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On an Orthogonal Method of Finding Approximate Solutions of Ill-Conditioned Algebraic Systems and Parallel Computation

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***Abstract*—A new method of finding approximate solutions of linear algebraic systems with ill-conditioned or singular matrices, using Schmidt orthogonalization, is presented. This method can be effectively used for arranging parallel computations for matrices of large size.**

***Index Terms*—ill conditioned matrices, eigenvalues, approximate solutions, parallel computation, Schmidt orthogonalization.**

This work is to continue [1], where we have considered an equation

$$Ax = f, \tag{1}$$

here A is a quadratic matrix of order n and f is n-dimensional vector. In in the paper [1] we solved a problem of parallelization of the problem (1) solving process and constructed sufficiently effective parallel algorithm for matrix A with bounded inverse.

In this paper we suggest a method for finding approximate solutions and parallel computation of the problem (1), when matrix A is noninvertible or ill-conditioned. Efficiency increase problem of solving large system of linear equations depends on development of high-effective calculating technics. Today multiprocessor systems and supercomputers is highly developed. Distribution of calculations into parallel branches implies the increase of solving general problem. Parallel computation of linear algebraic problems have been considered, for example, in monographs [2-4], and program implementation issues in [5].

The difference of an offered method from the known consists that existence of zero eigenvalues of a matrix A doesn't influence in any way efficiency of iterative process. Only small but nonzero eigenvalues of A^*A are important. Besides, estimates obtained by us in the theorem 3 for the solution doesn't depend on small and nonzero eigenvalues of a matrix A^*A. Also we will notice that the iterative formula (3), as far as we know, in computing practice wasn't applied earlier.

We denote by A^* adjoint matrix of A. Nonnegative square roots of eigenvalues of nonnegative matrix A^*A we denote by $s_j(A) = s_j, (j = 1, 2, ...)$ and numerate them in non-increasing order taking into account their multiplisities.

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Orthonormal eigenvectors of operator A^*A corresponding to s_j^2 we write as e_j $(j = 1, 2, ...; A^*Ae_j = s_j^2 e_j)$.

Numbers $s_1(A) \geq s_2(A) \geq ... \geq s_n(A)$ are called singular values of matrix A.

Note that the notion "ill-conditioned matrix" is relative, which often depends on hardware capabilities and its "boundary" moves away together with the power increase of computers. Let's say that the matrix A is ill-conditioned, if $\|A\|\|A^{-1}\|$ is large, when A is nonsingular.

Further vector's Euclidean norm and modulus of the number we write as $|\cdot|$, and operator's norm of matrix as $\|\cdot\|$, and scalar product as $\langle \cdot, \cdot \rangle$.

Let A and f be from (1), for $\varepsilon \geq 0$ consider functional

$$J_\varepsilon(x) = |Ax - f|^2 + \varepsilon |x|^2.$$

We will find $\mathring{x}$, which will be a solution of the problem

$$\inf J_\varepsilon(x) = J_\varepsilon(\mathring{x}). \tag{2}$$

In the left-hand side (2) $infimum$ is taken with respect to all vectors x in R^n. Since unit ball in R^n is compact, then solution of (2) exists.

Remark 1: If matrix A is invertible, then for $\varepsilon = 0$ problem (2) has unique solution $\mathring{x} = A^{-1}f$, if A is noninvertible, then $A\mathring{x}$ gives the best approximation f by elements Ax. If $\varepsilon \neq 0$ and A is invertible, then $\mathring{x}$ is the approximate solution of equation $Ax = f$.

If $\varepsilon = 0$ and matrix A is noninvertible, then the problem 1 may have several solutions, in this case we search for solution with minimal norm.

Lemma 1: If $\varepsilon \geq 0$ and $\mathring{x}$ is the solution of (2), then $A^*(A\mathring{x} - f) + \varepsilon \mathring{x} = 0$.

