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Adaptation and Optimization of Stochastic Airline Seat Inventory Control

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***Abstract*—The problem of adaptive stochastic airline seat inventory control lies at the heart of airline revenue management. This problem concerns the allocation of the finite seat inventory to the stochastic customer demand that occurs over time before the flight is scheduled to depart. The objective is to find the right combination of customers of various fare classes on the flight such that revenue is maximized. In this paper, the unbiased static and dynamic policies of stochastic airline seat inventory control (airline booking) are developed under parametric uncertainty of underlying models, which are not necessarily alternative. For the sake of simplicity, but without loss of generality, we consider (for illustration) the case of nonstop flights with two fare classes. The system developed is able to recognize a situation characterized by the number of reservations made by customers of the above fare classes at certain moment of time before departure. The proposed policies of the airline seat inventory control are based on the use of order statistics of cumulative customer demand, which have such properties as bivariate dependence and conditional predictability. Dynamic adaptation of the system to airline customer demand is carried out via the bivariate dependence of order statistics of cumulative customer demand. Dynamic optimization of the airline seat allocation is carried out via the conditional predictability of order statistics. The system makes on-line decisions as to whether to accept or reject any customer request using established decision rules based on order statistics of the current cumulative customer demand. The computer simulation results are promising.**

***Index Terms*—Airline booking, dynamic adaptation and optimization, stochastic demand**

I. Introduction

PASSENGER reservation systems have evolved from low level inventory control processes to major strategic information systems. Today, airlines and other transportation companies view revenue management systems and related information technologies as critical determinants of future success. Indeed, expectations of revenue gains that are possible with expanded revenue management capabilities are now driving the acquisition of new information technology. Each advance in information technology creates an opportunity for more comprehensive reservations control and greater integration with other important transportation planning functions. It is common practice for airlines to sell a pool of identical seats at different prices according to different booking classes to improve revenues in a very competitive market. In other words, airlines sell the same seat at different prices according to different types of travelers (first class, business and economy) and other conditions. The question then arises whether to offer seats at a relatively low price at a given time with a given number of seats remaining or to wait for the possible arrival of a higher paying customer. Assigning seats in the same compartment to different fare classes of customers in order to improve revenues is a major problem of airline seat inventory allocation. This problem has been considered in numerous papers. For details, the reader is referred to a review of yield management, as well as perishable asset revenue management, by Weatherford *et al*. [1], and a review of relevant mathematical models by Belobaba [2].

This paper deals with the airline seat allocation problem when customers for different fare levels are booked into a common seating pool in the aircraft. The following assumptions are made: (1) single-leg flight: bookings are made on the basis of a single departure and landing; no allowance is made for the possibility that bookings may be part of larger trip itineraries, (2) independent demands: the demands for different fare classes are stochastically independent, (3) low before high demands: the lowest fare reservations requests arrive first, followed by the next lowest, etc., (4) no cancellations: cancellations, no-shows and overbooking are not considered, (5) nested classes: any fare class can be booked into seats not taken by bookings in lower fare classes, (6) fare classes: the business and economy fare classes are considered.

The first purpose of this paper is to present the innovative information technologies for constructing the unbiased static and dynamic policies of the airline seat inventory allocation on the basis of the 'unbiasedness performance index'. The static and dynamic policies (unbiased) are more efficient (from the point of view of airline revenue management) as compared with the policies, where the unknown parameters of the airline customer demand models are estimated and then treated as if they were the true values. At the initial stage of airline booking it may be used the static policy of seat inventory allocation, and at the fundamental stage may be used the dynamic policy.

The second purpose of this paper is to introduce the idea of prediction of a future cumulative customer demand for the seats on a flight via the order statistics from the underlying

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distribution, where only the functional form of the distribution is specified, but some or all of its parameters are unspecified. This idea allows one to use the technique of invariant embedding of sample statistics in a performance index in order to eliminate the unknown parameters from the problem [3-5]. The technique represents a simple and computationally attractive statistical method based on the constructive use of the invariance principle in mathematical statistics. Unlike the Bayesian approach, an invariant embedding technique is independent of the choice of priors, i.e., subjectivity of investigator is eliminated from the problem. It allows one to find the improved invariant statistical decision rules, which have smaller risk than any of the well-known traditional statistical decision rules, and to use the previous and current sample data as completely as possible.

