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Fuzzy Delay Differential Equation in Predator-Prey Interaction: Analysis on Stability of Steady State

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Abstract—**In this paper, a fuzzy delay predator-prey (FDPP) system is proposed by adopting fuzzy parameter in a delay predator-prey (DPP) system. The steady state and linear stability of FDPP system are determined and analyzed. Here, we show that the trivial steady state is unstable for all value of delays. Mean while the semi trivial steady state is locally asymptotically stable for all values of delays under certain conditions. We prove that the steady state are fuzzy numbers. Several examples are considered to show the results.**

Index Terms—**Delay predator-prey (DPP), Steady states, Stability, Fuzzy delay predator-prey(FDPP).**

I. Introduction

IN real world, the study of population dynamics including (stable, unstable, and oscillatory behavior) has become very important since Volterra and Lotka proposed the seminal models of predator-prey models in 1920. Predator-prey models represent the basis of many models used today in the analysis of population dynamics and is one of the most popular in mathematical ecology. And the dynamics properties of the predator-prey models which have significant biological background has been paid a great attention. Some studies in the area of predator-prey interaction, that treat population can be extended by including time delay. The time delay is included into population dynamics when the rate of changes of population is not only a function of the present population but also depends on the pervious population.

In 2012, Changjin and Peiluan [1] explained the stability, the local Hopf bifurcation for the delay predator-prey model with two delays. In 2008, Toaha et.al [2] showed a deterministic and continuous model for predator-prey with time delay and constant rates of harvesting and studied the combined effects of harvesting and time delay on the dynamics of predator-prey model.

Although, the concepts of the steady states refer to the absence of changes in a system, In some cases, studying the stability of the steady state solutions become an important subject since, by examining what happens in a steady state, we can better understand the behavior of a system.

Forde et.al [3] had studied the stability analysis of the steady states of delay predator-prey interaction. They also considered the possibility of existence of the periodic solutions.

In our real life, we have learned to accept that we are actually dealed with uncertainty. Scientists also accepted the fact that uncertainty is very important study in most applications. Modeling the real life problems in such cases, usually involves vagueness or uncertainty in some of the parameters. The concept of fuzzy set and system was introduced by Zadeh [6] and its development has been growing rapidly to various situation of theory and application including the theory of differential equations with uncertainty. The later is known as fuzzy differential equation. It has been used to model a dynamical systems under possibility uncertainty [5].

In this paper the fuzzy approach is used to model an uncertainty in dynamical system which then can be represented as fuzzy delay differential equations. Specifically, the discussion on the theory and analysis of delay predator-prey differential equations with uncertainty parameters is considered.

The organization of this paper is as follows. In Section 2, the basic definitions regarding the fuzzy number, steady states and characteristic equation are briefly presented. In Section 3 delay predator-prey system is introduced, followed by the formulation of fuzzy delay predator-prey (FDPP) system. The analysis, of the steady state and linear stability are also given. And Section 4 presents some numerical examples, finally the conclusion of the finding is given in Section 5.

II. Preliminaries

Definition 1 [6] A fuzzy number is a function such as $u : R \to [0,1]$ satisfying the following properties:

1) u is normal, i.e $\exists\, x_0 \in R$ with $u(x_0) = 1$.
2) u is a convex fuzzy set i.e $u(\lambda x + (1-\lambda)y) \geq min\{u(x), u(y)\} \forall x, y \in R, \lambda \in [0,1]$.
3) u is upper semi-continuous on R.
4) $\overline{\{x \in R : u(x) > 0\}}$ is compact where $\bar{A}$ denotes the closure of A.

Definition 2 [5]

An $\alpha-$ cut, u_α, is a crisp set which contains all the elements of the universal set X that have a membership function at least to the degree of α and can be expressed as $u_\alpha = \{x \in X : \mu_u(x) \geq \alpha\}$

Definition 3 [4]

A fuzzy number u is completely determined by any pair $u = (\underline{u}, \overline{u})$ of functions $\underline{u}(\alpha), \overline{u}(\alpha) : [0,1] \to R$ satisfying the three conditions:

1) $\underline{u}(\alpha)$, $\overline{u}(\alpha)$ is a bounded, monotonic, (nondecreasing,nonincreasing) left- continuous function for all $\alpha \in (0,1]$ and right-continuous for $\alpha = 0$.
2) For all $\alpha \in (0,1]$ we have: $\underline{u}(\alpha) \leq \overline{u}(\alpha)$.