Proof. Let $\mathring{x}$ be the solution of (2) and $\omega = A^*(A\mathring{x} - f) + \varepsilon \mathring{x} \neq 0$. Consider $J_\varepsilon(\mathring{x} + \delta\omega)$. We have

$$\begin{aligned} J_\varepsilon(\mathring{x} + \delta\omega) &= J_\varepsilon(\mathring{x}) + 2\delta\langle A^*(A\mathring{x} - f) + \varepsilon\mathring{x}, \omega\rangle + \delta^2(|A\omega|^2 + \varepsilon|\omega|^2) \\ &= J_\varepsilon(\mathring{x}) + 2\delta|\omega|^2 + \delta^2(|A\omega|^2 + \varepsilon|\omega|^2). \end{aligned}$$

Let a number δ satisfy the following conditions

$$\delta < 0, -2\delta > \delta^2 \frac{|A\omega|^2 + \varepsilon|\omega|^2}{|\omega|^2}.$$

Such a choice is possible by assumption $\omega \equiv A^*(A\mathring{x} - f) + \varepsilon\mathring{x} \neq 0$. Then we get $J_\varepsilon(\mathring{x} + \delta\omega) < J_\varepsilon(\mathring{x})$ and it is contradiction. Lemma is proved.

Lemma 2: Let $\varepsilon \geq 0$ and $\mathring{x}$ be the solution of (2). Then for all $x \in H$ we have

$$\varepsilon x + A^*(Ax - f) = (\varepsilon + A^*A)(x - \mathring{x}).$$

Proof. It easily follows from Lemma 1.

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Now we define sequence x_j $(j = 1, 2, ...)$ by the following formula

$$x_j = \delta \sum_{k=0}^{j-1} [E - \delta(A^*A + \varepsilon E)]^k A^* f, \qquad (3)$$

where δ satisfies the condition

$$0 < \delta < \frac{2}{\|A^*A\| + \varepsilon}. \qquad (4)$$

Theorem 1: Let $\varepsilon \geq 0$, δ given by (4) and $\mathring{x}$ be the solution of (2), x_j be constructed by (3). Then

$$x_j - \mathring{x} = -[E - \delta(A^*A + \varepsilon E)]^j \mathring{x} \qquad (5)$$

and x_j converges to $\mathring{x}$ as $j \to +\infty$ at the geometric rate, i.e. there exists $\rho > 0$ and

$$|x_j - \mathring{x}| \leq C \cdot \rho^j, \qquad (6)$$

where C is the constant which depends on δ and ε.

Proof. By using lemma 1 we have

$$A^* f = \varepsilon \mathring{x} + A^*A\mathring{x}.$$

Substituting A^*f to (3) we get

$$\begin{aligned} x_j &= \delta \sum_{k=0}^{j-1} [E - \delta(A^*A + \varepsilon E)]^k [\varepsilon \mathring{x} + A^*A\mathring{x}] \\ &= \sum_{k=0}^{j-1} [E - \delta(A^*A + \varepsilon E)]^k [E - E + \delta(\varepsilon + A^*A)]\mathring{x} \\ &= -(E - \delta(A^*A + \varepsilon))^j \mathring{x} + \mathring{x}. \end{aligned}$$

It implies (5).

Further, since matrix $E - \delta(A^*A + \varepsilon E)$ is self-adjoint, then its norm is equal to maximum of modulus of eigenvalues. Its eigenvalues are $1 - \delta({s_j}^2 + \varepsilon)$, $(j = 1, 2, ..., n)$. If for each $j = 1, 2, ..., n$ eigenvalues satisfy

$$-1 < 1 - \delta({s_j}^2 + \varepsilon) < 1$$

we get

$$\|E - \delta(A^*A + \varepsilon E)\| < 1. \qquad (7)$$

These inequalities hold if conditions $\delta(\max_{j=1,2,...,n} {s_j}^2 + \varepsilon) < 2$ and $\delta > 0$ take place. But $\max_{j=1,2,...,n} {s_j}^2 = \|A^*A\|$. The condition (4) follows (7) and by (7) we get (6). The proof of theorem is complete.

Note that results similar to theorem 1 for linear ill-posed problems have been obtained in [6] (see [6], p. 238).

The space generated by eigenvectors of matrix A^*A corresponding to zero eigenvalues we denote by $R_0^{(n)}$, i. e. if $\mathring{x} \in R_0^{(n)}$ then $x = \sum_{k=j_0}^{n} x_j e_j$ and $A^*Ae_k = 0$ for $k = j_0, ..., n$. $R_0^{(n)}$ is the kernel of matrix A^*A.

If matrix A is invertible, then the space $R_0^{(n)}$ is empty.

Lemma 3: If $x \in R_0^{(n)}$ then $\langle A^*f, x\rangle = 0$, i. e. A^*f belongs to $R^{(n)} \ominus R_0^{(n)}$ which is the orthogonal complement of $R_0^{(n)}$.

