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Hybrid Orthonormal Bernstein and Block-Pulse Functions for Solving Fredholm Integral Equations

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***Abstract*—In this paper, we use a combination of orthonormal Bernstein and Block-Pulse functions on the interval [0,1], to solve the linear Fredholm integral equations of the second kind. We convert these integral equations, to a system of linear equations. Also we compared the result of the proposed method with true answers to show the convergence and advantages of the new method.**

***Index Terms*—Fredholm integral equation, Bernstein, Block-Pulse, orthonormal.**

I. Introduction

INTEGRAL equation is an equation in which an unknown function appears one or more integral sign. Naturally, in such an equation there can occur other terms as well. The general form of a fredholm integral equation is

$$u(x) = f(x) + \int_a^b K(x,t)u(t)dt, \qquad (1)$$

where u(x) is an unknown function, k(x,t) and f(x) are known function, a and b are known constant. integral equations are widely used for solving many problems in mathematical physics and engineering. In recent years, many different basic functions have been used to estimate the solution of integral equations, such as Block-Pulse functions $[1-6]$, Triangular functions $[7-9]$, Haar functions $[10]$, Hybrid Legendre and Block-Pulse functions $[11-13]$, Hybrid Chebyshev and Block-Pulse functions $[14, 15]$, Hybrid Taylor, Block-Pulse functions $[16]$, Hybrid Fourier and Block-Pulse functions $[17]$. In the first time, Block-Pulse functions were introduced to electrical engineering by Harmuth and several researchers discussed the Block-Pulse $[18-21]$. Bernstein polynomials play a prominent role in various areas of mathematics. These polynomials have been frequently used in the solution of integral equations, differentials and approximation theory $[22-25]$.

In this paper we used hybrid of orthonormal Bernstein and Block-Pulse functions for numerical solution of fredholm integral equations. The advantage of this method to other existing methods is its simplicity of implementation besides some other advantages.

This paper is organized as follows: In Section 2, we introduce Bernstein polynomials and their properties. Also we orthonormal these polynomials and hybrid them with Block-Pulse functions to obtain new basis. In Section 3, these new basis together with collocation method are used to reduce the linear fredholm integral equation to a linear system that can be solved by various method. Section 4 illustrates some applied models to show the convergence, accuracy and advantage of the proposed method and compares it with some other existed method. In Section 5, numerical experiments are conducted to demonstrate the viability and the efficiency of the proposed method computationally. Finally Section 6 concludes the paper.

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II. Basic Definition

In this section we introduce Bernstein polynomials and their properties to get better approximation, we orthonormal these polynomials and hybrid them with Block-Pulse functions.

A. *Definition of Bernstein polynomials*

The Bernstein basis polynomial of degree n are defined by [26]

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}. \qquad (2)$$

By using binomial expansion of $(1-x)^{n-i}$, we have

$$\binom{n}{i} x^i (1-x)^{n-i} = \sum_{k=0}^{n-i} (-1)^k \binom{n}{i}\binom{n-i}{k} x^{i+k}. \qquad (3)$$

Then $\{B_{0,n}, B_{1,n}, ..., B_{n,n}\}$ in Hilbert space $L^2[0,1]$ is a complete basis. Therefore, any polynomial of degree n can be expanded in terms of linear combination of $B_{i,n}(x)(i = 0, 1, ..., n)$.
By using Gram-schmidt algorithm we obtain orthonormal polynomials to construct new basis, these new basis are $OB_{i,n}(x)$. for example for $n = 4$ and $i = 3$

$$B_{3,4}(x) = 4x^3(1-x).$$

Our orthonormal polynomial is

$$OB_{3,4}(x) = \frac{\sqrt{3990}}{4}\left(4\,x^3\,(1-x) - \frac{4}{19} + \frac{68}{19}\,x\,(1-x)^3\right)$$
$$-\frac{\sqrt{3990}}{4}\left(\frac{96}{19}\,x^2\,(1-x)^2\right).$$

B. *Definition of Block-Pulse functions (BPFs) and their properties*

An M-set of Block-Pulse function is defined over the interval [0,T) as

$$b_i(x) = \begin{cases} 1 & \frac{iT}{M} \le x < \frac{(i+1)T}{M} \\ 0 & otherwise \end{cases}, \qquad (4)$$

where $i = 0, 1, ..., M-1$ with m as a positive integer. Also, $h = \frac{T}{M}$ and b_i is the ith BPF.

