkk}\delta_{ij}+2\left(C_{ikjk}-\frac{1}{3}C_{jkjk}\delta_{ij}\right)\right],$$

$$h_{ij}=\frac{2}{3}(C_{ijpp}-C_{ipjp})-\frac{2}{9}\delta_{ij}(C_{rrpp}-C_{rprp}).$$

The total scalar (isotropic) part (denoted by) of an elastic constant tensor (obtained from (8)) is

$$S=\frac{\delta_{ij}\delta_{kl}}{15}(2C_{ppqq}-C_{pqpq})+\frac{(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})}{30}(3C_{pqpq}-C_{ppqq}). \tag{9}$$

Furthermore the total deviatoric part or second rank traceless tensor is composed of summation of the linear combination of second order tensors (H_{ij} and h_{ij}) given in (8), which is (denoted as D)

$$D=\frac{1}{7}\delta_{kl}(5C_{ijpp}-4C_{ipjp})+\frac{1}{7}\delta_{ij}(5C_{klpp}-4C_{kplp})+\frac{1}{7}\delta_{ik}(3C_{jplp}-2C_{jlpp})+\frac{1}{7}\delta_{il}(3C_{jpkp}-2C_{jkpp})+\frac{1}{7}\delta_{jl}(3C_{ipkp}-2C_{ikpp})+\frac{1}{7}\delta_{jk}(3C_{iplp}-2C_{ilpp})+\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)\left(\frac{4}{21}C_{pprr}-\frac{2}{7}C_{rprp}\right)+\delta_{ij}\delta_{kl}(\frac{8}{21}C_{rprp}-\frac{10}{21}C_{pprr}). \tag{10}$$

From (8), harmonic part is

$$H=(C_{ijkl}+C_{iklj}+C_{iljk})/3-[(C_{ijmm}+2C_{imjm})\delta_{kl}+(C_{klmm}+2C_{kmlm})\delta_{ij}+(C_{ikmm}+2C_{imkm})\delta_{jl}+(C_{jlmm}+2C_{jmlm})\delta_{ik}+(C_{ilmm}+2C_{imlm})\delta_{jk}+(C_{jkmm}+C_{jmkm})\delta_{il}]/21+(C_{ppmm}+2C_{pmpm})(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})/105 \tag{11}$$

Moreover, the results for elastic constant tensor decomposition are given by [2], [3] and quoted by [4]-[9] in which elastic constant tensor is decomposed into two scalar, two deviatoric and one nonor parts. These decompositions are the same as harmonic decomposition method since scalar, deviatoric and nonor parts are common and they are identical with those obtained from harmonic decomposition method, only difference here is notations used for scalar, traceless symmetric second rank tensors and nonor parts.

According to these studies, decomposition of elastic constant tensor for anisotropic materials possessing triclinic symmetry is expressed as follows:

$$C_{ijkl}=\frac{1}{15}(2C_{ppqq}-C_{pqpq})\delta_{ij}\delta_{kl}+\frac{1}{30}(3C_{pqpq}-C_{ppqq})(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\delta_{ij}A_{kl}+\delta_{kl}A_{ij}+\delta_{ik}B_{jl}+\delta_{jl}B_{ik}+\delta_{il}B_{jk}+\delta_{jk}B_{il}+Z_{ijkl}, \tag{12}$$

From equation (8), total scalar part is

$$\frac{1}{15}(2C_{ppqq}-C_{pqpq})\delta_{ij}\delta_{kl}+\frac{1}{30}(3C_{pqpq}-C_{ppqq})(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}). \tag{13}$$

The total deviatoric part is

$$\delta_{ij}A_{kl}+\delta_{kl}A_{ij}+\delta_{ik}B_{jl}+\delta_{jl}B_{ik}+\delta_{il}B_{jk}+\delta_{jk}B_{il}. \tag{14}$$

The components of deviatoric part are

$$A_{ij}=(15C_{ijkk}-12C_{ikjk}-5\delta_{ij}C_{ppkk}+4\delta_{ij}C_{pkpk})/21, \tag{15}$$

$$B_{ij}=(-6C_{ijkk}+9C_{ikjk}+2\delta_{ij}C_{ppkk}-3\delta_{ij}C_{pkpk})/21. \tag{16}$$

Finally harmonic part is the same as $\boldsymbol{H_{ijkl}}$ given in (11). Furthermore the decomposition of elastic constant tensor given in [5] and [6] contain misprints in components of scalar part and total deviatoric part. In (13) and (14), these parts are corrected.