Proof. For $\varepsilon = 0$ by using lemmas 1 and 2 we obtain $A^*f = A^*A\mathring{x}$. Let $x \in R_0^{(n)}$, then

$$\langle A^*f, x\rangle = \langle A^*A\mathring{x}, x\rangle = \langle \mathring{x}, A^*Ax\rangle = 0.$$

This completes proof of the lemma.

Lemma 4: If $\varepsilon > 0$ and $\mathring{x}$ is the solution of (2), $x \in R_0^{(n)}$, then $\langle \mathring{x}, x\rangle = 0$, i.e. $\mathring{x}$ belongs to $R^{(n)} \ominus R_0^{(n)}$.

Proof. It easily follows from Lemmas 1 and 3.

Note that for $\varepsilon = 0$ the solution of (2) is determined up to a term, which is the solution of equation $Ax = 0$, but sequence x_j $(j = 1, 2, ...)$ by lemma 2 converges to the solution of (2) belonging to $R^{(n)} \ominus R_0^{(n)}$. In further for $\varepsilon = 0$ we take as $\mathring{x}(0)$ the limit of sequence x_j from (3).

Obviously the solution of (2) depends on ε. So sometimes we write $\mathring{x} = \mathring{x}(\varepsilon)$.

We have

Lemma 5: If $\mathring{x}(0)$ is the solution of (2), then for every $\varepsilon > 0$ and $\delta \geq 0$

$$\mathring{x}(\varepsilon) = (A^*A + \varepsilon E)^{-1} A^*A\mathring{x}(o) = (A^*A + \varepsilon E)^{-1} A^* f,$$

$$\mathring{x}(0) = (E + \varepsilon(A^*A)^{-1})\mathring{x}(\varepsilon),$$

$$\mathring{x}(\varepsilon) = (A^*A + \varepsilon E)^{-1}(A^*A + \delta E)\mathring{x}(\delta),$$

$$\mathring{x}(\varepsilon) - \mathring{x}(\delta) = (\delta - \varepsilon)(A^*A + \varepsilon E)^{-1}\mathring{x}(\delta).$$

Proof. It easily follows from Lemma 1. From proved lemmas and theorem 1 we state

Theorem 2: a) The solution $\mathring{x}(\varepsilon)$ of (2) continuously depends on $\varepsilon > 0$ and $\mathring{x}(\varepsilon) = (A^*A + \varepsilon E)^{-1} A^* f$.

b) For $j \to +\infty$ the limit of sequence $x_j(\varepsilon)$ from (3) continuously depends on $\varepsilon \geq 0$.

c) If $s_1 \geq s_2 \geq \ldots \geq s_{j_0} > 0, s_{j_0+1} = s_{j_0+2} = ... = 0$ are the eigenvalues of matrix A^*A and e_1, e_2, ..., e_n are corresponding orthonormal system of eigenvectors, $\mathring{x}(\varepsilon)$ $(\varepsilon > 0)$ is the solution of (2) and $x_j(\varepsilon)$ from (3), then $\mathring{x}(\varepsilon)$, $x_j(\varepsilon) \in R^{(n)} \ominus R_0^{(n)}$, $(j = 1, 2, \ldots)$

$$\begin{array}{rcll} x_{jk}(\varepsilon) - \mathring{x}_k(\varepsilon) &=& (1 - \delta(s_k^2 + \varepsilon))^j \mathring{x}_k(0), & 1 \leq k \leq j_0, \\ x_{jk}(\varepsilon) &=& \mathring{x}_k(\varepsilon), & j_0 + 1 \leq k. \end{array} \qquad (8)$$

Here $x_{jk}(\varepsilon) = \langle x_j(\varepsilon), e_k\rangle$, $\mathring{x}_k(\varepsilon) = \langle \mathring{x}(\varepsilon), e_k\rangle$.

d) Number $\rho > 0$ from theorem 1 is defined by

$$\rho = \max\{(1 - \delta(s_{j_0}^2 + \varepsilon)), (1 - \delta(\|A^*A\| + \varepsilon))\} < 1.$$

Note that if matrix A^*A hasn't zero eigenvalues then j_0 is taken as n.

The item c) of theorem 2 implies that vector $\mathring{x}(\varepsilon)$ for $\varepsilon = 0$ has minimal norm among all solutions of problem 1. Furthermore for each $\varepsilon \geq 0$ $\mathring{x}(\varepsilon)$ and $x_j(\varepsilon)$ $(j = 1, 2, \ldots)$ belong to subspace $R^{(n)} \ominus R_0^{(n)}$, where $R_0^{(n)}$ is the kernel of matrix A^*A.