In this paper it is assumed that $T = 1$, so BPFs are defined over $[0, 1)$ and $h = \frac{1}{M}$.

There are some properties for BPFs, the most important properties are disjointness, orthogonality, and completeness. The disjointness property can be clearly obtained from the definition of BPFs [27]:

$$b_i(x)b_j(x) = \begin{cases} b_i(x) & i = j \\ 0 & i \neq j \end{cases}, \quad (5)$$

where $i, j = 0, 1, ..., M-1$.

The other property is orthogonality. It is clear that [27]

$$\int_0^1 b_i(x)\, b_j(x)\, dx = h\delta_{ij}, \quad (6)$$

where δ_{ij} is Kronecker delta.

The third property is completeness. For every $f \in L^2[0,1]$ when m approaches to infinity, Parsevals identity holds [27]:

$$\int_0^1 f^2(x)dx = \sum_{i=0}^{\infty} f_i^2 \|b_i(x)\|^2, \quad (7)$$

where

$$f_i = \frac{1}{h}\int_0^1 f(x)b_i(x)dx. \quad (8)$$

C. *Definition Hybrid Orthonormal Bernstein Block-Pulse functions (OBH) functions*

We define OBH on the interval $[0,1]$ as follow:

$$OBH_{i,j}(x) = \begin{cases} B_{j,n}(Mx - i + 1) & \frac{i-1}{M} \leq x < \frac{i}{M} \\ 0 & othrewise \end{cases}, \quad (9)$$

where $i = 1, 2, ..., M$ and $j = 0, 1, ..., n$.
thus our new basis is $\{OBH_{1,0}, OBH_{1,1}, ..., OBH_{M,n}\}$ and we can approximate function with this base.

III. Function Approximation by using OBH functions

A function u(x), square integrable in $[0,1]$, maybe expressed in terms of the OBH basis as follow [28]:

$$u(x) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} c_{ij} \cdot OBH_{i,j}(x). \quad (10)$$

If we truncate the infinite series in (9), then we have

$$u(x) \approx \sum_{i=1}^{M}\sum_{j=0}^{n} c_{ij} \cdot OBH_{i,j}(x) = C^T OBH(x), \quad (11)$$

where

$$OBH(x) = [OBH_{1,0}, OBH_{1,1}, ..., OBH_{M,n}]^T, \quad (12)$$

and

$$C = [c_{1,0}, c_{1,1}, ..., c_{M,n}]^T. \quad (13)$$

Therefore we have

$$C^T < OBH(x), OBH(x) > = < u(x), OBH(x) >,$$

then

$$C = D^{-1} < u(x), OBH(x) >,$$

where

$$D = < OBH(x), OBH(x) > \quad (14)$$

$$= \int_0^1 OBH(x) \cdot OBH^T(x)dx$$

$$= \begin{pmatrix} D_1 & 0 & ... & 0 \\ 0 & D_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & D_M \end{pmatrix},$$

then by using (8), $D_i (i = 1, 2, ..., M)$ is defined as follow:

$$(D_n)_{i+1,j+1} = \int_{\frac{i-1}{M}}^{\frac{i}{M}} B_{i,n}(Mx - i + 1)B_{j,n}(Mx - j + 1)dx \quad (15)$$

$$= \frac{1}{M}\int_0^1 B_{i,n}(x)B_{j,n}(x)dx$$

$$= \frac{\binom{n}{i}\binom{n}{j}}{M(2n+1)\binom{2n}{i+j}}.$$

We can also approximate the function $k(x,t) \in L^2[0,1]$ as follow:

$$k(t,s) \approx OBH^T(x)KOBH(t), \quad (16)$$

where K is an $M(n+1)$ matrix that we can obtain as follows:

$$K = D^{-1} < OBH(x) < k(x,t), OBH(t) >> D^{-1}. \quad (17)$$

IV. Solving second kind Fredholm integral equation via OBH functions

Consider the following integral equation:

$$u(x) = f(x) + \int_0^1 k(x,t)u(t)dt. \quad (18)$$

where $f(x) \in L^2[0,1)$, $k(x,t) \in L^2[0,1) \times [0,1)$. $u(x)$ is an unknown function which can be expanded into OBH functions with nM terms.

$$u(x) = U^T OBH(x), \quad (19)$$

where U is an unknown nM–vector and $OBH(x)$ is given by Eq.(9). Likewise, $k(x,t)$ and $u(x)$ are also expanded into the OBH functions

$$k(x,t) = OBH^T(x)KOBH(t), \quad (20)$$

$$f(x) = F^T OBH(x), \quad (21)$$

where K is a known $nM \times nM$–matrix and F is a known nM–vector. Substituting Eq.(19)–(21) into Eq.(18) produces

$$OBH^T(x)U = OBH^T(x)F + \quad (22)$$

$$+ \int_0^1 OBH^T(x)KOBH(t)OBH^T(t)Uds.$$

Applying Eq.(14), Eq(22) becomes

$$OBH^T(x)U = OBH^T(x)F + OBH^T(x)KDU. \quad (23)$$

Therefore

$$U = F + KDU, \quad (24)$$

where the dimensional subscripts have been dropped to simplify the notation. Rewriting Eq.(24), we have

$$U = (I - KD)^{-1}F, \quad (25)$$

where I is $nM \times nM$–identity matrix. The unknown vector U can be obtained by solving Eq.(25). Thus the solution $u(x)$ can be calculated in the OBH function expansion by using U and Eq.(19).

TABLE I
MAXIMUM ERROR IN EXAMPLE 1 WITH $h = .01$

h	$\|E\|_\infty for \quad n=3; M=2$	$\|E\|_\infty for \quad n=3; M=4$
0.1	4.4×10^{-6}	4.7×10^{-6}
0.2	8.4×10^{-7}	5.1×10^{-6}
0.3	2.1×10^{-6}	7.5×10^{-6}
0.4	9.7×10^{-8}	5.3×10^{-6}
0.5	8.8×10^{-6}	3.4×10^{-6}
0.6	9.0×10^{-6}	9.0×10^{-6}
0.7	7.1×10^{-6}	1.4×10^{-5}
0.8	4.5×10^{-6}	1.2×10^{-5}
0.9	9.1×10^{-6}	9.8×10^{-7}

TABLE II
MAXIMUM ERROR IN EXAMPLE 2 WITH $h = 0.01$

h	$\|E\|_\infty for \quad n=4; M=2$	$\|E\|_\infty for \quad n=5; M=2$
0.1	5.5×10^{-6}	9.0×10^{-6}
0.2	2.1×10^{-4}	2.1×10^{-5}
0.3	3.7×10^{-5}	9.7×10^{-6}
0.4	2.2×10^{-4}	6.6×10^{-8}
0.5	3.1×10^{-4}	7.5×10^{-6}
0.6	4.1×10^{-4}	1.2×10^{-5}
0.7	5.1×10^{-4}	1.7×10^{-5}
0.8	5.2×10^{-4}	2.1×10^{-5}
0.9	5.6×10^{-4}	2.4×10^{-5}

V. NUMERICAL EXAMPLE

We applied the presented schemes to the following fredholm integral equation of second kind. For this purpose, we consider three examples.