A. For Isotropic Materials

There are two independent components for isotropic elastic constant tensor. So it must have two harmonic decomposed parts. By considering the symmetry conditions in (2) and matrix structure of isotropic symmetry, (9) is rearranged. In matrix form, the first and second scalar parts are represented as:

$$S_{mn}^1 = \begin{bmatrix} \frac{(3C_{11}+12C_{12}-6C_{44})}{15} & \frac{(3C_{11}+12C_{12}-6C_{44})}{15} & \frac{(3C_{11}+12C_{12}-6C_{44})}{15} & 0 & 0 & 0 \\ \frac{(3C_{11}+12C_{12}-6C_{44})}{15} & \frac{(3C_{11}+12C_{12}-6C_{44})}{15} & \frac{(3C_{11}+12C_{12}-6C_{44})}{15} & 0 & 0 & 0 \\ \frac{(3C_{11}+12C_{12}-6C_{44})}{15} & \frac{(3C_{11}+12C_{12}-6C_{44})}{15} & \frac{(3C_{11}+12C_{12}-6C_{44})}{15} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$S_{mn}^2 = \begin{bmatrix} \frac{a}{15} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{a}{15} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{a}{15} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{a}{30} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{a}{30} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{a}{30} \end{bmatrix} \quad (17)$$

Where $a = 6C_{11} - 6C_{12} + 18C_{44}$

The sum of the parts in (17) gives total scalar part for isotropic materials.

B. For Transversely Isotropic Materials

There are five irreducible parts for transversely isotropic materials which are two scalars and two deviators and a harmonic part. Since a transversely isotropic material has five independent components of elastic constant tensor. In matrix form, the first and second scalar parts are represented as:

$$S_{mn}^1 = \begin{bmatrix} \frac{a}{15} & \frac{a}{15} & \frac{a}{15} & 0 & 0 & 0 \\ \frac{a}{15} & \frac{a}{15} & \frac{a}{15} & 0 & 0 & 0 \\ \frac{a}{15} & \frac{a}{15} & \frac{a}{15} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, S_{mn}^2 = \begin{bmatrix} \frac{b}{15} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{b}{15} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{b}{15} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{b}{30} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{b}{30} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{b}{30} \end{bmatrix} \quad (18)$$

Where $a = C_{11} + C_{33} + 5C_{12} + 8C_{13} - 4C_{44}$, $b = 7C_{11} - 5C_{12} - 4C_{13} + 2C_{33} + 12C_{44}$

In matrix form, the first and second deviatoric parts are denoted as follows

$$D_{mn}^1 = \begin{bmatrix} \frac{4a}{15} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{4b}{15} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4c}{15} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{d}{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{e}{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{f}{15} \end{bmatrix}, \quad (19)$$

Where $a = \left(\frac{3C_{11}-C_{12}}{2} - C_{33} - 3C_{44}\right)$,

$b = \left(\frac{3C_{11}-C_{12}}{2} - C_{33} - C_{44}\right)$,

$c = (-3C_{11} + C_{12} + 2C_{33} + 2C_{44})$,

$d = \left(\frac{-3C_{11}+C_{12}}{2} + C_{33} + C_{44}\right)$,

$e = \left(\frac{-3C_{11}+C_{12}}{2} + C_{33} + C_{44}\right)$,

$f = (3C_{11} - C_{12} - 2C_{33} - 3C_{44})$,

$$D_{mn}^2 = \begin{bmatrix} h - \frac{2b}{105} & n - \frac{10b}{105} & o - \frac{10b}{105} & 0 & 0 & 0 \\ n - \frac{10b}{105} & i - \frac{2b}{105} & p - \frac{10b}{105} & 0 & 0 & 0 \\ o - \frac{10b}{105} & p - \frac{10b}{105} & j - \frac{2b}{105} & 0 & 0 & 0 \\ 0 & 0 & 0 & k + \frac{4b}{105} & 0 & 0 \\ 0 & 0 & 0 & 0 & l + \frac{4b}{105} & 0 \\ 0 & 0 & 0 & 0 & 0 & m + \frac{4b}{105} \end{bmatrix} \quad (20)$$

Where $b=-2C_{11}+C_{33}-16C_{44}+6C_{12}+20C_{13}$,
$h=2/7(-C_{11}+7C_{12}+5C_{13}-4C_{44})-8/35(-C_{11}+7C_{12}+5C_{13}-4C_{44})$,
$i=2/7(-C_{11}+3C_{12}+5C_{13}-4C_{44})-8/35(-C_{11}+3C_{12}+5C_{13}-4C_{44})$,
$j=2/7(C_{33}+10C_{13}-8C_{44})-8/35(C_{33}+10C_{13}-8C_{44})$,
$k=-2/35(-C_{11}+C_{33}+3C_{12}+15C_{13}-12C_{44})$,
$l=-2/35(-C_{11}+C_{33}+3C_{12}+15C_{13}-12C_{44})$,
$m=-2/35(-2C_{11}+6C_{12}+10C_{13}-8C_{44})$,
$n=1/7(-2C_{11}+6C_{12}+10C_{13}-8C_{44})$,
$o=1/7(-3C_{11}+7C_{12}+15C_{13}+C_{33}-12C_{44})$,
$p=1/7(-C_{11}+7C_{12}+15C_{13}+C_{33}-12C_{44})$.