For every $u = (\underline{u}, \overline{u}), v = (\underline{v}, \overline{v})$ and $k > 0$,
$(\underline{u+v})(\alpha) = \underline{u}(\alpha) + \underline{v}(\alpha)$
$(\overline{u+v})(\alpha) = \overline{u}(\alpha) + \overline{v}(\alpha)$
$(\underline{ku})(\alpha) = k\underline{u}(\alpha), (\overline{ku})(\alpha)) = k\overline{u}(\alpha)$

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Fuzzy sets is a mapping from a universal set into $[0,1]$. Conversely, every function $\mu : X \longrightarrow [0,1]$ can be represented as a fuzzy set ([6]). We can define a set $F_1 = \{x \in \Re x, \text{ is about } a_2\}$ with triangular membership function as below

Definition 4 [6]

$$\mu_{F_1}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & x \in [a_1, a_2) \\ 1 & x = a_2 \\ \frac{-x+a_3}{a_3-a_2} & x \in (a_2, a_3] \\ 0 & otherwise \end{cases}$$

So the Fuzzy set F can be written as any ordinary function $F = \{(x, \mu_F(x)) : x \in X\}$.

Consider the linear fuzzy delay system as follows:

$$\begin{aligned} \dot{\underline{\mathbf{x}}}_\alpha(t) &= \mathbf{A}_\alpha \underline{\mathbf{x}}_\alpha(t) + \mathbf{B}_\alpha \underline{\mathbf{x}}_\alpha(t-\tau) \\ \dot{\overline{\mathbf{x}}}_\alpha(t) &= \mathbf{A}_\alpha \overline{\mathbf{x}}_\alpha(t) + \mathbf{B}_\alpha \overline{\mathbf{x}}_\alpha(t-\tau) \qquad 0 \le \alpha \le 1 \\ \underline{\mathbf{x}}_\alpha(t) &= \underline{\mathbf{x}}_{\alpha 0} \qquad\qquad t \in [t_0-\tau, t_0] \\ \overline{\mathbf{x}}_\alpha(t) &= \overline{\mathbf{x}}_{\alpha 0} \end{aligned} \tag{1}$$

Suppose $(a_{ij})_\alpha = [(a_{ij})^-_\alpha, (a_{ij})^+_\alpha]$, $\mathbf{A}_\alpha = [\mathbf{A}^-_\alpha, \mathbf{A}^+_\alpha]$ where $\mathbf{A}^-_\alpha = [(a_{ij})^-_\alpha]_{n\times n}$ $\mathbf{A}^+_\alpha = [(a_{ij})^+_\alpha]_{n\times n}$ and $(b_{ij})_\alpha = [(b_{ij})^-_\alpha, (b_{ij})^+_\alpha]$, $\mathbf{B}_\alpha = [\mathbf{B}^-_\alpha, \mathbf{B}^+_\alpha]$ where $\mathbf{B}^-_\alpha = [(b_{ij})^-_\alpha]_{n\times n}$, $\mathbf{B}^+_\alpha = [(b_{ij})^+_\alpha]_{n\times n}$. Then we introduce the following definition :

Definition 5 Let $\mathbf{A}(\mu,\alpha) = [a_{ij}(\mu,\alpha)]_{n\times n} = (1-\mu){\mathbf{A}_\alpha}^- + \mu{\mathbf{A}_\alpha}^+$, $\mathbf{B}(\mu,\alpha) = [b_{ij}(\mu,\alpha)]_{n\times n} = (1-\mu){\mathbf{B}_\alpha}^- + \mu{\mathbf{B}_\alpha}^+$, for $\mu \in [0,1]$. The solution of (1) is $(\underline{\mathbf{x}}_\alpha(t), \overline{\mathbf{x}}_\alpha(t))$, if $(\underline{\mathbf{x}}_\alpha(t), \overline{\mathbf{x}}_\alpha(t))$ is also a solution of the problem below:

$$\begin{aligned} \dot{\underline{\mathbf{x}}}_\alpha(t) &= \bigcup_{\mu=0}^{1} \mathbf{C}(\mu,\alpha)\underline{\mathbf{x}}_\alpha(t) + \bigcup_{\mu=0}^{1} \mathbf{D}(\mu,\alpha)\underline{\mathbf{x}}_\alpha(t-\tau), \\ \dot{\overline{\mathbf{x}}}_\alpha(t) &= \bigcup_{\mu=0}^{1} \mathbf{C}(\mu,\alpha)\overline{\mathbf{x}}_\alpha(t) + \bigcup_{\mu=0}^{1} \mathbf{D}(\mu,\alpha)\overline{\mathbf{x}}_\alpha(t-\tau) \\ \underline{\mathbf{x}}_\alpha(t) &= \underline{\mathbf{x}}_{\alpha 0} \qquad t \in [t_0-\tau, t_0], \qquad 0 \le \alpha \le 1 \\ \overline{\mathbf{x}}_\alpha(t) &= \overline{\mathbf{x}}_{\alpha 0} \end{aligned} \tag{2}$$

The elements of the matrices **C** and **D** are determined from of $\mathbf{A}(\mu,\alpha)$ and $\mathbf{B}(\mu,\alpha)$ as follws:

$$c_{ij} = \begin{cases} e a_{ij}(\mu,\alpha), & a_{ij} \ge 0 \\ g a_{ij}(\mu,\alpha), & a_{ij} < 0 \end{cases}$$

and

$$d_{ij} = \begin{cases} e b_{ij}(\mu,\alpha), & b_{ij} \ge 0 \\ g b_{ij}(\mu,\alpha), & b_{ij} < 0 \end{cases}$$

where e is the identity operation and g corresponds to negative value in $\Re$ and $\forall z, w \in \Re$,
$e : (z,w) \to (z,w)$,
$g : (z,w) \to (w,z)$.

Consider the delay predator-prey system (DPP) of equations as follows:

$$\begin{aligned} \frac{dx(t)}{dt} &= x(1-x) - yp(x) \\ \frac{dy(t)}{dt} &= be^{-d_j\tau}y(t-\tau)p(x(t-\tau)) - dy \end{aligned} \tag{3}$$

where x is prey population, y is a predator population, d is a death rate of predator, $p(x)$ is a predator functional response to prey and τ is time necessary to change prey biomass into predator biomass.

A. Fuzzy Delay Predator-Prey System

We propose a new model of system (3) by, first let $p(x) = cx$ which is the standard mass action or linear response. Then we fuzzify the linear part of the system (3) by symmetric triangular fuzzy number and let $x(t)$, $y(t)$ are non negative fuzzy functions.

Let
$\widetilde{1} = (1-(1-\alpha)\sigma_1, 1+(1-\alpha)\sigma_1)$
$\widetilde{d} = (d-(1-\alpha)\sigma_2, d+(1-\alpha)\sigma_2)$ where $0 \le \alpha \le 1$.
By using Definition 5 system (3) can be written as follows:

$$\begin{bmatrix} \dot{\underline{x}}_\alpha(t) \\ \dot{\overline{x}}_\alpha(t) \\ \dot{\underline{y}}_\alpha(t) \\ \dot{\overline{y}}_\alpha(t) \end{bmatrix} = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & -a_2 \\ 0 & 0 & -a_2 & 0 \end{bmatrix} \begin{bmatrix} \underline{x}_\alpha \\ \overline{x}_\alpha \\ \underline{y}_\alpha \\ \overline{y}_\alpha \end{bmatrix} + \begin{bmatrix} -\underline{x}^2_\alpha(t) - c\underline{x}_\alpha(t)\underline{y}_\alpha(t) \\ -\overline{x}^2_\alpha(t) - c\overline{x}_\alpha(t)\overline{y}_\alpha(t) \\ cbe^{-d_j\tau}\underline{x}_\alpha(t-\tau)\underline{y}_\alpha(t-\tau) \\ cbe^{-d_j\tau}\overline{x}_\alpha(t-\tau)\overline{y}_\alpha(t-\tau) \end{bmatrix}, \tag{4}$$

where

$a_1 = (1-\mu)(1-(1-\alpha)\sigma_1) + \mu(1+(1-\alpha)\sigma_1)$,
$a_2 = (1-\mu)(d-(1-\alpha)\sigma_2) + \mu(d+(1-\alpha)\sigma_2)$,
$0 \le \mu \le 1$.