Example1:
Consider the following linear fredholm integral equation

$$u(x) = sin(x) - x + (x+1)cos(1) - sin(1) +$$

$$+\int_0^1 (x+t)u(t)dt,$$

with the exact solution $u(x) = sin(x)$ for $0 \le x \le 1$. The numerical results for grid points $x_i = ih$ with h=0.01, $n = 3$ $M = 4$ and $n = 3$ $M = 2$ are shown in table 1. $\|E\|_\infty$ shows the maximum error at each grid point. In this table we indicate that by low degree of hybrid Block-Pulse function, we can get good approximation.

Example2:
Consider the following linear fredholm integral equation

$$u(x) = e^{2x+\frac{1}{3}} + \int_0^1 (-\frac{1}{3}e^{2x-\frac{5}{3}t})u(t)dt,$$

with the exact solution $u(x) = e^{2x}$ for $0 \le x \le 1$. The numerical results for grid points $x_i = ih$ with h=0.01, $n = 4$ $M = 2$ and $n = 5$ $M = 2$ are shown in table 2. $\|E\|_\infty$ shows the maximum error at each grid point. This table shows that by increasing the value of n we get better results.

Example3:
Consider the following linear fredholm integral equation

$$u(x) = sin(x) + \int_0^1 (1 - xcos(xt))u(t)dt,$$

TABLE III
MAXIMUM ERROR IN EXAMPLE 3 WITH $h = 0.01$

h	$\|E\|_\infty for \quad n=2; M=2$	$\|E\|_\infty for \quad n=3; M=2$
0.1	1.0×10^{-10}	4.4×10^{-7}
0.2	1.0×10^{-10}	3.4×10^{-7}
0.3	1.0×10^{-10}	1.0×10^{-6}
0.4	1.0×10^{-10}	4.0×10^{-7}
0.5	3.7×10^{-7}	5.2×10^{-6}
0.6	2.4×10^{-8}	7.5×10^{-7}
0.7	2.0×10^{-7}	1.9×10^{-6}
0.8	1.9×10^{-7}	1.1×10^{-6}
0.9	3.1×10^{-8}	1.0×10^{-6}

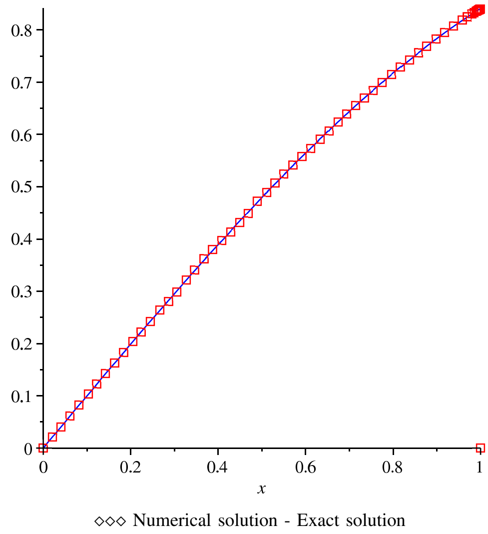

Fig. 1. Results for example 1 with $n = 3, M = 4$.

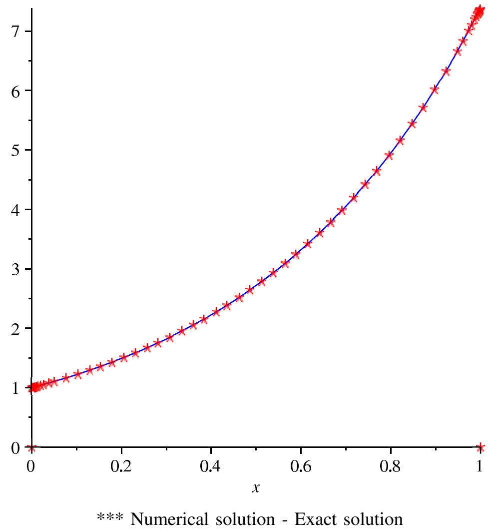

Fig. 2. Results for example 2 with $n = 5, M = 2$.

with the exact solution $u(x) = 1$ for $0 \le x \le 1$. The numerical results for grid points $x_i = ih$ with h=0.01, $n = 2$ $M = 2$ and $n = 3$ $M = 2$ are shown in table 3. $\|E\|_\infty$ shows the maximum error at each grid point. As it is clear form this table we get good results by low degree Bernstein polynomials.