The harmonic part for transversely isotropic materials in harmonic decomposition method.

$$H_{ijkl} = \frac{e}{35}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \delta_{ij}\delta_{kl} - 5\delta_{3j}\delta_{3i}\delta_{kl} - 5\delta_{3k}\delta_{3l}\delta_{ij} - 5\delta_{3i}\delta_{3k}\delta_{jl} - 5\delta_{3j}\delta_{3k}\delta_{il} - 5\delta_{3i}\delta_{3l}\delta_{jk}\right) \frac{e}{35}\left(-5\delta_{3j}\delta_{3l}\delta_{ik} + 35\delta_{3i}\delta_{3j}\delta_{3k}\delta_{3l}\right), \quad (21)$$

Where $a = 2S_{11} + 2S_{12} + 4S_{13} + S_{33}$, $b = 7S_{11} - 5S_{12} + 2S_{33} + 12S_{44} - 4S_{13}$, $c = 3S_{11} - 2S_{33} - 2S_{44} - S_{12}$, $d = 7S_{12} - S_{33} - S_{11} - 5S_{13} + 4S_{44}$ and $e = S_{11} + S_{33} - 2S_{13} - 4S_{44}$.

By summing up those five parts, elastic constant tensor for transversely isotropic material is obtained.

C. For Orthotropic Materials

There are nine independent components of elastic constant tensor. Since the maximum number of decomposed parts is five in harmonic method. There are five harmonic parts which are two scalars, two deviators and one harmonic parts for these type of materials. In matrix form, the first and second scalar parts are denoted as:

$$S_{mn}^1 = \begin{bmatrix} \frac{a}{15} & \frac{a}{15} & \frac{a}{15} & 0 & 0 & 0 \\ \frac{a}{15} & \frac{a}{15} & \frac{a}{15} & 0 & 0 & 0 \\ \frac{a}{15} & \frac{a}{15} & \frac{a}{15} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, S_{mn}^2 = \begin{bmatrix} \frac{b}{15} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{b}{15} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{b}{15} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{b}{30} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{b}{30} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{b}{30} \end{bmatrix} \quad (22)$$

Where $a = C_{11} + C_{22} + C_{33} + 4C_{12} + 4C_{13} + 4C_{23} - 2C_{44} - 2C_{55} - 2C_{66}$, $b = 2C_{11} + 2C_{22} + 2C_{33} + 6C_{44} + 6C_{55} + 6C_{66} - 2C_{12} - 2C_{13} - 2C_{23}$

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In matrix form, the first and second deviatoric parts are obtained as:

$$D_{mn}^{1}=\begin{bmatrix} \frac{4a}{15} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{4b}{15} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4c}{15} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{d}{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{e}{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{f}{15} \end{bmatrix}, \tag{23}$$

Where $a=(2C_{11}-C_{22}-C_{33}-2C_{44}+C_{55}+C_{66})$,

$b=(-C_{11}+2C_{22}-C_{33}+C_{44}-2C_{55}+C_{66})$,

$c=(-C_{11}-C_{22}+2C_{33}+C_{44}+C_{55}-2C_{66})$,

$d=(C_{22}+C_{33}-C_{55}+2C_{44}-C_{66}-2C_{11})$,

$e=(C_{11}-2C_{22}+C_{33}-C_{44}+2C_{55}-C_{66})$,

$f=(C_{11}+C_{22}-2C_{33}-C_{44}-C_{55}+2C_{66})$.

$$D_{mn}^{2}=\begin{bmatrix} h-\frac{2b}{105} & n-\frac{10b}{105} & o-\frac{10b}{105} & 0 & 0 & 0 \\ n-\frac{10b}{105} & i-\frac{2b}{105} & p-\frac{10b}{105} & 0 & 0 & 0 \\ o-\frac{10b}{105} & p-\frac{10b}{105} & j-\frac{2b}{105} & 0 & 0 & 0 \\ 0 & 0 & 0 & k+\frac{4b}{105} & 0 & 0 \\ 0 & 0 & 0 & 0 & l+\frac{4b}{105} & 0 \\ 0 & 0 & 0 & 0 & 0 & m+\frac{4b}{105} \end{bmatrix} \tag{24}$$

Where $b=C_{11}+C_{22}+C_{33}-8C_{44}-8C_{55}-8C_{66}+10C_{12}+10C_{13}+10C_{23}$,
$h=2/7(C_{11}+5C_{12}+5C_{13}-4C_{66}-4C_{55})-8/35(C_{11}+5C_{12}+5C_{13}-4C_{66}-4C_{55})$,
$i=2/7(C_{22}+5C_{12}+5C_{23}-4C_{66}-4C_{44})-8/35(C_{22}+5C_{12}+5C_{23}-4C_{66}-4C_{44})$,
$j=2/7(C_{33}+5C_{13}+5C_{23}-4C_{44}-4C_{55})-8/35\ (C_{33}+5C_{13}+5C_{23}-4C_{44}-4C_{55})$,
$k=-2/35(C_{22}+C_{33}+5C_{12}+5C_{13}+10C_{23}-8C_{44}-4C_{55}-4C_{66})$,
$l=-2/35(C_{11}+C_{33}+5C_{12}+10C_{13}+5C_{23}-4C_{44}-8C_{55}-4C_{66})$,
$m=-2/35.*(C_{11}+C_{22}+10C_{12}+5C_{13}+5C_{23}-4C_{44}-4C_{55}-8C_{66})$,
$n=1/7(C_{11}+10C_{12}+5C_{13}+5C_{23}+C_{22}-8C_{66}-4C_{44}-4C_{55})$,
$o=1/7\ (C_{11}+5C_{12}+10C_{13}+C_{33}+5C_{23}-4C_{66}-4C_{44}-8C_{55})$,
$p=1/7\ (C_{22}+5C_{12}+5C_{13}+10C_{23}+C_{33}-4C_{66}-8C_{44}-4C_{55})$.