Then (4) is known as fuzzy delay predator-prey (FDPP) system.

B. Steady States

To find the steady states of the system (4), we assume that the constant $(\underline{x}^*, \overline{x}^*, \underline{y}^*, \overline{y}^*)_\alpha$, is a solution and we will determine the values of these constant. The equations for determining steady states are

$$\begin{aligned} &\underline{x}_\alpha(a_1 - \underline{x}_\alpha - c\underline{y}_\alpha) = 0 \\ &\overline{x}_\alpha(a_1 - \overline{x}_\alpha - c\overline{y}_\alpha) = 0 \\ &-a_2\overline{y}_\alpha + cbe^{-d_j\tau}\underline{x}_\alpha\underline{y}_\alpha = 0 \\ &-a_2\underline{y}_\alpha + cbe^{-d_j\tau}\overline{x}_\alpha\overline{y}_\alpha = 0. \end{aligned} \tag{5}$$

If $\underline{x}_\alpha = 0$ and $\overline{x}_\alpha = 0$, then the first and the second equations of (5) are satisfied, from third and the fourth equations we obtain $(0,0,0,0)_\alpha$ as trivial steady state.

If we consider $\underline{y}_\alpha = \overline{y}_\alpha = 0$, then the third and fourth equations of (5) are satisfied, and the first and second equations gives $\underline{x}_\alpha = \underline{a_1}$ and $\overline{x}_\alpha = \overline{a_1}$, where
$\underline{a_1} = (1-(1-\alpha)\sigma_1)$ and $\overline{a_1} = (1+(1-\alpha)\sigma_1)$.

If $\underline{y}_\alpha$ and $\overline{y}_\alpha$ are not equal zero then the steady state equations are:

$$\begin{aligned} a_1 - \underline{x}_\alpha - c\underline{y}_\alpha &= 0 \\ a_1 - \overline{x}_\alpha - c\overline{y}_\alpha &= 0. \end{aligned} \tag{6}$$

So, if the equation (6) are satisfied, then the system (4) has a nontrivial steady state$(\underline{x}^*, \overline{x}^*, \underline{y}^*, \overline{y}^*)_\alpha$. Thus, the system (4) has three steady state solutions such that ;

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$(0,0,0,0)_\alpha$, $(\underline{a_1},\overline{a_1},0,0)_\alpha$ and the nontrivial steady state $(\underline{x}^*,\overline{x}^*,\underline{y}^*,\overline{y}^*)_\alpha$.

Theorem 1 *Consider the DPP system (3), if the coefficients of linear part of x and y are symmetric triangular fuzzy numbers then the trivial steady state $(0,0,0,0)_\alpha$ is a fuzzy number and the semi trivial steady state $(\underline{a_1},\overline{a_1},0,0)_\alpha$ $\forall\alpha\in[0,1]$, is also fuzzy number.*

Proof: The proof of Theorem 1 is trivial. ∎

Now, we test the stability of the steady states.

C. Linear Stability

The linearization of the fuzzy system (4) about the trivial steady state $(0,0,0,0)_\alpha$ is

$$\begin{bmatrix}\dot{\underline{x}}_\alpha(t)\\ \dot{\overline{x}}_\alpha(t)\\ \dot{\underline{y}}_\alpha(t)\\ \dot{\overline{y}}_\alpha(t)\end{bmatrix}=\begin{bmatrix}a_1&0&0&0\\0&a_1&0&0\\0&0&0&-a_2\\0&0&-a_2&0\end{bmatrix}\begin{bmatrix}\underline{x}_\alpha\\ \overline{x}_\alpha\\ \underline{y}_\alpha\\ \overline{y}_\alpha\end{bmatrix}+\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}\underline{x}_{\alpha t}\\ \overline{x}_{\alpha t}\\ \underline{y}_{\alpha t}\\ \overline{y}_{\alpha t}\end{bmatrix}. \tag{7}$$

Where $x_{t\alpha}=x_\alpha(t-\tau)$ and similarly for y from linearized model we obtain the characteristic equation

$$(a_1-\lambda)^2(\lambda^2-a_2^2)=0. \tag{8}$$

Clearly the linear system has eigenvalues a_1 and $\pm a_2$ which are two positive and one negative fuzzy numbers. Hence, the trivial steady state is unstable for all values of τ.