Below we suggest one method of parallel computation for solving problem (2) based on theorems 1 and 2.

Let n be large enough integer and we have $N+1$-processor system. Let $k_0, k_1, \ldots, k_N$ are integers such that $k_{m-1} + 1 < k_m$, $m = 0, 1, \ldots, N$, $k_0 = 0$, $k_N = n$. We define matrices A_m

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 \\ a_{k_{m-1}+1,1} & a_{k_{m-1}+1,2} & a_{k_{m-1}+1,3} & \ldots & a_{k_{m-1}+1,n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{k_m,1} & a_{k_m,2} & a_{k_m,3} & \ldots & a_{k_m,n} \\ 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0. \end{pmatrix}$$

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and $(A^*)_m$

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \\ \tilde{a}_{k_{m-1}+1,1} & \tilde{a}_{k_{m-1}+1,2} & \tilde{a}_{k_{m-1}+1,3} & \dots & \tilde{a}_{k_{m-1}+1,n} \\ \dots & \dots & \dots & \dots & \dots \\ \tilde{a}_{k_m,1} & \tilde{a}_{k_m,2} & \tilde{a}_{k_m,3} & \dots & \tilde{a}_{k_m,n} \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0. \end{pmatrix},$$

$m = 1, 2, \dots, N$, in such a way that lines numerated from $k_{m-1}+1$ to k_m coincide with those of matrices A and A^* respectively. Here $\tilde{a}_{kj}$ and a_{kj} are elements of A^* and A such that $\tilde{a}_{kj} = a_{jk}$. Also we use vectors $\omega^j = [E - \delta(A^*A + \varepsilon E)]\,\omega^{j-1}$, $\omega^0 = \delta A^* f$, $j = 1, 2, \dots$. Then formula (3) can be written in the following way

$$x_{j+1} = x_j + \omega^j, x_1 = \omega^0, j = 1, 2, \dots$$

Before computation matrices A_m and $(A^*)_m$ are passed to processors C_m $(m = 1, 2, \dots, N)$ and vector $\omega^0 = \delta A^* f$ to root processor C_{N+1}. Algorithm works as follows

1) Processor C_{N+1} forms j-th approximation x_j and passes vector ω^{j-1} to processors C_m $(m = 1, 2, \dots, N)$. Each processor C_m calculates $A_m \omega^{j-1}$ spending $(k_m - k_{m-1})n$ multiplications, $(k_m - k_{m-1})(n-1)$ additions and sends vector to C_{N+1}.
2) C_{N+1} forms vector $A\omega^{j-1} = \sum_{m=1}^{N} A_m \omega^{j-1}$ and spends $(n-1)N$ additions. C_{N+1} transmits vector $A\omega^{j-1}$.
3) Processor C_m calculates $(A^*)_m A\omega^{j-1}$ and sends it to C_{N+1}, spending $(k_m - k_{m-1})n$ multiplications and $(k_m - k_{m-1})(n-1)$ additions. C_m transmits $(A^*)_m A\omega^{j-1}$ to processor C_{N+1}.
4) Summing up recieved vectors C_{N+1} gets $A^*(A\omega^{j-1}) = \sum_{m=1}^{N} (A^*)_m A\omega^{j-1}$. Root processor calculates $\omega^j = (1 - \delta\varepsilon)\omega^{j-1} - \delta(A^*)_m A\omega^{j-1}$ and forms approximate solution $x_{j+1} = x_j + \omega^j$, $j = 1, 2, \dots$. It spends $2n$ multiplications and $(n-1)N + 2n$ additions.
   Amount of operations per cycle which root processor C_{N+1} carries out consists of $2n$ multiplications and $2(n-1)N + 2n$ additions. Each processor C_m $(m = 1, 2, \dots, N)$ spends $2(k_m - k_{m-1})n$ multiplications and $2(k_m - k_{m-1})(n-1)$ additions per cycle. All processors spend $2n^2$ multiplications and $2n(n-1)$ additions per cycle. Since s iterations all computers spend $2sn^2 + n^2 + n$ multiplications and $2sn(n-1) + n(n-1)$ additions. Division is absent.

It follows from theorem 2 that for effectivicity of iteration formula (3) the existence of small but nonzero eigenvalues of matrix A^*A are important, but existence of zero eigenvalues of the matrix A doesn't play any role! Therefore we come to the question: Is it possible to reduce "noises" due to nonzero small eigenvalues of matrix A^*A? It turns out that it is possible. We demonstrate it by simple example.