II. State-of-the-art and Progress Beyond

Airline seat allocation is a very profitable tool in the airline industry. A major problem of airline seat allocation is to sell the same seat at different prices according to different types of travelers (first class, business and economy) and other conditions in order to improve revenues. This problem has been considered in numerous papers. Littlewood [6] was the first to propose a solution method of the airline seat allocation problem for a single-leg flight with two fare classes. The idea of his scheme is to equate the marginal revenues in each of the two fare classes. He suggests closing down the low fare class when the certain revenue from selling low fare seat is exceeded by the expected revenue of selling the same seat at the higher fare. That is, low fare booking requests should be accepted as long as

$$c_2 \geq c_1 \Pr\{Y_1 > u_1\}, \tag{1}$$

where c_1 and c_2 are the high and low fare levels respectively, Y_1 denotes the demand for the high fare (or business) class, u_1 is the number of seats to protect for the high fare class and $\Pr\{Y_1 > u_1\}$ is the probability of selling more than u_1 protected seats to high fare class customers. The smallest value of u_1 that satisfies the above condition is the number of seats to protect for the high fare class, and is known as the protection level of the high fare class customers. The concept of determining a protection level for the high fare class can also be seen as setting a booking limit, a maximum number of bookings, for the low fare class. Both concepts restrict the number of bookings for the low fare class in order to accept bookings for the high fare class.

It should be remarked that there is no protection level for the low fare (or economy) class; u_2 is the booking limit, or number of seats available, for the low fare class; the low fare class is open as long as the number of bookings in this class remains less than this limit. Thus, (u_1+u_2) is the booking limit or number of seats available for the high fare class at time. The high fare class is open as long as the number of bookings in this and low classes remain less than this limit.

Richter [7] gave a marginal analysis, which proved that (1) gives an optimal allocation (assuming certain continuity conditions). Optimal policies for more than two classes have been presented independently by Curry [8], Wollmer [9], Brumelle & McGill [10], and Nechval *et al.* [11].

III. Airline Booking Policies Known from Practice

A. Static Airline Booking Policy under Certainty

It will be noted that (1) represents the static policy of airline seat allocation (or airline booking) under complete information. If F_θ, the probability distribution function of Y_1 with the parameter θ (in general, vector), is continuous and strictly increasing, the definition (1) of u_1 is equivalent to

$$u_1 = \arg(\overline{F}_\theta(u_1) = \gamma) \tag{2}$$

where

$$\gamma = c_2 / c_1, \tag{3}$$

$$\overline{F}_\theta(u_1) = 1 - F_\theta(u_1). \tag{4}$$

B. Static Airline Booking Policy under Uncertainty

In practice, under parametric uncertainty, i.e. when the parameter θ is unknown, the performance index,

$$\overline{F}_{\widehat{\theta}}(u_1) = \gamma, \tag{5}$$

is usually used to construct the static policy given by

$$u_1 = \arg(\overline{F}_{\widehat{\theta}}(u_1) = \gamma), \tag{6}$$

where $\widehat{\theta}$ represents the maximum likelihood estimator of θ. The performance index (5) is named as ‘maximum likelihood performance index’. The static policy (6) based on (5) is named as ‘*static maximum likelihood airline booking policy*’.

C. Dynamic Airline Booking Policy

The static policy of airline booking is optimal as long as no change in the probability distributions of the customer demand is foreseen. However, information on the actual customer demand process can reduce the uncertainty associated with the estimates of demand. Hence, repetitive use of a static policy over the booking period, based on the most recent demand and capacity information, is the general way to proceed.

IV. Airline Booking Policies Proposed in the Paper

A. Static Airline Booking Policy under Uncertainty

This policy is based on the performance index,

$$E_\theta\{\overline{F}_\theta(u_1)\} = \gamma, \tag{7}$$

which takes into account (2) and the previous data of cumulative customer demand Y_1 for the seats on a flight. It allows one to construct the static unbiased airline booking policy given by

$$u_1^{(\mathrm{unb})} = \arg(E_\theta\{\overline{F}_\theta(u_1)\} = \gamma), \tag{8}$$

where $u_1 \equiv u_1(\widehat{\theta})$, $\widehat{\theta}$ represents either the maximum likelihood estimator of θ or sufficient statistic S for θ, i.e., $u_1 \equiv u_1(S)$. The performance index (7) is named as ‘unbiasedness performance index’. The static policy (8), which is based on (7), is named as ‘*static unbiased airline*

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booking policy'.