We can conclude the following proposition:

Proposition 1 A trivial steady state $(0,0,0,0)_\alpha$ with characteristic equation (8) is unstable for all values of delay.

Similarly, for the semi trivial steady state $(\underline{a_1},\overline{a_1},0,0)$ where

$$\begin{bmatrix}\dot{\underline{x}}_\alpha(t)\\ \dot{\overline{x}}_\alpha(t)\\ \dot{\underline{y}}_\alpha(t)\\ \dot{\overline{y}}_\alpha(t)\end{bmatrix}=\begin{bmatrix}a_1-2\underline{a_1}&0&-c\underline{a_1}&0\\0&a_1-2\overline{a_1}&0&-c\overline{a_1}\\0&0&0&-a_2\\0&0&-a_2&0\end{bmatrix}\begin{bmatrix}\underline{x}_\alpha\\ \overline{x}_\alpha\\ \underline{y}_\alpha\\ \overline{y}_\alpha\end{bmatrix}+\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&cbe^{-d_j\tau}\underline{a_1}&0\\0&0&0&cbe^{-d_j\tau}\overline{a_1}\end{bmatrix}\begin{bmatrix}\underline{x}_{\alpha t}\\ \overline{x}_{\alpha t}\\ \underline{y}_{\alpha t}\\ \overline{y}_{\alpha t}\end{bmatrix}. \tag{9}$$

The characteristic equation for (9) is

$$\begin{aligned}&\lambda^4+A\lambda^3+B\lambda^2+C\lambda+D+\\&e^{-(d_j+\lambda)\tau}(E\lambda^3+F\lambda^2+G\lambda)+\\&e^{-2(d_j+\lambda)\tau}(H\lambda^2+I\lambda+J)=0,\end{aligned} \tag{10}$$

where

$$\begin{aligned}A=&-2a_1+4,\ \ B=a_1^2-4a_1+4(1-(1-\alpha)^2\sigma_1^2),\\C=&a_2^2,\ \ D=2a_2^2(1+(1-\alpha)\sigma_1)-a_2^2a_1,\\E=&-2cb,\ \ F=4a_1cb-8cb,\\G=&-2a_1^2cb+8a_1cb-8cb(1-(1-\alpha)^2\sigma_1^2),\\H=&c^2b^2(1-(1-\alpha)^2\sigma_1^2),\\I=&-2a_1c^2b^2(1-(1-\alpha)^2\sigma_1^2)+\\&4c^2b^2(1-(1-\alpha)^2\sigma_1^2),\\J=&a_1^2c^2b^2(1-(1-\alpha)^2\sigma_1^2)-\\&4a_1c^2b^2(1-(1-\alpha)^2\sigma_1^2)+\\&4c^2b^2(1-(1-\alpha)^2\sigma_1^2)^2.\end{aligned} \tag{11}$$

The steady state is stable in the absence of delay if the roots of $\lambda^4+(A+E)\lambda^3+(B+F+H)\lambda^2+(C+G+I)\lambda+(D+J)=0$ have negative real parts. This occurs if and only if

$$\begin{aligned}&(A+E)>0,\ (C+G+I)>0,\ (D+J)>0\\&\text{and } (A+E)(B+F+H)(C+G+I)>\\&(C+G+I)^2+(A+E)^2(D+J).\end{aligned} \tag{12}$$

Hence, in the absence of time delay, the steady state $(\underline{a}_1,\overline{a}_1,0,0)$ is stable if and only if (12) are satisfied.

Now for increasing τ, $\tau\neq0$, we first assume that the root of the characteristic equation (10) is $\lambda=i\mu$ and $\mu>0$. Substitute $\lambda=i\mu$ in (10), we obtain,

$$\mu^4-Ai\mu^3-B\mu^2+Ci\mu+D+e^{-d_j\tau}\Big(cos(\mu\tau)-isin(\mu\tau)\Big)\Big(-iE\mu^3-F\mu^2+iG\mu\Big)+e^{-2d_j\tau}\Big(cos(2\mu\tau)-isin(2\mu\tau)\Big)\Big(-H\mu^2+iI\mu+J\Big)=0.$$