The corresponding harmonic part is

$$\begin{aligned} H_{ijkl}=&\tfrac{1}{35}\Big(-(n+\imath)\big(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\big)+(3n+2\imath)\delta_{ij}\delta_{kl}+\\ &(\imath-g)\big(\delta_{1i}\delta_{1k}\delta_{jl}+\delta_{1j}\delta_{1k}\delta_{il}+\delta_{1i}\delta_{1l}\delta_{jk}+\delta_{1j}\delta_{1l}\delta_{ik}\big)\Big)\\ &+\tfrac{1}{35}\Big((\imath-h)\big(\delta_{2i}\delta_{2k}\delta_{jl}+\delta_{2j}\delta_{2k}\delta_{il}+\delta_{2i}\delta_{2l}\delta_{jk}+\delta_{2j}\delta_{2l}\delta_{ik}\big)+\\ &(2m-3n+3g)\big(\delta_{2l}\delta_{2k}\delta_{ij}+\delta_{2i}\delta_{2j}\delta_{kl}\big)\Big)+\\ &\tfrac{1}{35}\Big((2l-3n+3h)\big(\delta_{1l}\delta_{1k}\delta_{ij}+\delta_{1i}\delta_{1j}\delta_{kl}\big)+35(S_{12}+S_{33}+\\ &2S_{66}-S_{23}-S_{13}-2S_{44}-2S_{55})\Big)+\\ &\tfrac{1}{35}\Big(\big(\delta_{1i}\delta_{1j}\delta_{2k}\delta_{2l}+\delta_{2i}\delta_{2j}\delta_{1k}\delta_{1l}\big)+35(S_{11}+S_{33}-2S_{13}-\\ &4S_{55})\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l}\Big)+\\ &(S_{33}+S_{22}-2S_{23}-4S_{44})\delta_{2i}\delta_{2j}\delta_{2k}\delta_{2l}, \end{aligned} \tag{25}$$

where $a=S_{11}+S_{22}+S_{33}+2S_{12}+2S_{13}+2S_{23}$,
$b=S_{11}+S_{22}+S_{33}-S_{12}-S_{13}-S_{23}+3S_{44}+3S_{55}+3S_{66}$,
$c=2S_{11}-S_{22}-S_{33}-2S_{44}+S_{55}+S_{66}$, $d=2S_{22}-S_{11}-S_{33}+S_{44}-2S_{55}+S_{66}$, $e=2S_{11}-S_{22}-S_{33}+5S_{12}+5S_{13}-10S_{23}+8S_{44}-4S_{55}-4S_{66}$, $f=2S_{22}-S_{11}-S_{33}+5S_{12}-10S_{13}+5S_{23}-5S_{44}+8S_{55}-4S_{66}$, $g=S_{11}-4S_{22}-4S_{33}-S_{12}-S_{13}+9S_{23}+18S_{44}-2S_{55}-2S_{66}$, $h=S_{22}-4S_{11}-4S_{33}-S_{12}+9S_{13}-S_{23}-2S_{44}+18S_{55}-2S_{66}$,
$l=S_{33}-4S_{11}-4S_{22}+9S_{12}-S_{13}-S_{23}-2S_{44}-2S_{55}+18S_{66}$, $j=-(c+d)$, $k=-(e+f)$, $l=-(h+\imath)$,
$m=-(j+\imath)$, $n=-(j+h)$

Elastic constant tensor for orthorhombic material is obtained by adding those five parts.

III. Numerical Analyses

This method can be carried out for various anisotropic materials such that fiber-reinforced composites, polysytrene and olivine, examples for isotropic, transversely isotropic and orthotropic materials. Under specific couplings of the elastic constants of orthotropic media, a very important family of orthotropic materials degenerates into the class of either transversely isotropic or isotropic media. Most of the engineering composites, especially fiber-reinforced are transversely isotropic. Decomposition process is performed by simple computer programs written in MATLAB in order to ease the computations. All units are in Gpa(10^{10} dyn $\cdot$ cm $^{-2}$) .

A. Fiber-Reinforced Composites

In recent years fiber reinforced composite materials have been paid considerable attention due to the search for materials of light weight, great strength and stiffness. Consequently the determination of their mechanical properties i.e. stiffness effect, becomes important.

