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Stability and Convergence of a Difference Scheme for a Singular Cauchy Problem

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***Abstract*—In the work we consider a model of difference scheme for a numerical solution of Cauchy problem for first order differential equation with the singularity at infinity. We build a sequence of discrete operators for the difference scheme and prove that the sequence is stable. Approximation and convergence theorems for the approximative scheme are proved on the solutions to the Cauchy problem.**

***Index Terms*—difference scheme, stability, approximation and convergence of approximative scheme.**

I. Introduction

Systematic research of an initial value problems for the ordinary differential equations with singularities of rather independent variable or one of phase variables began in the first half of last century. Though in applications such problems began to arise rather long ago. Further the theory and especially methods of the solution of problems with some types of singularities have intensive development and now are studied with sufficient completeness even for the differential equations and systems of the highest orders [1-9].

Problems with time parameter strives for some critical values are of interest that naturally causes difficulties in the solution [10-17]. Problems with such feature are a little studied. At the same time rather often meet in various applications [18]. We consider Cauchy problem on an infinite interval. In the numerical solution it is important to construct a constructive grid on this interval with finite number of grid points. So, in the work one difference scheme for a considered problem is offered and investigated.

II. Construction of the approximate scheme

We consider Cauchy problem

$$\begin{cases} Ly \equiv y' + \upsilon(t)y = z(t), \quad t > 0, \\ y(0) = \alpha, \end{cases} \tag{1}$$

on functions of class $\dot{H}_\upsilon$.

Now we introduce some notation. By $C^l = C^l(I)$ $(l = 0, 1, ...)$ we denote the space of l times continuously differentiable on $I = [0, \infty)$ functions. By $\dot{C}^l = \dot{C}^l(I)$ $(l = 0, 1, ...)$ we denote the space of all functions $y \in C^l$ such that

$$\lim_{t\to\infty} y^{(k)}(t) = 0 \quad (0 \le k < l).$$

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Denote by $\dot{C} = \dot{C}^0$. Norm in $\dot{C}$ is given by

$$\|y\|_{\dot{C}} = \sup_{t\ge 0}|y(t)|.$$

The class $\dot{H}_\upsilon$ we define as completion of the linear manifold

$$\dot{C}^1 H_\upsilon = \left\{ y \in \dot{C}^1 : \|y\|^2_{2,\upsilon} = \int\limits_0^\infty \left(|y''|^2 + |\upsilon^2(t)y|^2\right) dt < \infty \right\}$$

with respect to the norm $\|\cdot\|_{2,\upsilon}$. We impose on weight function $\upsilon(t)$ the following condition: $\upsilon(t)$ is continuous and $|\upsilon(t)| > 0$ on $I = [0, \infty)$ and

$$\int\limits_0^\infty |\upsilon(t)|^{-2} dt < \infty. \tag{2}$$

Note that the condition (2) specifies convergence rate $y(x) \to 0$ $(x \to \infty)$ for $y \in \dot{H}_\upsilon$. That is to say

$$\dot{H}_\upsilon \subset \dot{C}^2.$$

Moreover,

$$L\left(\dot{H}_\upsilon\right) \subset \dot{C}.$$

At first we will construct the approximate scheme for the numerical solution of the equation (1). The following definitions are given in [19].

Let X, F are Banach spaces, $\mathcal{L}(X, F)$ be a space of all continuous linear operators acting from X into F. Let $A \in \mathcal{L}(X, F)$. By $D(A)$ we denote the domain of operator A.