Separating the real and imaginary parts, we get

$$\begin{aligned}\mu^4-B\mu^2+D=&\,e^{-d_j\tau}\Big(cos(\mu\tau)(F\mu^2)\\&+sin(\mu\tau)(E\mu^3-G\mu)\Big)\\&-e^{-2d_j\tau}\Big(cos(2\mu\tau)(-H\mu^2+J)+\\&sin(2\mu\tau)(I\mu)\Big),\\-A\mu^3+C\mu=&\,e^{-d_j\tau}\Big(cos(\mu\tau)(E\mu^3-G\mu)+\\&sin(\mu\tau)(-F\mu^2)\Big)+e^{-2d_j\tau}\Big(cos(2\mu\tau)(-I\mu)+\\&sin(2\mu\tau)(-H\mu^2+J)\Big).\end{aligned}$$

Squaring and adding both sides gives the polynomial of degree eight as follows:

$$\begin{aligned}&(\mu^4-B\mu^2+D)^2+(-A\mu^3+C\mu)^2=\\&\Big(e^{-d_j\tau}(cos(\mu\tau)(F\mu^2)+sin(\mu\tau)(E\mu^3+G\mu))\\&-e^{-2d_j\tau}(cos(2\mu\tau)(-H\mu^2+J)+sin(2\mu\tau)(I\mu))\Big)^2\\&+\Big(e^{-d_j\tau}(cos(\mu\tau)(E\mu^3-G\mu)+sin(\mu\tau)(-F\mu^2))\\&-e^{-2d_j\tau}(cos(2\mu\tau)(I\mu)+sin(2\mu\tau)(-H\mu^2+J))\Big)^2.\end{aligned} \tag{13}$$

As $\tau\to\infty$, the right hand side of (13)$\to0$ and let $\gamma=\mu^2$ the equation (13) can be written in terms of γ as follows:

$$\begin{aligned}S(\gamma)=&\,\gamma^4+(A^2-2B)\gamma^3+(B^2+2D-2AC)\gamma^2\\&+(C^2-2BD)\gamma+D^2=0.\end{aligned} \tag{14}$$

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This can be simplified by substituting the known values of A, B, C and D. For the γ^3 coefficient, we have

$$\begin{aligned} A^2 - 2B &= 4a_1^2 - 16a_1 + 16 - 2(a_1^2 - 4a_1 \\ &\quad +4(1-(1-\alpha)^2\sigma_1^2)) \\ &= 2(a_1-2)^2 + 8(1-\alpha)^2\sigma_1^2 \end{aligned}$$

which is always positive.

Further, for the γ^2 and γ coefficients, we have

$$B^2 + 2D - 2AC = \Big(a_1^2 + 4(1-(1-\alpha)^2\sigma_1^2)\Big)^2 + \Big((a_1-2) - 2(1-\alpha)\sigma_1\Big)\Big(2a_2^2 - 8a_1((a_1-2) + 2(1-\alpha)\sigma_1)\Big), \tag{15}$$

$$C^2 - 2BD = a_2^4 + 2a_2^2\Big((a_1-2) - 2(1-\alpha)\sigma_1\Big)^2 \Big((a_1-2) + 2(1-\alpha)\sigma_1\Big) \tag{16}$$

respectively. (15) and (16) are positive coefficient if the right hand side of (15) and (16) are greater than zero for certain value of α. Finally, the constant term D^2 is always positive.

Therefore all the coefficients of the polynomial (14) are positive and it has no positive real roots. In other words $i\mu$ is not a root of the characteristic equation (10) for increasing delay. Hence, the system (4) cannot lead to a bifurcation. It means that the semi trivial steady state is locally asymptotically stable for all values of delay [7]. We conclude the following proposition:

Proposition 2 A semi trivial steady state $(\underline{a_1}, \overline{a_1}, 0, 0)$ with characteristic equation (10) is locally asymptotically stable for all values of delay if and only if

- $(A+E) > 0$, $(C+G+I) > 0$, $(D+J) > 0$ and $(A+E)(B+F+H)(C+G+I) > (C+G+I)^2 + (A+E)^2(D+J)$. A, B, C and D are given by (11).
- $\Big((a_1^2+4(1-(1-\alpha)^2\sigma_1^2))^2 + ((a_1-2)-2(1-\alpha)\sigma_1)(2a_2^2 - 8a_1((a_1-2)+2(1-\alpha)\sigma_1))\Big) > 0$ for certain value of α.
- $\Big(a_2^4 + 2a_2^2((a_1-2)-2(1-\alpha)\sigma_1)^2((a_1-2)+2(1-\alpha)\sigma_1)\Big) > 0$ for certain value of α.