The relative bias of the static airline booking policy is given by

$$r(u_1) = \frac{|E_\theta\{\overline{F}_\theta(u_1)\} - \gamma|}{\gamma} 100\%. \tag{9}$$

B. Dynamic Airline Booking Policy under Certainty

In this section, we consider a flight for a single departure date with m predefined reading dates at which the dynamic policy is to be updated, i.e., the booking period before departure is divided into m readings periods: $(0, \tau_1]$, $(\tau_1, \tau_2]$, …, $(\tau_{m-1}, \tau_m]$ determined by the m reading dates: τ_1, τ_2, …, τ_m. These reading dates are indexed in increasing order: $0<\tau_1<\tau_2< \cdots <\tau_m$, where $(\tau_{m-1}, \tau_m]$ denotes the reading period immediately preceding departure, and τ_m is at departure. Typically, the reading periods that are closer to departure cover much shorter periods of time than those further from departure. For example, the reading period immediately preceding departure may cover 1 day whereas the reading period 1-month from departure may cover 1 week.

Let us suppose that the cumulative passenger demand for the high fare class at the kth reading date (time τ_k, $1\le k\le m$) is Y_{1k} representing the kth order statistic from the underlying distribution with the probability distribution function $G_\theta(y_{1k})$, where θ is a parameter (in general, vector). In other words, Y_{1k} represents the number of seats sold for the customers of the high fare class at the kth reading date. We assume that the cumulative passenger demands for the high and low fare classes are stochastically independent. Each booking of a seat of the high fare class generates average revenue of c_1. Each booking of a seat of the low fare class generates average revenue of c_2, where $c_2<c_1$. Let u_{1k} be an individual protection level for the high fare class at time τ_k (the kth reading date). This many seats are protected for the high fare class from the low fare class. There is no protection level for the low fare class; u_{2k} is the booking limit for the low fare class at time τ_k; the low fare class is open as long as the number of bookings in this class remains less than this limit. Thus, $(u_{1k}+u_{2k})$ is the booking limit for the high fare class at time τ_k. The high fare class is open as long as the number of bookings in this and low classes remain less than this limit. The maximum number of seats that may be booked by fare classes in the next at time τ_k prior to flight departure is the number of unsold seats u_k°.

Under the complete information, the dynamic airline booking policy is given by

$$u_{1k} = \arg(\overline{G}_\theta(u_{1k} \mid y_{1k}) = \gamma), \quad k = 1, 2, ..., m-1, \tag{10}$$

where

$$\overline{G}_\theta(u_{1k} \mid y_{1k}) = 1 - G_\theta(u_{1k} \mid y_{1k}), \tag{11}$$

$G_\theta(u_{1k} \mid y_{1k})$ represents the conditional probability distribution function of the mth order statistic Y_{1m}. The number of unsold seats protected for the high fare class from the low fare class in the next at time τ_k prior to flight departure is the number of unsold seats, u_{1k}°, which is given by

$$u_{1k}^\circ = \min(u_k^\circ, u_{1k} - y_{1k}). \tag{12}$$

C. Dynamic Airline Booking Policy under Uncertainty

Under the parametric uncertainty, the dynamic unbiased airline booking policy is given by

$$u_{1k}^{(\text{unb})} = \arg(E_\theta\{\overline{G}_\theta(u_{1k} \mid y_{1k})\} = \gamma), \quad k = 1, 2, ..., m-1, \tag{13}$$

where $u_{1k} \equiv u_{1k}(\hat{\theta})$, $\hat{\theta}$ represents either the maximum likelihood estimator of θ or sufficient statistic S for θ, i.e., $u_{1k} \equiv u_{1k}(S)$. The number of unsold seats protected for the high fare class from the low fare class in the next at time τ_k prior to flight departure is the number of unsold seats, which is given by

$$u_{1k}^{\circ(\text{unb})} = \min(u_k^\circ, u_{1k}^{(\text{unb})} - y_{1k}). \tag{14}$$