Typical numerical examples for E-glass/epoxy fiber-reinforced composite materials shown for two cases; the first case is a homogeneous material in which the properties of the fiber and resin are equal [15] and given data are

$$\nu_f=0.2 \tag{26}$$

$$\nu_m=0.2 \tag{27}$$

$$G_f=13.79\ Gpa \tag{28}$$

$$G_m=13.79\ Gpa \tag{29}$$

where ν_f,ν_m are Poisson's ratio of fibers and matrix respectively and G_f,G_m are the shear modulus of fibers and matrix respectively. The structure of elastic constant tensor for isotropic case is:

$$C_{pq}=\begin{bmatrix} 73.567 & 18.409 & 18.409 & 0 & 0 & 0 \\ 18.409 & 73.567 & 18.409 & 0 & 0 & 0 \\ 18.409 & 18.409 & 73.567 & 0 & 0 & 0 \\ 0 & 0 & 0 & 27.58 & 0 & 0 \\ 0 & 0 & 0 & 0 & 27.58 & 0 \\ 0 & 0 & 0 & 0 & 0 & 27.58 \end{bmatrix} \tag{30}$$

<u>For isotropic case, harmonic representation of elastic constant tensor:</u>

$$=18.4\times\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}+\begin{bmatrix} 55.16 & 0 & 0 & 0 & 0 & 0 \\ 0 & 55.16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 55.16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 27.58 & 0 & 0 \\ 0 & 0 & 0 & 0 & 27.58 & 0 \\ 0 & 0 & 0 & 0 & 0 & 27.58 \end{bmatrix} \tag{31}$$

The second case is a non-homogeneous material. For a fiber-reinforced composite material in transversely isotropic media; where the fibers and matrix are not equal, volume of fibers are approximately 63%.

$$\nu_f=0.2 \tag{32}$$

$$\nu_m=0.23 \tag{33}$$

$$G_f=28.75\ Gpa \tag{34}$$

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$$G_m = 1.28\ Gpa \tag{35}$$

The structure of elastic constant tensor for transversely isotropic case is:

$$C_{pq} = \begin{bmatrix} 15.44 & 6.14 & 5.24 & 0 & 0 & 0 \\ 6.14 & 15.44 & 5.24 & 0 & 0 & 0 \\ 5.24 & 5.24 & 47.30 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4.75 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4.75 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.65 \end{bmatrix} \tag{36}$$

<u>For transversely isotropic case, harmonic representation of elastic constant tensor:</u>

$$= \begin{bmatrix} 7.757 & 7.757 & 7.757 & 0 & 0 & 0 \\ 7.757 & 7.757 & 7.757 & 0 & 0 & 0 \\ 7.757 & 7.757 & 7.757 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 13.868 & 0 & 0 & 0 & 0 & 0 \\ 0 & 13.868 & 0 & 0 & 0 & 0 \\ 0 & 0 & 13.868 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6.934 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6.934 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6.934 \end{bmatrix} +$$

$$\begin{bmatrix} -8.523 & 0 & 0 & 0 & 0 & 0 \\ 0 & -8.523 & 0 & 0 & 0 & 0 \\ 0 & 0 & 17.045 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.131 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.131 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4.261 \end{bmatrix} + \begin{bmatrix} -0.514 & -2.568 & 1.284 & 0 & 0 & 0 \\ -2.568 & -0.514 & 1.284 & 0 & 0 & 0 \\ 1.284 & 1.284 & 1.027 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.514 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.514 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.027 \end{bmatrix} +$$

$$\begin{bmatrix} 2.865 & 0.935 & -3.799 & 0 & 0 & 0 \\ 0.935 & 2.865 & -3.799 & 0 & 0 & 0 \\ -3.799 & -3.799 & 7.599 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3.799 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.799 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.935 \end{bmatrix} \tag{37}$$

B. *Polystyrene [16]*

Polystyrene is an example for transversely isotropic material. The following matrix exhibits the matrix structure of elastic constant tensor for polystyrene.

$$C_{ij} = \begin{bmatrix} 5.20 & 2.75 & 2.75 & 0 & 0 & 0 \\ 2.75 & 5.20 & 2.75 & 0 & 0 & 0 \\ 2.75 & 2.75 & 5.70 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.30 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.30 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.225 \end{bmatrix} \tag{38}$$

<u>Harmonic representation of elastic constant tensor of polystyrene</u>:

$$= \begin{bmatrix} 2.7633 & 2.7633 & 2.7633 & 0 & 0 & 0 \\ 2.7633 & 2.7633 & 2.7633 & 0 & 0 & 0 \\ 2.7633 & 2.7633 & 2.7633 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2.577 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2.577 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2.577 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.288 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.288 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.288 \end{bmatrix} +$$

$$\begin{bmatrix} -0.1533 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1533 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.3067 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0383 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0383 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.0767 \end{bmatrix} +$$

$$\begin{bmatrix} -0.0038 & -0.0190 & 0.0095 & 0 & 0 & 0 \\ -0.0190 & -0.0038 & 0.0095 & 0 & 0 & 0 \\ 0.0095 & 0.0095 & 0.0076 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0038 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0038 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0076 \end{bmatrix} +$$

$$\begin{bmatrix} 0.0171 & 0.0057 & -0.0229 & 0 & 0 & 0 \\ 0.0057 & 0.0171 & -0.0229 & 0 & 0 & 0 \\ -0.0229 & -0.0229 & 0.0457 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0229 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0229 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0057 \end{bmatrix}, \tag{39}$$