Definition 1. *We say that sequence of the equations*

$$A_n x = f_n \quad (n \ge 1) \tag{3}$$

is an approximate scheme of the equation

$$Ay = f, \tag{4}$$

if $A_n \in \mathcal{L}(X_n, F_n)$*,* X_n *is in agreement with* X *by means of operator* $T_n \in \mathcal{L}(X, X_n)$*,* F_n *is in agreement with* F *by means of operator* $T'_n \in \mathcal{L}(F, F_n)$*.*

We will write the equation (1) in an operator form (4), where we will take

$$Ay = \left(y' + \upsilon(t)y,\ y(0)\right). \tag{5}$$

We consider the operator A in (5) as an operator from $X = \dot{H}_\upsilon$ into $F = \dot{C} \times \mathrm{R}$, where $\mathrm{R} = (-\infty, \infty)$. Norm for pair $f = (z, a) \in F$ is given by

$$\|f; F\| = \left\|z; \dot{C}\right\| + |a|.$$

To construct of approximate scheme (3) for problem (4), (5) at the beginning we will consider a sampling operator

$$S : \dot{C}(I) \to c_0,$$

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where c_0 is a space of numerical sequences $(x_j)_{j\geq 0}$, which converge to zero. Norm in c_0 is given by

$$\|x\|_{c_0} = \sup_{j\geq 0}|x_j|.$$

We define characteristic measure with respect to the function $\upsilon^4(t)$ as

$$\mu(t) = \sup_{h>0}\left\{h:\ h^3\int_t^{t+h}\upsilon^4(\zeta)d\zeta \leq 1\right\}.$$

Function $\mu(t)$ is continuous, positive and bounded function in I.
The following characteristic equality

$$h^3\int_t^{t+h}\upsilon^4(\zeta)d\zeta = 1, \quad \text{if} \quad h = \mu(t), \tag{6}$$

takes place. Besides

$$\mu(0) > 0, \ \lim_{t\to\infty}\mu(t) = 0,$$

see [20]. Let $M = \sup_{t\geq 0}\mu(t)$. It is clear, that $M < \infty$.We take small enough ε, $0 < \varepsilon < 1$. Then

$$\infty > T_\varepsilon = \inf_{t>0}\{t : \mu(t) \leq \varepsilon\} > 0.$$

There is the following finite disjoint cover of interval $[0, T_\varepsilon)$

$$[0, T_\varepsilon) \subset \bigcup_{k=1}^m \triangle_k, \ \ t_m < T_\varepsilon,$$

by intervals

$$\Delta_k = [t_k, t_{k+1}) \ \ (k = 1, 2, ..., m),$$

where we take $t_{k+1} = t_k + \mu_k$ $(k \geq 1)$, $\mu_k = \mu(t_k)$ $(t_1 = 0)$.
Denote by $\bar{t}_k = (t_{ki})_{i=1}^n$ a uniform grid on Δ_k:

$$t_{ki} = t_{k0} + i\delta_k, \quad \delta_k = \mu_k/n, \quad t_{k0} = t_{k-in},$$

$$(t_{10} = 0).$$

Let

$$\Delta_{ki} = [t_{ki-1}, t_{ki}) \ \ (i = 1, ..., n).$$

We set

$$Sy = (\bar{y}, \tilde{y}), \ \tilde{y} = (y_{m+j})_{j\geq 1}, \ y_{m+j} = y(t_{m+j}),$$

$$\bar{y} = (\bar{y}_1, ..., \bar{y}_m), \ \bar{y}_k = (y_{ki})_{i=1}^n, \ y_{ki} = y(t_{ki}).$$

Now we introduce some spaces. Let l_2^m be a space of vectors $\bar{a} = (a_1, ..., a_m) \in \mathrm{R}^m$ with norm:

$$\|\bar{a}; l_2^m\| = \left(\sum_{j=1}^m |a_j|^2\right)^{1/2},$$

$\bar{X}_k$ $(k = 1, ..., m)$ be a space of vectors $\bar{x}_k = (x_{k1}, ..., x_{kn}) \in \mathrm{R}^n$ with the spherical norm:

$$\|\bar{x}_k\|_{sph} = \left(\delta_k\sum_{i=1}^n x_{ki}^2\right)^{1/2},$$

$\bar{F}_k$ $(k = 1, ..., m)$ be a space of vectors $\bar{x}_k = (x_{k1}, ..., x_{kn}) \in \mathrm{R}^n$ with norm:

$$\|\bar{x}_k\|_c = \max_{1\leq i\leq n}|x_{ki}|.$$

Here R^k is k-dimensional arithmetical space.
Let $X_n = \bar{X}_1 \times \bar{X}_2 \times ... \times \bar{X}_m$. And let

$$\|\overline{x}; X_n\| = \left(\sum_{k=1}^m \|\bar{x}_k\|_{sph}^2\right)^{1/2}, \ \ \overline{x} = (\overline{x}_1, ..., \overline{x}_m).$$

Let's set for pair $(\overline{x}, a) \in X_n \times \mathrm{R}$

$$\|(\overline{x}, a)\| = \|\overline{x}; X_n\| + |a|.$$

Furthermore, let $F_n = \bar{F}_1 \times \bar{F}_2 \times ... \times \bar{F}_m$ be a space with the following norm

$$\|\overline{x}; F_n\| = \max_{1\leq k\leq m}\|\bar{x}_k\|_c.$$

Let's put for $(\overline{x}, \bar{a}) \in F_n \times l_2^m$

$$\|(\overline{x}, \bar{a})\| = \|\overline{x}; F_n\| + \|\bar{a}; l_2^m\|.$$

Let $T_n : X \to X_n$, $T_n' : F \to F_n$ are correlation operators such that:

$$T_n y = (\bar{y}_1, ..., \bar{y}_m), \quad \bar{y}_k = \bar{T}_k y = (y_{k1}, ..., y_{kn}), \tag{7}$$

$$y_{ki} = y(t_{ki}) \ \ (k = 1, ..., m);$$

$$T_n'(z, a) = T_n z.$$

In (7) $X = \dot{H}_\upsilon$, $F = \dot{C} \times \mathrm{R}$.
On n-th step we will consider approximate operator

$$A_n : X_n \times \mathrm{R} \to F_n \times l_2^m,$$

defined by the following equalities:

$$A_n(\overline{x}, a) = (\bar{A}_1\bar{x}_1, ..., \bar{A}_m\bar{x}_m; G(\overline{x}, a)), \tag{8}$$

where

$$\left(\bar{A}_k\bar{x}_k\right)_i = \frac{x_{ki} - x_{ki-1}}{\delta_k} + \upsilon_{ki}x_{ki-1} \ (i = 1, 2, ..., n), \tag{9}$$

$$x_{k0} = x_{k-1n}, \quad \text{if} \quad 2 \leq k \leq m, \ x_{10} = 0, \tag{10}$$

$$G(\overline{x}, a) = (a, x_{20}, ..., x_{m0}); \tag{11}$$

$$\upsilon_{ki} = \upsilon(t_{ki}) \ \ (k = 1, ..., m). \tag{12}$$

Further let $P' : \mathrm{R}^{mn+m} \to \mathrm{R}^{mn}$, $P'' : \mathrm{R}^{mn+m} \to \mathrm{R}^m$ are standard projectors. We define approximate scheme of equation (1) as follows: on n-th step we set

$$P'\left(A_n(\overline{x}, a)\right) = T_n'(z, a), \tag{13}$$

$$P''\left(A_n(\overline{x}, a)\right) = G(\overline{x}, a). \tag{14}$$

III. Properties of Approximate Scheme

A. Stability

Let X, Y are Banach spaces. We say that X is *embedded* into Y if $X \subset Y$ and $\exists K > 0 : \|x\|_Y \leq K\|x\|_X$ for all $x \in X$.
We say that function $\upsilon(t)$ satisfies *slow change condition* (with respect to the characteristic measure $\mu(t)$) if there exist $0 < \beta_1 < \beta_2 < \infty$ such that

$$\beta_1|\upsilon(t)| < |\upsilon(\zeta)| < \beta_2|\upsilon(t)| \ \textit{whenever}\ 0 < \zeta - t \leq \mu(t). \tag{15}$$

Example. Function $\upsilon(x) = (1+x)\, e^{sin x^2}$ $(x \geq 0)$ satisfies condition (15). For $h = \mu(t)$

$$1 = h^3 \int_x^{x+h} \upsilon^4(t)dt \geq e^{-4} (1+x)^4 h^4.$$

It is easy to see that the condition (15) holds if $\beta_2 = e^2(1+e)$, $\beta_1 = \beta_2^{-1}$.