V. Mathematical Preliminaries

Theorem 1. Let $X_1 \le ... \le X_k$ be the first k ordered observations (order statistics) in a sample of size m from a continuous distribution with some probability density function $f_\theta(x)$ and distribution function $F_\theta(x)$, where θ is a parameter (in general, vector). Then the joint probability density function of $X_1 \le ... \le X_k$ and the lth order statistics X_l $(1 \le k < l \le m)$ is given by

$$g_\theta(x_1, ..., x_k, x_l) = g_\theta(x_1, ..., x_k) g_\theta(x_l \mid x_k), \tag{15}$$

where

$$g_\theta(x_1,..., x_k) = \frac{m!}{(m-k)!}\prod_{i=1}^{k} f_\theta(x_i)[1-F_\theta(x_k)]^{m-k}, \tag{16}$$

$$g_\theta(x_l \mid x_k) = \frac{(m-k)!}{(l-k-1)!(m-l)!}\left[\frac{F_\theta(x_l)-F_\theta(x_k)}{1-F_\theta(x_k)}\right]^{l-k-1}$$

$$\times\left[1-\frac{F_\theta(x_l)-F_\theta(x_k)}{1-F_\theta(x_k)}\right]^{m-l}\frac{f_\theta(x_l)}{1-F_\theta(x_k)}$$

$$= \frac{(m-k)!}{(l-k-1)!(m-l)!}\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}$$

$$\times(-1)^j\left[\frac{1-F_\theta(x_l)}{1-F_\theta(x_k)}\right]^{m-l+j}\frac{f_\theta(x_l)}{1-F_\theta(x_k)}$$

$$= \frac{(m-k)!}{(l-k-1)!(m-l)!}\sum_{j=0}^{m-l}\binom{m-l}{j}$$

$$\times(-1)^j\left[\frac{F_\theta(x_l)-F_\theta(x_k)}{1-F_\theta(x_k)}\right]^{l-k-1+j}\frac{f_\theta(x_l)}{1-F_\theta(x_k)} \tag{17}$$

represents the conditional probability density function of X_l given $X_k=x_k$.

Proof. The joint density of $X_1 \le ... \le X_k$ and X_l is given by

$$g_\theta(x_1,...,x_k, x_l) = \frac{m!}{(l-k-1)!(m-l)!}\prod_{i=1}^{k} f_\theta(x_i)$$

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$$\times[F_\theta(x_l)-F_\theta(x_k)]^{l-k-1} f_\theta(x_l)[1-F_\theta(x_l)]^{m-l}$$

$$= g_\theta(x_1,...,x_k)g_\theta(x_l \mid x_k). \qquad (18)$$

It follows from (18) that

$$g_\theta(x_l \mid x_1,...,x_k)=\frac{g_\theta(x_1,...,x_k,x_l)}{g_\theta(x_1,...,x_k)}=g_\theta(x_l \mid x_k), \quad (19)$$

i.e., the conditional distribution of X_l, given $X_i = x_i$ for all $i = 1, \dots, k$, is the same as the conditional distribution of X_l, given only $X_k = x_k$, which is given by (17). This ends the proof. □

Corollary 1.1. The conditional probability distribution function of X_l given $X_k=x_k$ is

$$P_\theta\{X_l \le x_l \mid X_k = x_k\}=1-\frac{(m-k)!}{(l-k-1)!(m-l)!}$$

$$\times\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}\frac{(-1)^j}{m-l+1+j}\left[\frac{1-F_\theta(x_l)}{1-F_\theta(x_k)}\right]^{m-l+1+j}$$

$$=\frac{(m-k)!}{(l-k-1)!(m-l)!}\sum_{j=0}^{m-l}\binom{m-l}{j}\frac{(-1)^j}{l-k+j}$$

$$\left[\frac{F_\theta(x_l)-F_\theta(x_k)}{1-F_\theta(x_k)}\right]^{l-k+j}. \qquad (20)$$