Let $\varepsilon = 0.01$ and

$$A = \begin{pmatrix} 1 & 1 \\ 3 & 3.001 \end{pmatrix}, \ f = \begin{pmatrix} 2 \\ 6.006 \end{pmatrix} \tag{9}$$

Matrix

$$A^*A = \begin{pmatrix} 1 & 3 \\ 1 & 3.001 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & 3.001 \end{pmatrix}$$

has a small nonzero eigenvalue. Iterative process by formula (8) may last long. However, if matrix A is replaced with its approximation

$$\tilde{A} = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}, \ \tilde{f} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \tag{10}$$

then we have

$$\tilde{A}^*\tilde{A} = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 10 & 10 \\ 10 & 10 \end{pmatrix}$$

The eigenvalues of this matrix are equal to $\lambda_1^2 = 20$, $\lambda_2^2 = 0$ and its norm is 20. So δ may be taken from interval $(0, \frac{1}{10})$. Let's take $\delta = \frac{1}{20} < \frac{1}{10}$. Then by theorem 2 we obtain $\rho = 0$. Therefore the problem 1 with matrix $\tilde{A}$ and vector $\tilde{f}$ from (10) is solved in one step.

The solution of problem 1 is vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Equation

$$\tilde{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

has solution vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that $x_1 + x_2 = 2$. Vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ satisfies it and has a minimal norm among all vectors. The vector found $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ will be the approximate solution of (2) with matrix A and vector f from (9). Indeed

$$\begin{pmatrix} 1 & 1 \\ 3 & 3.001 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 6.006 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.006 \end{pmatrix}.$$

We have $|A\mathring{x} - f| = 0.006 \approx 0$. (Recall that we try to reduce a norm $|Ax - f|$ increasing the norm $|x|$ not too much.)

The vector $\tilde{x} = \begin{pmatrix} \tilde{x_1} \\ \tilde{x_2} \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$ is the actual solution of system

$$\begin{pmatrix} 1 & 1 \\ 3 & 3.001 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 6.006 \end{pmatrix}.$$

For $\mathring{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ we get

$$|A\mathring{x} - f|^2 + \varepsilon|\mathring{x}|^2 = \left| \begin{pmatrix} 0 \\ 0.006 \end{pmatrix} \right|^2 + 0.01 \left| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right|^2$$
$$= (0.006)^2 + 0.01 \approx 0.01.$$

And for $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$

$$|A\tilde{x} - f|^2 + \varepsilon|\tilde{x}|^2 = 0 + 0.01(16 + 24) = 0.4.$$

Therefore for $\varepsilon = 0.01$ vector $\mathring{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is closer to solution of problem 1 than $\tilde{x} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$.

This simple idea tracked on simple example we will develop in the next work with matrices arisen in solving numerically ill-posed direct and inverse problems of mathematical physics.

In general this effect is not always possible. But we have

Theorem 3: Let $\varepsilon \geq 0$ and x_j $(j = 1, 2, \dots)$ be the sequence of vectors from (3). Then

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a) If $\gamma > 0$ and j is chosen to satisfy $(1-\delta(\gamma+\varepsilon))^{2j} \leq \gamma$, then the following inequality

$$|Ax_j(\varepsilon) - f| \leq 2|f|\sqrt{\gamma} + \gamma|\mathring{x}(\varepsilon)| + |A\mathring{x}(\varepsilon) - f|$$

holds

b) If j is given by

$$(1-\delta(\gamma+\varepsilon))^{2j} \leq \gamma^2,$$

then

$$|A^*A(x_j(\varepsilon) - \mathring{x}(\varepsilon))|^2 \leq \gamma^2 \left[|A^*A\mathring{x}|^2 + |\mathring{x}|^2\right];$$

c) If j is given by

$$(1-\delta(\gamma+\varepsilon))^{2j} \leq \frac{2}{5\|A^*\|}\gamma,$$

then

$$|A^*A(x_j(\varepsilon) - \mathring{x}(\varepsilon))|^2 \leq 8\gamma|f|^2;$$

d) For $\varepsilon = 0$, if j is given by

$$(1-\delta\gamma)^{2j} \leq \frac{2}{5\|A^*\|}\gamma,$$

then

$$|A^*A(x_j(0) - f)|^2 \leq 8\gamma|f|^2. \quad (11)$$

Proof. Let $\varepsilon \geq 0$ and

$$\inf_{\{x\}}\left(|Ax-f|^2 + \varepsilon|x|^2\right) = |A\mathring{x}(\varepsilon) - f|^2 + \varepsilon|\mathring{x}(\varepsilon)|^2$$