C. *Olivine [17]*

As an example to orthotropic material, the elastic constant tensor of olivine is presented below.

$$C_{ij} = \begin{bmatrix} 192 & 66 & 60 & 0 & 0 & 0 \\ 66 & 160 & 56 & 0 & 0 & 0 \\ 60 & 56 & 272 & 0 & 0 & 0 \\ 0 & 0 & 0 & 60 & 0 & 0 \\ 0 & 0 & 0 & 0 & 62 & 0 \\ 0 & 0 & 0 & 0 & 0 & 49 \end{bmatrix} \tag{40}$$

Harmonic representation for elastic constant tensor of Olivine:

$$= \begin{bmatrix} 67.33 & 67.33 & 67.33 & 0 & 0 & 0 \\ 67.33 & 67.33 & 67.33 & 0 & 0 & 0 \\ 67.33 & 67.33 & 67.33 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 127.33 & 0 & 0 & 0 & 0 & 0 \\ 0 & 127.33 & 0 & 0 & 0 & 0 \\ 0 & 0 & 127.33 & 0 & 0 & 0 \\ 0 & 0 & 0 & 63.67 & 0 & 0 \\ 0 & 0 & 0 & 0 & 63.67 & 0 \\ 0 & 0 & 0 & 0 & 0 & 63.67 \end{bmatrix} +$$

$$\begin{bmatrix} -15.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -42.40 & 0 & 0 & 0 & 0 \\ 0 & 0 & 57.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -14.4 \end{bmatrix} + \begin{bmatrix} 1.10 & -0.762 & 3.52 & 0 & 0 & 0 \\ -0.762 & -1.41 & -2.76 & 0 & 0 & 0 \\ 3.52 & -2.76 & 0.305 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.10 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.41 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.305 \end{bmatrix} +$$

$$\begin{bmatrix} 11.43 & -0.57 & -10.857 & 0 & 0 & 0 \\ -0.57 & 9.14 & -8.57 & 0 & 0 & 0 \\ -10.857 & -8.57 & 19.43 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8.57 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10.857 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.57 \end{bmatrix}. \tag{41}$$

Elastic constant data for transversely isotropic and orthotropic materials are given in Table II and III respectively. These data (except Canine femora [18]) are taken from [19].

TABLE II
ELASTIC CONSTANT DATA FOR TRANSVERSELY ISOTROPIC MATERIALS

Materials	C_{11}	C_{12}	C_{13}	C_{33}	C_{44}
Polystyrene	5.20	2.75	2.75	5.70	1.30
Zinc(Zn)	165	31.1	50	61.8	39.6
Tool steel(Normal)	289	116	117	284	84.5
Tool steel(Hardened)	277	113	112	272	80.3

TABLE III
ELASTIC CONSTANT DATA FOR ORTHOTROPIC MATERIALS

Materials	C_{11}	C_{12}	C_{13}	C_{22}	C_{23}	C_{33}	C_{44}	C_{55}	C_{66}
Pine (Softwood)	1.24	0.74	0.76	17.1	0.94	1.79	1.18	0.079	0.91
Olivinite	232	93	92	210	82	199	73.3	70.9	68.6
Canine femora	19	9.73	11.9	22.2	11.9	29.7	6.67	5.67	4.67
Marble	119	51	52	110	47	104	29.7	30.7	32.6

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IV. The Norm, Norm Ratio and Anisotropy Degree

The norm concept for elastic constant tensor is described, norm and norm ratios as well as the measure of `nearness' of the nearest isotropic tensor are computed for several examples from various anisotropic materials exhibiting elastic symmetries such as transversely isotropic and orthotropic. These computations are used to compare and assess the anisotropy in various anisotropic materials by means of strength or magnitude and also determine the `nearness' of the nearest isotropic tensor for the materials with lower symmetry types. Norm is an invariant of the material. Generalizing the concept of the modulus of a vector, norm of a Cartesian tensor (or the modulus of a tensor) is defined as the square root of the contracted product over all indices with itself:

$$N = \|C\| = \{C_{ijkl\ldots}C_{ijkl\ldots}\}^{\frac{1}{2}} \qquad (42)$$

Denoting rank n Cartesian $C_{ijkl\ldots\ldots}$ by C_n the square of the norm is expressed as [10]:

$$N^2 = \|C\|^2 = \sum_{j,q}\left\|C^{(j;q)}\right\|^2 = \sum_{(n)} C_{(n)}C_{(n)} = \sum_{(n),j,q} C_{(n)}^{(j;q)}C_{(n)}^{(j;q)} \qquad (43)$$

This definition is consistent with the reduction of the tensor in Cartesian formulation when all the irreducible parts are embedded in the original rank n tensor space. In this work, (43) is changed and rearranged by using harmonic tensors and it becomes

$$N^2 = \sum_{mn}(S_{mn}^1)^2 + \sum_{mn}(S_{mn}^2)^2 + 2\sum_{mn}(S_{mn}^1)(S_{mn}^2) + \sum_{mn}\left(D_{mn}^1\right)^2 +$$

$$+\sum_{mn}(D_{mn}^2)^2 + 2\sum_{mn}(D_{mn}^1)(D_{mn}^2) + \sum_{mn}(H_{mn})^2 \qquad (44)$$