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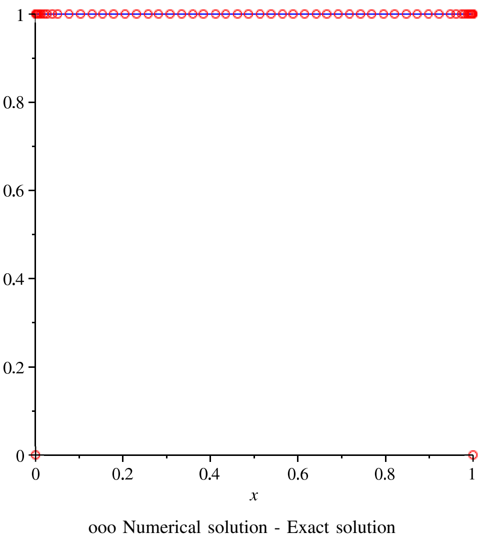

Fig. 3. Results for example 3 with $n = 2, M = 2$.

VI. Conclusion

In this paper by use of the combination of orthonormal Bernstein and Block-Pulse functions we solved linear Fredholm integral equations. The method is based upon reducing the system into a set of algebraic equations. The generation of this system needs just sampling of functions multiplication and addition of matrices and needs no integration. The main advantage of this method is its efficiency and simple applicability . The matrix D is sparse; hence is much faster than other functions and reduces the CPU time and the computer memory, at the same time keeping the accuracy of the solution. The numerical examples support this claim.

References

[1] K. Maleknejad, M. Shahrezaee, H. Khatami, Numerical solution of integral equations system of the second kind by Block-Pulse functions, Applied Mathematics and Computation, 166 (2005) 15-24.

[2] E. Babolian, Z. Masouri, Direct method to solve Volterra integral equation of the first kind using operational matrix with block-pulse functions, Applied Mathematics and Computation, 220 (2008) 51-57.

[3] K. Maleknejad, S. Sohrabi, B. Berenji, Application of D-BPFs to nonlinear integral equations, Commun Nonlinear Sci Numer Simulat, 15 (2010) 527-535

[4] K. Maleknejad, K. Mahdiani, Solving nonlinear mixed Volterra-Fredholm integral equations with two dimensional block-pulse functions using direct method, Commun Nonlinear Sci Numer Simulat, Article in press.

[5] K. Maleknejad, B. Rahimi, Modification of Block Pulse Functions and their application to solve numerically Volterra integral equation of the first kind, Commun Nonlinear Sci Numer Simulat, 16 (2011) 2469-2477.

[6] K. Maleknejad, M. Mordad, B. Rahimi, A numerical method to solve Fredholm-Volterra integral equations I two dimensional spaces using Block Pulse Functions and operational matrix, Journal of Computational and Applied Mathematics, 10.1016/j.cam.2010.10.028

[7] E. Babolian, K. Maleknejad, M. Roodaki, H. Almasieh, Two-dimensional triangular functions and their applications to nonlinear 2D Volterra-Fredholm integral equations, Computer and Mathematics with Applications, 60 (2010) 1711-1722.

[8] K. Maleknejad, H. Almasieh, M. Roodaki, Triangular functions (TF) method for the solution of nonlinear Volterra-Fredholm integral equations, Commun Nonlinear Sci Numer Simulat, 15 (2010) 3293-3298.

[9] F. Mirzaee, S. Piroozfar, Numerical solution of the linear two-dimensional Fredholm integral equations of the second kind via two-dimensional triangular orthogonal functions, Journal of King Saud University, 22 (2010) 185-193 .

[10] Y. Ordokhani, Solution of nonlinear Volterra-Fredholm-Hammerstein integral equations via rationalized Haar functions, Applied Mathematics and Computation, 180 (2006) 436-443.