For any vector u we have

$$|Au|^2 = \langle Au, Au\rangle = \langle A^*Au, u\rangle = \left|(A^*A)^{\frac{1}{2}}u\right|^2.$$

Therefore, as $(A^*A)^{\frac{1}{2}}e_k = s_k e_k$ and using (8) from theorem 2 we have

$$\begin{aligned}|A(x_j(\varepsilon) - \mathring{x}(\varepsilon))|^2 &= \left|(A^*A)^{\frac{1}{2}}(x_j(\varepsilon) - \mathring{x}(\varepsilon))\right|^2 \\ &= \sum_{k=1}^{n} s_k^2 (x_{jk}(\varepsilon) - \mathring{x}_k(\varepsilon))^2 \\ &= \sum_{k=1}^{n} s_k^2 \left(1-\delta(s_k^2+\varepsilon)\right)^{2j} |\mathring{x}_k(\varepsilon)|^2.\end{aligned}$$

Hence for all $\gamma > 0$ we get

$$\begin{aligned}|A(x_j(\varepsilon) - \mathring{x}(\varepsilon))|^2 &= \sum_{s_k^2>\gamma} s_k^2\left(1-\delta(s_k^2+\varepsilon)\right)^{2j}|\mathring{x}_k(\varepsilon)|^2 \\ &+ \sum_{s_k^2\leq\gamma} s_k^2\left(1-\delta(s_k^2+\varepsilon)\right)^{2j}|\mathring{x}_k(\varepsilon)|^2 \\ &\leq (1-\delta(\gamma+\varepsilon))^{2j}\sum_{s_k^2>\gamma} s_k^2|\mathring{x}_k(\varepsilon)|^2 + \gamma\sum_{s_k^2\leq\gamma}|\mathring{x}_k(\varepsilon)|^2 \\ &\leq (1-\delta(\gamma+\varepsilon))^{2j}\sum_{k=1}^{n} s_k^2|\mathring{x}_k(\varepsilon)|^2 + \gamma\sum_{k=1}^{n}|\mathring{x}_k(\varepsilon)|^2 \\ &= (1-\delta(\gamma+\varepsilon))^{j}\left|(A^*A)^{\frac{1}{2}}\mathring{x}(\varepsilon)\right|^2 + \gamma|\mathring{x}(\varepsilon)|^2 \\ &= (1-\delta(\gamma+\varepsilon))^{j}|A\mathring{x}(\varepsilon)|^2 + \gamma|\mathring{x}(\varepsilon)|^2 \quad (12)\end{aligned}$$

But

$$\begin{aligned}|A\mathring{x}(\varepsilon)|^2 &= |A\mathring{x}(\varepsilon) - f + f|^2 \leq 2\left(|A\mathring{x}(\varepsilon) - f|^2 + |f|^2\right) = \\ &= 2\left[\left(\inf_{\{x\}}\left(|Ax-f|^2 + \varepsilon|x|^2\right)\right) + |f|^2\right] \leq \\ &\leq 2(|f|^2 + |f|^2) = 4|f|^2.\end{aligned} \quad (13)$$

Using this estimate and (12) we arrive at the estimate

$$|A(x_j(\varepsilon) - \mathring{x}(\varepsilon))|^2 \leq 4[1-\delta(\gamma+\varepsilon)]^{2j}|f|^2 + \gamma|\mathring{x}(\varepsilon)|^2.$$

Then

$$\begin{aligned}|Ax_j(\varepsilon) - f| &= |A(x_j(\varepsilon) - \mathring{x}(\varepsilon)) + A\mathring{x}(\varepsilon) - f| \\ &\leq [|A(x_j(\varepsilon) - \mathring{x}(\varepsilon))| + |A\mathring{x}(\varepsilon) - f|] \\ &\leq 2|f|(1-\delta(\gamma+\varepsilon))^{j} + |\mathring{x}(\varepsilon)|\sqrt{\gamma} + |A\mathring{x}(\varepsilon) - f|\end{aligned}$$

If it take place conditions

$$(1-\delta(\gamma+\varepsilon))^{2j} \leq \gamma$$

then we have from the last inequality

$$|Ax_j(\varepsilon) - f| \leq 2|f|\sqrt{\gamma} + \sqrt{\gamma}|\mathring{x}(\varepsilon)| + |A\mathring{x}(\varepsilon) - f|.$$

It implies item a) of theorem.