Corollary 1.2. Let $X_1 \le \dots \le X_k$ be the first k order statistics in a sample of size m from the two-parameter Weibull distribution with the probability density function

$$f_\theta(x)=\frac{\delta}{\beta}\left(\frac{x}{\beta}\right)^{\delta-1}\exp\left[-\left(\frac{x}{\beta}\right)^{\delta}\right] \quad (x>0), \qquad (21)$$

where $\theta = (\beta,\delta)$, $\beta>0$ and $\delta>0$ are the scale and shape parameters, respectively. Then the conditional probability distribution function of X_l given $X_k=x_k$ is

$$P_\theta\{X_l \le x_l \mid X_k = x_k\}=1-\frac{(m-k)!}{(l-k-1)!(m-l)!}$$

$$\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}\frac{(-1)^j}{m-l+1+j}\left[\exp\left(-\frac{x_l^\delta-x_k^\delta}{\beta^\delta}\right)\right]^{m-l+1+j}. \qquad (22)$$

Theorem 2. If in (22) the scale parameter β is unknown, then the predictive probability distribution function of X_l based on (x_k,δ) is given by

$$P_\delta\left\{\left(\frac{X_l}{X_k}\right)^\delta \le \left(\frac{x_l}{x_k}\right)^\delta\right\}=1-\frac{m!}{(l-k-1)!(m-l)!}$$

$$\times\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}\frac{(-1)^j}{m-l+1+j}$$

$$\times\left(\prod_{s=0}^{k-1}\left[\left(\left(\frac{x_l}{x_k}\right)^\delta-1\right)(m-l+1+j)+(m-k+1+s)\right]\right)^{-1}. \quad (23)$$

Proof. We reduce (22) to

$$P_\theta\left\{\left(\frac{X_l}{X_k}\right)^\delta \le \left(\frac{x_l}{x_k}\right)^\delta \middle| \left(\frac{X_k}{\beta}\right)^\delta=\left(\frac{x_k}{\beta}\right)^\delta\right\}$$

$$=1-\frac{(m-k)!}{(l-k-1)!(m-l)!}\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}$$

$$\times\frac{(-1)^j}{m-l+1+j}\left[\exp(-w[v^\delta-1])\right]^{m-l+1+j}$$

$$=P_\delta\left\{V^\delta \le v^\delta \mid W=w\right\}, \qquad (24)$$

where $V = X_l / X_k$ is the ancillary statistic whose distribution does not depend on the parameter β. Since X_k does not depend on V, $W=(X_k/\beta)^\delta$ is the pivotal quantity, whose distribution is known and does not depend on the parameters β and δ, we eliminate the parameter β from the problem as

$$P_\delta\{X_l \le x_l\}=\int_0^\infty P_\theta\{X_l \le x_l \mid X_k = x_k\}g_\theta(x_k)dx_k, \quad (25)$$

where

$$g_\theta(x_k)=\frac{m!}{(k-1)!(m-k)!}F_\theta^{k-1}(x_k)$$

$$\times[1-F_\theta(x_k)]^{m-k} f_\theta(x_k), \quad x_k \in (0,\infty), \qquad (26)$$

represents the probability density function of the kth order statistic X_k. Indeed, it follows from (26) that

$$g_\theta(x_k)dx_k=\frac{m!}{(k-1)!(m-k)!}\left[1-\exp\left(-\left(\frac{x_k}{\beta}\right)^\delta\right)\right]^{k-1}$$

$$\times\exp\left(-\left(\frac{x_k}{\beta}\right)^{\delta(m-k)}\right)\exp\left(-\left(\frac{x_k}{\beta}\right)^\delta\right)d\left(\frac{x_k}{\beta}\right)^\delta$$

$$=\frac{m!}{(k-1)!(m-k)!}[1-e^{-w}]^{k-1}e^{-w(m-k+1)}dw=g(w)dw. \quad (27)$$

It follows from (24) and (27) that

$$P_\delta\{V^\delta \le v^\delta\}=\int_0^\infty P_\delta\{V^\delta \le v^\delta \mid W=w\}g(w)dw$$

$$=1-\frac{m!}{(l-k-1)!(m-l)!}\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}\frac{(-1)^j}{m-l+1+j}$$

$$\times\left(\prod_{s=0}^{k-1}\left[\left(v^\delta-1\right)(m-l+1+j)+(m-k+1+s)\right]\right)^{-1}. \qquad (28)$$