[11] K. Maleknejad, M.T. Kajani, Solving second kind integral equations by Galerkin methods with hybrid Legendre and Block-Pulse functions, Applied Mathematics and Computation, 145 (2003) 623-629.

[12] E. Hashemzadeh, K. Maleknejad, B. Basirat, Hybrid functions approach for the nonlinear Volterra-Fredholm integral equations, Procedia Computer Sience,3 (2011) 1189-1194.

[13] H.R. Marzban, H.R. Tabrizidooz, M. Razzaghi, A composite collection method for the nonlinear mixed Volterra-Fredholm-Hammerstein integral equation, Commun Nonlinear Sci Numer Simulat,16 (2011) 1186-1194.

[14] M.T. Kajani, A. H. Vencheh, Solving second kind integral equations with Hybrid Chebyshev and Block-Pulse functions, Applied Mathematics and Computation, 163 (2005) 71-77.

[15] X.T. Wang, Y.M. Li, Numerical solutions of integrodifferential systems by hybrid of general block-pulse functions and the second Chebyshev polynomials, Applied Mathematics and Computation, 209 (2009) 266-272.

[16] K. Maleknejad, Y. Mahmoudi, Numerical solution of linear Fredholm integral equation by using hybrid Taylor and Block-Pulse functions, Applied Mathematics and Computation, 149 (2004) 799-806.

[17] B. Asady, M.T. Kajani, A.H. Vencheh, A. Heydari, Solving second kind integral equations with hybrid Fourier and block-pulse functions, Applied Mathematics and Computation, 160 (2005) 517-522.

[18] G.P. Rao, Piecewise Constant Orthogonal Functions and their Application to System and Control, Springer-Verlag, 1983.

[19] B.M. Mohan, K.B. Datta, Orthogonal Function in Systems and Control, 1995.

[20] C.F. Chen, Y.T. Tsay, T.T. Wu, Walsh operational matrices or fractional calculus and their application to distributed systems, Journal of the Franklin Institute, 303 (1977) 267-284.

[21] P. Sannuti, Analysis and synthesis of dynamic systems via block pulse functions, IEEE Proceeding, 124 (1977) 569-571.

[22] K. Maleknejad, E. Hashemzadeh, R. Ezzati, A new approach to the numerical solution of Volterra integral equations by using Bernstein approximation, Commun Nonlinear Sci Numer Simulat, 16 (2011) 647-655.

[23] E.H. Doha, A.H. Bhrawy, M.A. Saker, Integrals of Bernstein polynomials: an application for the solution of high even-order differential equations, applied Mathematics Letter, 24 (2011) 559-565.

[24] B.N. Mandal, S. Bhattacharya, Numerical solution of some classes of integral equations using Bernstein polynomials, Applied Mathematics and Computation, 190 (2007) 1707-1706.

[25] K. Maleknejad, E. Hashemizadeh, B. Basirat, Computational method based on Bernestein operational matrices for nonlinear Volterra-Fredholm-Hammerstein integral equations, Commun. Nonlinear. Sci. Numer. Simulat., 17 (1) (2012) 52–61.

[26] K. Maleknejad, B. Basirat, E. Hashemizadeh, A Bernstein operational matrix approach for solving a system of high order linear Volterra-Fredholm integro-differential equations, Math. Comput. Model., 55 (2012) 1363–1372.

[27] T.J. Rivlin, An introduction to the approximation of functions, New York, Dover Publications, 1969.

[28] S.A. Yousefi, M. Behrooozifar, Operational matrices of Bernstein polynomials and their applications, International Journal of System Science, 41 (2010) 709-716.

[29] Z.H. Jiang, W. Schaufelberger, Block Pulse Functions and Their Applications in Control Systems, Springer-Verlag Berlin Heidelberg 1992.

[30] K. Maleknejad , S. Sohrabi, Numerical solution of Fredholm integral equations of the first kind by using Legendre wavelets, Applied Mathematics and Computation 186 (2007) 836843.

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