Let $L_2 = L_2(I)$ be a Lebesgue space with norm

$$\|y\|_2 = \left(\int_0^\infty |y(t)|^2 dt \right)^{1/2}.$$

Statement 1. Let $\upsilon(t)$ satisfies condition (15). Then
1) Operator L in (1) belongs to $\mathcal{L}(\dot{H}_\upsilon, \dot{C})$;
2) The space $\dot{H}_\upsilon$ is embedded into L_2.

Statement 2. Let $\upsilon(t)$ satisfies condition (15). Then
1) $T_n \in \mathcal{L}(\dot{H}_\upsilon, X_n)$.
2) $T_n' \in \mathcal{L}(F, F_n)$, where $F = \dot{C} \times \mathrm{R}$.

Statement 3. Let $\upsilon(t)$ satisfies the condition (15). Then the estimate

$$\left\| \tilde{y}^{(k)} \right\|_{c_0} \leq \dot{c}_k \varepsilon^{k+\frac{1}{2}} \|y; H_\upsilon\|, \quad k = 0, 1, \tag{16}$$

holds for all functions $y \in \dot{H}_\upsilon$, where $\dot{c}_k > 0$ does not depend on y.

Essential role in convergence of (3) plays stability property of this scheme.

Definition 2. *We say that approximate scheme (3) is stable if there exist a constant* $\gamma > 0$ *and integer* $n_0 > 0$ *such that*

$$\|A_n x\|_{F_n} \geq \gamma \|x\|_{X_n} \quad (x \in D(A_n), \quad n \geq n_0). \tag{17}$$

Theorem 1. Let the condition (15) holds. Let A_n be an operator defined by equalities (8)-(12). Then the approximate scheme (13)-(14) with the right hand side $f_n = (z, \bar{x}_0)$ is stable.

B. Approximation and convergence

Now we investigate a question about approximation and convergence of the difference scheme (13)-(14) on solutions to the problem (1).

It is well-known that the problem (1) for right hand side $z \in \dot{C} \times \mathrm{R}$ is uniquely solvable (see [21]).

Definition 3. *We say that the difference scheme (13)-(14) approximate the equation (1) on the solution* y^* *if:*

$$\|P' A_n(T_n y^*, y^*(0)) - T_n'(z, y^*(0)); F_n\| \to 0, \quad n \to \infty.$$

Theorem 2. Let $\upsilon(t)$ satisfies the condition (15). Then:
1) The problem (1) has solution $y^* \in \dot{H}_\upsilon$ if and only if $z(t) \in \dot{C}$.
2) The difference scheme (13)-(14) approximate the equation (1) on y^*.

Let $\{x^{*n}\}_{n \geq 1}$ be a sequence of solutions to the equation

$$P' A_n(\overline{x}, y^*(0)) = T_n'(z, y^*(0)).$$

We say that approximate scheme (13)-(14) *convergent on* y^* if

$$\|x^{*n} - T_n y^*; X_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Theorem 3. Let $\upsilon(t)$ satisfies the condition (15). Then the difference scheme (13)-(14) convergent on y^*.

All statements obtained in this work in fact follow from the below reasoning and estimates.