Now (23) follows from (28). This ends the proof. □

Corollary 2.1. If the parameter $\delta=1$, i.e. we deal with the exponential distribution, then the predictive probability distribution function of X_l based on x_k is given by

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$$P\left\{\left(\frac{X_l}{X_k}\right)\le\left(\frac{x_l}{x_k}\right)\right\}=1-\frac{m!}{(l-k-1)!(m-l)!}\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}\frac{(-1)^j}{m-l+1+j}$$

$$\times\left(\prod_{s=0}^{k-1}\left[\left(\frac{x_l}{x_k}-1\right)(m-l+1+j)+(m-k+1+s)\right]\right)^{-1}. \quad (29)$$

Theorem 3. Let $X_1 \le ... \le X_k$ be the first k ordered observations from a sample of size m from the two-parameter Weibull distribution (21). Then the joint probability density function of the pivotal quantities

$$W_2=\frac{\delta}{\widehat{\delta}},\quad W_3=\left(\frac{\widehat{\beta}}{\beta}\right)^{\widehat{\delta}}, \quad (30)$$

conditional on fixed $\mathbf{z}^{(k)}=(z_i, \ldots, z_k)$, where $Z_i=(X_i/\widehat{\beta})^{\widehat{\delta}},\ i=1,\ldots,k,$ are ancillary statistics, any k–2 of which form a functionally independent set, $\widehat{\beta}$ and $\widehat{\delta}$ are the estimators of β and δ, based on the first k ordered observations ($X_1\le ... \le X_k$) from a sample of size m from the two-parameter Weibull distribution (21), such that W_2 and W_3 are the pivotal quantities (in particular, the maximum likelihood estimators of β and δ,

$$\widehat{\beta}=\left(\left[\sum_{i=1}^{k}x_i^{\widehat{\delta}}+(m-k)x_k^{\widehat{\delta}}\right]\Big/k\right)^{1/\widehat{\delta}} \quad (31)$$

and

$$\widehat{\delta}=\left[\left(\sum_{i=1}^{k}x_i^{\widehat{\delta}}\ln x_i+(m-k)x_k^{\widehat{\delta}}\ln x_k\right)\left(\sum_{i=1}^{k}x_i^{\widehat{\delta}}+(m-k)x_k^{\widehat{\delta}}\right)^{-1}-\frac{1}{k}\sum_{i=1}^{k}\ln x_i\right]^{-1} \quad (32)$$

respectively, lead to the pivotal quantities W_2 and W_3) is given by

$$f(w_2,w_3\,|\,\mathbf{z}^{(k)})=\vartheta^{\bullet}(\mathbf{z}^{(k)})w_2^{k-1}\prod_{i=1}^{k}z_i^{w_2}w_3^{kw_2-1}$$

$$\times\exp\left(-w_3^{w_2}\left[\sum_{i=1}^{k}z_i^{w_2}+(m-k)z_k^{w_2}\right]\right)$$

$$=\vartheta^{\bullet}(\mathbf{z}^{(k)})w_2^{k-2}\prod_{i=1}^{k}z_i^{w_2}w_3^{w_2(k-1)}$$

$$\times\exp\left(-w_3^{w_2}\left[\sum_{i=1}^{k}z_i^{w_2}+(m-k)z_k^{w_2}\right]\right)w_2w_3^{w_2-1}$$

$$=f(w_2\,|\,\mathbf{z}^{(k)})f(w_3\,|\,w_2,\mathbf{z}^{(k)}),\quad w_2\in(0,\infty),\quad w_3\in(0,\infty), \quad (33)$$

where

$$\vartheta^{\bullet}(\mathbf{z}^{(k)})=\left[\int_0^{\infty}\Gamma(k)w_2^{k-2}\prod_{i=1}^{k}z_i^{w_2}\left(\sum_{i=1}^{k}z_i^{w_2}+(m-k)z_k^{w_2}\right)^{-k}dw_2\right]^{-1} \quad (34)$$

is the normalizing constant,

$$f(w_2\,|\,\mathbf{z}^{(k)})=\vartheta(\mathbf{z}^{(k)})w_2^{k-2}\prod_{i=1}^{k}z_i^{w_2}\left(\sum_{i=1}^{k}z_i^{w_2}+(m-k)z_k^{w_2}\right)^{-k},$$