Further, using (8) we have

$$\begin{aligned}|A^*A(x_j(\varepsilon) - \mathring{x}(\varepsilon))|^2 &= \sum_{k=1}^{n} s_k^4\left(1-\delta(s_k^2+\varepsilon)\right)^{2j}|\mathring{x}_k(\varepsilon)|^2 \\ &= \sum_{s_k^2>\gamma} s_k^4\left(1-\delta(s_k^2+\varepsilon)\right)^{2j}|\mathring{x}_k(\varepsilon)|^2 \\ &+ \sum_{s_k^2\leq\gamma} s_k^4\left(1-\delta(s_k^2+\varepsilon)\right)^{2j}|\mathring{x}_k(\varepsilon)|^2 \\ &\leq \sum_{s_k^2>\gamma} s_k^4\left(1-\delta(s_k^2+\varepsilon)\right)^{2j}|\mathring{x}_k(\varepsilon)|^2 \\ &+ \sum_{s_k^2\leq\gamma} s_k^4\left(1-\delta(s_k^2+\varepsilon)\right)^{2j}|\mathring{x}_k(\varepsilon)|^2\end{aligned} \quad (14)$$

It follows from the inequality above that

$$|A^*A(x_j(\varepsilon) - \mathring{x}(\varepsilon))|^2 \leq (1-\delta(\gamma+\varepsilon))^{2j}|A^*A\mathring{x}|^2 + \gamma|\mathring{x}|^2.$$

If j is taken from

$$(1-\delta(\gamma+\varepsilon))^{2j} \leq \gamma^2,$$

we obtain

$$|A^*A(x_j(\varepsilon) - \mathring{x}(\varepsilon))|^2 \leq \gamma^2\left[|A^*A\mathring{x}|^2 + |\mathring{x}|^2\right].$$

It implies assertion of item b) of theorem.

By (14) we have also

$$\begin{aligned}|A^*A(x_j(\varepsilon) - \mathring{x}(\varepsilon))|^2 &\leq \sum_{k=1}^{n}(1-\delta(\gamma+\varepsilon))^{2j}\left|s_k^2\mathring{x}_k(\varepsilon)\right|^2 \\ &+ \sum_{k=1}^{n} s_k^2|\mathring{x}_k(\varepsilon)|^2 \\ &= (1-\delta(\gamma+\varepsilon))^{2j}|A^*A\mathring{x}_k(\varepsilon)|^2 + \gamma|A^*A\mathring{x}_k(\varepsilon)|^2 \\ &= (1-\delta(\gamma+\varepsilon))^{2j}|A^*(A\mathring{x}_k(\varepsilon) - f) + A^*f|^2 + \gamma|A\mathring{x}_k(\varepsilon)|^2 \\ &\leq 2(1-\delta(\gamma+\varepsilon))^{2j}\left(|A^*(A\mathring{x}_k(\varepsilon) - f)|^2 + |A^*f|^2\right) \\ &+ \gamma|A\mathring{x}_k(\varepsilon)|^2.\end{aligned}$$

But in view of (13) we get following inequalities

$$|A^*(A\mathring{x}_k - f)|^2 \leq \|A^*\|^2\left(|A\mathring{x}_k|^2 + |f|^2\right) \leq 5\|A^*\|^2|f|^2,$$

$$|A\mathring{x}_k(\varepsilon)|^2 \leq 4|f|^2.$$

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Therefore

$$\begin{aligned}|A^*A(x_j(\varepsilon)-\mathring{x}(\varepsilon))|^2 \le& 2(1-\delta(\gamma+\varepsilon))^{2j}\\ &\times\left[5\|A^*\|^2|f|^2+|A^*f|^2\right]+4\gamma|f|^2\\ \le& 10(1-\delta(\gamma+\varepsilon))^{2j}\\ &\times\|A^*\|^2|f|^2+4\gamma|f|^2.\end{aligned}$$

Choosing j from

$$(1-\delta(\gamma+\varepsilon))^{2j}\,10\,\|A^*\|^2\le 4\gamma,$$

we get

$$|A^*A(x_j(\varepsilon)-\mathring{x}(\varepsilon))|^2\le 8\gamma|f|^2.$$

This completes assertion of item c) of the theorem.