III. Numerical Examples

To show the behavior and properties of our analysis of the steady states, two examples will be given in this section.

Example 1

Consider the model (4) with parameters $b = 0.2$, $c = 0.5$, $d = 1.2$, $d_j = 1$, $\sigma_1 = 1.4$, $\sigma_2 = 0.1$, $\sigma_4 = 0.2$, $\sigma_5 = 0.5$, $\mu = 1$ with three initial conditions $(\underline{x}_\alpha, \overline{x}_\alpha) = (m_1 - (1-\alpha)\sigma_4, m_1 + (1-\alpha)\sigma_5)$ and $(\underline{y}_\alpha, \overline{y}_\alpha) = (m_2 - (1-\alpha)\sigma_4, m_2 + (1-\alpha)\sigma_5)$ where $m_1 = 1, 3, 2$ and $m_2 = 2, 2, 2$. For $\alpha = 1$ and $\tau = 0$, the semi trivial steady state of the model is $(1, 1, 0, 0)$ and it is stable. Hence, we conclude that for $\tau \geq 0$, it is locally asymptotically stable. The results is shown in Figures 1 and 2.

Example 2

Consider the same conditions of Example 1 but for $\alpha = 0.8$, the semi trivial steady state $(0.63, 1.63, 0, 0)$ is locally asymptotically stable for all values of τ. It means that the conditions of Proposition 2 are satisfied and it is shown in Figure 3.

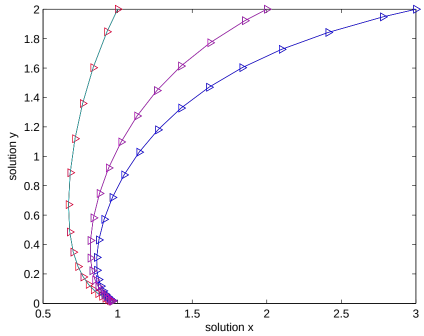

Fig. 1. The Steady State Converges to $(1, 0)$ for Different Initial Conditions and $\alpha = 1$, $\tau = 0$

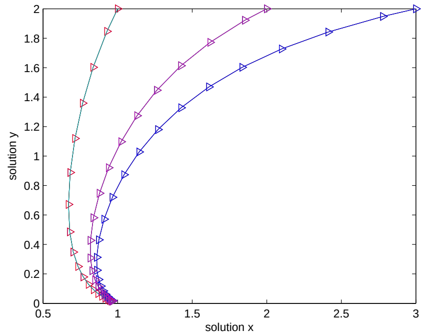

Fig. 2. The Steady State for $\alpha = 1$ and $\tau = 2$

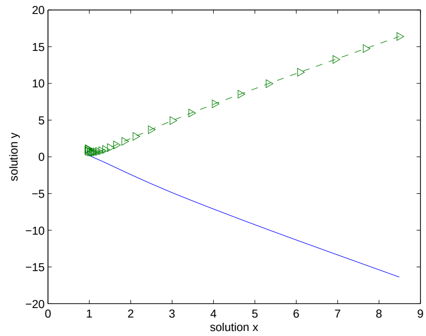

Fig. 3. The Steady State for $\alpha = 0.8$ and $\tau = 2$

IV. Conclusion

In this paper, we proposed a system of fuzzy delay predator-prey equations by using symmetric triangular fuzzy number. The crisp delay predator-prey of (2×2) system is extended to a FDPP of (4×4) system by using parametric form of $\alpha-$ cut. The FDPP system has trivial, semi trivial and nontrivial steady states. In this case the characteristic equation is of degree 4. The fuzzy system proposed leads to the difficulty of locating the roots of the characteristic equation since the system becomes larger compare with the crisp system. Generally, the situation is more complex to arrive at general conditions on the coefficients of characteristic equation such that it describes a locally asymptotically stable for semi trivial steady state for all values of delay, and the trivial steady state is always unstable. We conclude the results as in Propositions 1 and 2. We provide two examples to demonstrate the results.

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