$$w_2\in(0,\infty), \quad (35)$$

$$\vartheta(\mathbf{z}^{(k)})=\left[\int_0^{\infty}w_2^{k-2}\prod_{i=1}^{k}z_i^{w_2}\left(\sum_{i=1}^{k}z_i^{w_2}+(m-k)z_k^{w_2}\right)^{-k}dw_2\right]^{-1}, \quad (36)$$

$$f(w_3\,|\,w_2,\mathbf{z}^{(k)})=\frac{\left[\sum_{i=1}^{k}z_i^{w_2}+(m-k)z_k^{w_2}\right]^k}{\Gamma(k)}w_3^{w_2(k-1)}$$

$$\times\exp\left(-w_3^{w_2}\left[\sum_{i=1}^{k}z_i^{w_2}+(m-k)z_k^{w_2}\right]\right)w_2w_3^{w_2-1},\quad w_3\in(0,\infty). \quad (37)$$

Proof. The joint density of $X_1 \le ... \le X_k$ is given by

$$f_\theta(x_1,\ldots,x_k)=\frac{m!}{(m-k)!}\prod_{i=1}^{k}\frac{\delta}{\beta}\left(\frac{x_i}{\beta}\right)^{\delta-1}$$

$$\times\exp\left(-\left(\frac{x_i}{\beta}\right)^{\delta}\right)\exp\left(-(m-k)\left(\frac{x_k}{\beta}\right)^{\delta}\right). \quad (38)$$

Using $\widehat{\beta}$ and $\widehat{\delta}$ (the maximum likelihood estimators of β and δ obtained from solution of (31) and (32)) and the invariant embedding technique [3-5], we transform (38) as follows:

$$f_\theta(x_1,\ldots,x_k)\,d\widehat{\beta}\,d\widehat{\delta}=\frac{m!}{(m-k)!}\prod_{i=1}^{k}x_i^{-1}\delta^k$$

$$\times\prod_{i=1}^{k}\left(\frac{x_i}{\beta}\right)^{\delta}\exp\left(-\sum_{i=1}^{k}\left(\frac{x_i}{\beta}\right)^{\delta}-(m-k)\left(\frac{x_k}{\beta}\right)^{\delta}\right)d\widehat{\beta}\,d\widehat{\delta}$$

$$=-\frac{m!}{(m-k)!}\widehat{\beta}\widehat{\delta}^k\prod_{i=1}^{k}x_i^{-1}w_2^{k-2}\prod_{i=1}^{k}z_i^{w_2}w_3^{w_2(k-1)}$$

$$\times\exp\left(-w_3^{w_2}\left[\sum_{i=1}^{k}z_i^{w_2}+(m-k)z_k^{w_2}\right]\right)w_2w_3^{w_2-1}dw_2dw_3. \quad (39)$$

Normalizing (39), we obtain (33). This ends the proof. □

Theorem 4. If in (8) both parameters β and δ are unknown, then the predictive probability distribution function of X_l based on $(x_k,\widehat{\delta})$ and conditional on fixed $\mathbf{z}^{(k)}$ is given by

$$P\left\{\left(\frac{X_l}{X_k}\right)^{\widehat{\delta}}\le\left(\frac{x_l}{x_k}\right)^{\widehat{\delta}}\,\middle|\,\mathbf{z}^{(k)}\right\}=1-\frac{m!}{(l-k-1)!(m-l)!}$$

$$\times\int_0^{\infty}\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}\frac{(-1)^j}{m-l+1+j}$$

$$\times\left(\prod_{s=0}^{k-1}\left[\left(\left(\left[\frac{x_l}{x_k}\right]^{\widehat{\delta}}\right)^{w_2}-1\right)(m-l+1+j)+(m-k+1+s)\right]\right)^{-1}f(w_2\,|\,\mathbf{z}^{(k)})dw_2. \quad (40)$$

Proof. We reduce (23) to

$$P_\delta\left\{(X_l/X_k)^{\widehat{\delta}\left(\frac{\delta}{\widehat{\delta}}\right)}\le(x_l/x_k)^{\widehat{\delta}\left(\frac{\delta}{\widehat{\delta}}\right)}\right\}=P\left\{V_2^{W_2}\le v_2^{w_2}\right\}$$