Due to this new concept of norm, following rules are suggested:

Rule 1. It can be used as a parameter representing and comparing the overall effect of a certain property of anisotropic materials of the same or different symmetry. If the norm value of a material is large, it has more effective property than the other materials of the same symmetry type. It is known that the anisotropy of the materials, for instance the symmetry group of the material and the anisotropy of the measured property depicted in the same materials may be quite different. Obviously, the property, tensor must show, at least, the symmetry of the material. For example, a property, which is measured in a material, can almost be isotropic but the material symmetry group itself may have very few symmetry elements. For isotropic materials, the elastic constant tensor has two decomposed parts which are scalar parts, so the norm of the elastic constant tensor for isotropic materials depends only on the norm of the scalar parts,

i.e., $N = N_s$ Hence, the ratio $N_s/N = 1$ for isotropic materials. For anisotropic materials, the elastic constant tensor additionally contains two deviator parts and one harmonic part, so N_d/N for the deviator parts and N_h/N for nonor part. Generalizing this to harmonic tensors to elastic constant tensor, N_s/N for scalar parts, N_d/N for deviator parts and N_h/N for harmonic part. Here these ratios of different harmonic parts represent the anisotropy of that particular decomposed part, they can also be used to assess the anisotropy degree of a material property as a whole, so two more rules are also suggested:

Rule 2. When N_s is the largest among norms of the decomposed parts, if the norm ratio N_s/N is closer to one, the material property is closer to isotropic.

Rule 3. When N_s is not the largest or not present, norm of the other parts can be used as a criterion. But in this case the situation is reverse; if the norm ratio is larger than the others, the material property is more anisotropic.

The norm and norm ratios for transversely and orthorhombic materials are calculated in order to determine the effect of anisotropy in other words which one is more anisotropic or isotropic. The results for norm, norm ratios are summarized in Table IV and V.

TABLE IV
THE NORMS AND NORM RATIOS FOR TRANSVERSELY ISOTROPIC MATERIALS

Materials	N_s	N_d	N_h	N	N_s/N	N_d/N	N_h/N
Polystyrene	10.561	0.382	0.0618	11.697	0.911	0.0327	0.0528
Zinc(Zn)	172.01	88.578	12.150	278.97	0.617	0.318	0.044
composite	41.558	22.969	12.782	54.461	0.751	0.426	0.235
Tool steel (Normal)	556.89	0.735	0.384	592.56	0.940	0.001	0.0006
Tool steel(Hardened)	432.45	4.653	1.461	568.13	0.761	0.008	0.003

The larger ratio $\boldsymbol{N_d/N}$ and $\boldsymbol{N_h/N}$, the more anisotropic property exist for a transversely isotropic material and in reverse manner, the smaller ratio $\boldsymbol{N_s/N}$ a transversely isotropic material possesses the more anisotropic property of norm ratio for the scalar part. From Table 4, normal tool steel is the most isotropic material with the largest $\boldsymbol{N_s/N}$ ratio. Whereas Zinc is the most anisotropic material with the least value. Furthermore, it can be concluded that for the material of orthotropic media, with specific couplings, it can be degenerate into either transversely isotropic or isotropic media and are able to study the overall stiffness effect by means of the norm; it is concluded that the E-glass/epoxy undertaking the properties of fiber and matrix (homogeneous) to be equal is stronger than those when the properties of fibers and matrix are not equal (non-homogeneous) which is already understood from the theory of mechanics and experiments. Using the norm concept, this method will enable us to reveal the effect of the fiber orientations and the material properties of fiber and matrix of the composite.

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TABLE V
THE NORMS AND NORM RATIOS FOR ORTHOTROPIC MATERIALS

Materials	N_s	N_d	N_h	N	N_s/N	N_d/N	N_h/N
Olivine	373.42	133.9	32.91	410.46	0.91	0.33	0.080
Pine	9.456	11.76	4.894	17.418	0.54	0.66	0.281
Olivinite	426.091	25.22	16.89	447.55	0.95	0.06	0.038
Canine femora	45.902	8.352	1.076	50.894	0.90	0.16	0.021
Marble	213.6	12.18	1.449	234.50	0.91	0.05	0.006

Considering the rules, from Table V, olivinite is the most isotropic material with the largest $\frac{N_s}{N}$ ratio. Whereas pine is the most anisotropic material with the least value.

V. CONCLUSIONS

Harmonic decomposition method has many applications in different subjects of physics and engineering (atomic and molecular physics and the physics of condensed matter). In the mechanics of continuous media (for instance; in elasticity studies, the strain and the stress tensors are decomposed into spherical and deviatoric parts), the hydrostatic pressure is connected to the change of volume without change of shape through the bulk modulus whereas the change of shape is connected to the deviatoric part of the stress tensor through the shear modulus.