Let $W_2^2(\Omega)$ $(\Omega = (a,b), -\infty < a < b < \infty)$ be a Sobolev space with norm

$$\left| y; W_2^2(\Omega) \right| = \left(\int_\Omega \left(|y''|^2 + |y|^2 \right) dx \right)^{1/2}.$$

The embedding

$$W_2^2(\Omega) \to C^k\left(\overline{\Omega}\right) \quad (k = 0, 1), \tag{18}$$

$\overline{\Omega} = [a, b]$, holds. That is each $z \in W_2^2(\Omega)$ is equivalent to function $y \in C^k\left(\overline{\Omega}\right)$, and also

$$\max_{\overline{\Omega}} \left| y^{(k)} \right| \leq c_k \left| y; W_2^2(\Omega) \right|. \tag{19}$$

See. [22]. c_k $(k = 0, 1)$ are exact constants of the embedding (18) for $\Omega = (0, 1)$.

Let $y \in \dot{H}_\upsilon$. Function y may be considered as finite function on I. It follows from (15) and characteristic equality (6) that for all $i = 1, 2, ..., n$, $k = 1, 2, ..., m$

$$\beta_1 \leq \mu_k \upsilon_{ki} = \left(\mu_k^3 \int_{\Delta_k} \upsilon_{ki}^4 dt \right)^{1/4} \leq \beta_2. \tag{20}$$

Note that

$$y \in W_2^2(\Delta_k) \quad (k \geq 1).$$

It is easy to obtain this through the estimate

$$\infty > \int_{\Delta_k} \left(|y''|^2 + |y|^2 \upsilon(x)^4 \right) dx \geq$$

$$\geq \frac{\upsilon(t_k)}{\upsilon(t_k) + \beta} \int_{\Delta_k} \left(|y''|^2 + |y|^2 \right) dx.$$

Now we can consider that $y \in C^1\left(\overline{\Delta}_k\right)$. In addition

$$\max_{\overline{\Delta}_k} |y| \leq c_0 \mu_k^{3/2} \left(\int_{\Delta_k} |y''|^2 d\zeta + \mu_k^{-4} \int_{\Delta_k} |y|^2 d\zeta \right)^{1/2} \leq$$

$$\leq \widetilde{c}_0 \mu_k^{3/2} \left[\int_{\Delta_k} |y''|^2 d\zeta + |\upsilon^2(\zeta) y(\zeta)|^2 d\zeta \right]^{1/2}; \tag{21}$$

$$\max_{\overline{\Delta}_k} |y^{(1)}| \leq \widetilde{c}_1 \mu_k^{1/2} \left(\int_{\Delta_k} \left(|y''|^2 + |\upsilon^2(\zeta) y(\zeta)|^2 \right) d\zeta \right)^{1/2}, \tag{22}$$

where $\tilde{c}_k = \left(1 + \beta^8\right)^{1/4} c_k$.

Let $x \to t_k$. Let us take $\Delta = \Delta_{k-1}$ if $x \to t_k - 0$, $\Delta = \Delta_{k+1}$ if $x \to t_k + 0$. Then

$$|y(x) - y(t_k)| = \left| \int_x^{t_k} y^{(k+1)}(t)\, dt \right| \leq |x - t_k| \left(\widetilde{c}_0 \, |y; H_\upsilon| \right),$$

$$|y'(t) - y'(t_k)| = \left| \int_x^{t_k} y''(t)\, dt \right| \leq |x - t_k|^{1/2} |y; H_\upsilon|,$$

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whence it follows that y, y' are continuous also in a t_k points. Because of $\mu_k \to 0$ as $k \to \infty$ there exist limits

$$\lim_{x\to\infty} y(x) = 0, \quad \lim_{x\to\infty} y'(x) = 0,$$

as follows from (21) and (22). From (20) and (21) we obtain $(x \in \Delta_k)$

$$|\upsilon(x)\, y(x)| \le \beta\upsilon(t_k) \max_{\overline{\Delta}_k} |y| \le \left(\widetilde{c}_0 \beta^2 \,|y; H_\upsilon|\right) \mu_k^{1/2}.$$

Hence,

$$\lim_{x\to\infty} |\upsilon(x)\, y(x)| = 0.$$

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