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$$=1-\frac{m!}{(l-k-1)!(m-l)!}\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}\frac{(-1)^j}{m-l+1+j}$$
$$\times\left(\prod_{s=0}^{k-1}[(v_2^{w_2}-1)(m-l+1+j)+(m-k+1+s)]\right)^{-1}, \quad (41)$$

where $V_2=(X_l/X_k)^{\hat{\delta}}$ is the ancillary statistic whose distribution does not depend on the parameters β and δ. Since the pivotal quantity W_2, whose distribution is given by (35), does not depend on V_2, it follows from (41) and (35) that

$$P\left\{V_2\le v_2 \mid \mathbf{z}^{(k)}\right\}=\int_0^\infty P\left\{V_2^{W_2}\le v_2^{w_2}\right\} f(w_2\mid \mathbf{z}^{(k)})dw_2, \quad (42)$$

where the unknown parameters β and δ are eliminated from the problem. Now (40) follows from (42). This ends the proof. □

VI. Illustrative Example of Airline Booking Policies

Let X_1, …, X_n be the random sample of the previous independent observations of the cumulative customer demand for the high fare class, which follow the exponential distribution with the probability density function (21) (δ=1), where the parameter β is unknown. Then the static policies of airline booking under parametric uncertainty are given as follows.

The static maximum likelihood airline booking policy follows from (6):

$$u_1^{(\mathrm{ml})}=\ln\gamma^{-S/n}, \quad (43)$$

where $S=\sum_{i=1}^n X_i$ is the sufficient statistic for β, with

$$V=S/\beta \sim f(v)=\frac{1}{\Gamma(n)}v^{n-1}\exp(-v), \quad v\ge 0, \quad (44)$$

and the relative bias,

$$r(u_1^{(\mathrm{ml})})=\frac{|E_\theta\{\overline{F}_\theta(u_1^{(\mathrm{ml})})\}-\gamma|}{\gamma}100\%=\frac{|(1+\ln\gamma^{-1/n})^{-1}-\gamma|}{\gamma}100\%. \quad (45)$$

If, say, n=1 and γ=0.4, then $r_{\mathrm{rb}}(u_1^{(\mathrm{ml})})=30\%$. Thus, in this example the static maximum likelihood airline booking policy has the relative bias equal to 30%. It follows that the protection level for customers of the high fare class will be determined incorrectly. This may lead to serious loss.

The static unbiased airline booking policy follows from (8):

$$u_1^{(\mathrm{unb})}=[\gamma^{-1/n}-1]S, \quad (46)$$

where the relative bias $r(u_1^{(\mathrm{unb})})=0$.

The dynamic unbiased airline booking policy follows from (13) and (29):

$$u_{1k}^{(\mathrm{unb})}=\arg\left(\begin{array}{l}\dfrac{m!}{(m-k-1)!}\displaystyle\sum_{j=0}^{m-k-1}\binom{m-k-1}{j}\frac{(-1)^j}{1+j}\\ \times\left(\displaystyle\prod_{s=0}^{k-1}\left[\left(\frac{u_{1k}}{y_{1k}}-1\right)(1+j)+(m-k+1+s)\right]\right)^{-1}=\gamma\end{array}\right),$$

$$k=1,2,\dots,m-1. \quad (47)$$

$$u_{1k}^{\circ(\mathrm{unb})}=\min(u_k^{\circ},u_{1k}^{(\mathrm{unb})}-y_{1k}). \quad (48)$$

VII. Conclusion and Directions for Future Research

In this paper, we develop a new frequentist approach to improve predictive statistical decisions for airline seat allocation problems. The methodology, which is developed in this paper for the use in the airline industry under parametric uncertainty of airline customer demand models, may be found to be useful in other industries such as hotels, car rental companies, shipping companies, etc. While the details of problems considered in the paper can change significantly from one industry to the next, the focus is always on making better demand decisions − and not manually with guess work and intuition − but rather scientifically with models and technology, all implemented with disciplined processes and systems.

The methodology described here can be extended in several different directions to handle various problems that arise in practice. We have illustrated the proposed methodology for scale distributions (such as the exponential distribution). Application to other distributions could follow directly.

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