For very valuable materials like diamond or quartz used in mining, it is difficult to measure its elastic constants because of its small samples. As another application for harmonic decomposition, it is possible to decide which type of symmetry a material has when the elastic constants are measured relative to an arbitrary coordinate system. A second rank symmetric tensor associated to the elastic constant tensor can be used to verify if the coordinate axes are the symmetry axes of the material and determine a symmetry coordinate system. So comprehending the decomposition method is considerable to understand the idea behind these decomposition methods as well as the physical properties of anisotropic materials.

Furthermore, in this study, comparing the irreducible methods in literature with harmonic decomposition reveals the following major result:

Components of scalar and deviatoric parts in irreducible method are not equal to those in harmonic decomposition method, so it proves that there is not a unique decomposition for both deviatoric and scalar parts, in other words total deviatoric and scalar parts can be decomposed into infinitely many independent components. This case also indicates that total scalar, deviatoric and nonor parts of elastic constant tensor obtained from irreducible decomposition methods are the same as those of harmonic decomposition method.

Decomposition of elastic constant tensor into harmonic parts provides a deeper understanding about elastic and mechanical behavior of anisotropic materials. It also has more significant effects on many applications in different fields such as:comparing the anisotropic properties of materials, examining the material symmetry types in detail, determination of materials possessing same crystal symmetry type which are highly anisotropic or close to isotropy, understanding the mechanical and elastic behaviour of natural composites such as Bone and Wood types

REFERENCES

[1] J.F. Nye, *Physical Properties of Crystals; their representation by tensors and matrices*. Oxford: Clarendon Press, 1957.
[2] E.T. Onat, "Effective properties of elastic materials that contain penny shaped voids", Journal *of Elasticity*, vol. 22, 1984, pp. 1013-1021.
[3] E.T Onat, J.P. Boehler and Jr. Kirillov, "On the polynomial invariants of the elasticity tensor", *Journal of Elasticity*, 34, 1994, pp. 97-110.
[4] S. Cowin, "Properties of the anisotropic elasticity tensor", *Quarterly Journal of Mechanics and Applied Mathematics*, vol. 42, 1989, pp. 249-267.
[5] S. Forte, M. Vianello, "Symmetry classes for elasticity tensors", *Journal of Elasticity*, vol. 43, 1996, pp. 81-108.
[6] S. Forte, M. Vianello, "Restricted invariants on the space of elasticity tensors", *Mathematics and Mechanics of Solids*, vol. 11, 2006, pp. 48-82.
[7] T.C.T. Ting and Q.C. He, "Decomposition of elasticity tensors and tensors that are structurally invariant in three dimensions" *Quarterly Journal of Mechanics and Applied Mathematics*, vol. 59, 2006, pp. 323-341.
[8] X. Zheng, "Identification of symmetry planes in weakly anisotropic elastic media", *Geophys.J.Int*., vol. 170, 2007, pp. 468-478.
[9] B.D. Annin, N.I. Ostrosablin, "Anisotropy of elastic properties of materials". *Journal of Applied Mechanics and Technical Physics*, 49, 2008, pp. 998-1014.
[10] J. Jerphagnon, D. Chemla and R. Bonneville, "The description of the physical properties of condensed matter using irreducible tensors", *Advances in Physics*, vol. 27, 1978, pp. 609-650.
[11] D.L. Andrews, W.A. Ghoul, "Irreducible fourth-rank cartesian tensors", *Physical Review A*, vol. 25, 1982, pp. 2647-2657.
[12] G. Backus, "A geometrical picture of anisotropic elastic tensors". *Reviews of Geophysics and Space Physics*, 8, 1970, pp. 633-671.
[13] Yu.I. Sirotin "Decomposition of Material Tensors into Irreducible Parts", *Sov. Phys. Crystallogr.*, vol. 19, 1975, pp. 565-568.
[14] R. Baerheim, "Harmonic decomposition of the anisotropic elasticity tensor", *Quarterly Journal of Mechanics and Applied Mathematics*, vol. 46, 1993, pp. 391-418.
[15] C. H. Chen, S. Cheng, "Mechanical properties of anisotropic fiber-reinforced composites", *Journal of Applied Mechanics*, 37 No.1, Trans. ASME, 92 Series E, 1970, pp. 186-189.
[16] A. Wright, C. S. N. Faraday, E. F. T. White, "Elastic constants of oriented glassy polymers", *Journal of Physics D-Applied Physics*, vol. 4, 1971, pp. 2002.
[17] S. Chevrot, J. T. Browaeys, "Decomposition of the elastic tensor and geophysical applications", *Geophys. J. Int*., vol. 159, 2004, pp. 667-678.
[18] S. C. Cowin and W. C . Van Buskirk, "Thermodynamics restrictions on the elastic constants of bone", *Journal of Biomechanics*, vol. 19, 1986, pp. 85-87.
[19] Landolt-Börnstein, *Numerical Data and Functional Relationships in Science and Technology, New Series, Group III, (Crystal and Solid State Physics)*. 11, Springer-Verlag, Berlin, 1979.