For $\varepsilon=0$ using lemma 2 we have

$$A^*A\mathring{x}=Af.$$

Therefore the item c) of the theorem implies the item d). This completes proof of the theorem.

Usually it is important in practice to reduce difference $Ax-f$ (less important to find solution of equation $Ax=f$!). Thus the theorem allows to solve problem (2) effectively. Note that usage of formula for x_j doesn't require $\varepsilon>0$. Much more suitable case is $\varepsilon=0$.

Now we can suggest the next numerical algorithm for the problem (2) based on the theorem 3. We can form sufficiently effective process of solving problem (2) with ill-conditioned or non-invertible matrix. The algorithm will distinguished be from above one only by these points:

It is chosen $\gamma>0$ (stands for accuracy). The number ε is chosen to be zero. Item d) condition of the theorem 3 is verified after every cycle iteration. Computation is finished when condition (11) holds.

Now we will describe in brief Schmidt orthogonalization process.

Let $e_1,\dots,e_n$ be a basis in H and A is ill-conditioned matrix. Then

$$Ae_j=(a_{1j},a_{2j},\dots,a_{nj}),\quad A=\{a_{ij}\}.$$

We will orthogonalize $\{Ae_j\}_{j=1}^n$ by Schmidt. Fix a number $\varepsilon>0$. If $|Ae_1|>\varepsilon$, then we put $\psi_1=(Ae_1)|Ae_1|^{-1}$. Otherwise, if $|Ae_1|\le\varepsilon$, then put $\psi_1=0$.

Let vectors$\psi_1,\dots,\psi_j$ are constructed. We define

$$\widetilde{\psi}_{j+1}=Ae_{j+1}-\sum_{k=1}^{j}\alpha_k\psi_k,$$

where

$$\alpha_k=\begin{cases}0, & \text{if } \psi_k=0\\ \langle Ae_{j+1},\psi_k\rangle, & \text{if } \psi_k\ne 0.\end{cases}$$

Now we define ψ_{j+1}. If

$$\widetilde{\psi}_{j+1}\ne 0\qquad \left|\widetilde{\psi}_{j+1}\right|\left|e_{j+1}-\sum_{k=1}^{j}\alpha_kA^{-1}\psi_k\right|^{-1}>\varepsilon,$$

then we put

$$\psi_{j+1}=\widetilde{\psi}_{j+1}\left|\widetilde{\psi}_{j+1}\right|^{-1}.$$

In the case of

$$\widetilde{\psi}_{j+1}=0\quad\text{or}\quad\left|\widetilde{\psi}_{j+1}\right|\left|e_{j+1}-\sum_{k=1}^{j}\alpha_kA^{-1}\psi_k\right|^{-1}\le\varepsilon$$

then put $\psi_{j+1}=0$.

Calculation of $A^{-1}\psi_k$ doesn't meet difficulty. If $\psi_k=0$, we get $A^{-1}\psi_k=0$, otherwise $\psi_k\ne 0$, in view of formula

$$\widetilde{\psi}_{k+1}=Ae_{k+1}-\sum_{l=1}^{k}\alpha_l\psi_l,$$

we obtain the representation

$$\widetilde{\psi}_{k+1}=A\Theta_{k+1},$$

where Θ_{k+1} is defined recurrently.

Finally we get $\psi_1,\dots,\psi_n$, which satisfy the conditions $\langle\psi_i,\psi_j\rangle=0$, as $i\ne j$, $|\psi_j|=1$ or $\psi_j=0$.

Now put

$$\widetilde{f}=\sum_{j=1}^{n}\langle f,\psi_j\rangle\psi_j=\sum_{\psi_j\ne 0}\langle f,\psi_j\rangle\frac{A\Theta_j}{\left|\widetilde{\psi}_j\right|}=A\sum_{\psi_j\ne 0}\langle f,\psi_j\rangle\frac{\Theta_j}{\left|\widetilde{\psi}_j\right|}.$$

We get now as the approximate solution $\mathring{\widetilde{x}}$ to the problem

$$\inf_{x\in H}|Ax-f|=\left|A\mathring{x}-f\right|$$

the vector

$$\widetilde{x}=\sum_{\psi_j\ne 0}\langle f,\psi_j\rangle\frac{\Theta_j}{\left|\widetilde{\psi}_j\right|}.$$

It can be shown that the inequality

$$|A\widetilde{x}-f|\le C(\varepsilon),$$

holds with $C(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Some of results of this work have been announced in [7] (see also [8]).

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