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Existence and Uniqueness of the Navier-Stokes Problem in Infinite Space

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***Abstract*—This paper is devoted to the mathematical theory of the existence and uniqueness of weak solutions for incompressible three dimensional Navier-Stokes equations. We presented our vision in developing the mathematical concept and the physical description for the turbulent fluid motion. Using rotor operator and a well-known formula of vector analysis we obtained nonlinear Volterra-Fredholm integral equation in a matrix form which contained only three components of velocity vector. Due to the theory of Volterra-Fredholm matrix operators and the successive approximation method were defined the velocity components. Considering the pressure gradient as a potential field we determined the distribution pressure. According to the defined estimation for the velocity vector we proved the uniqueness theorem for the Navier-Stokes problem in a Hilbert spaces.**

***Index Terms*— Navier-Stokes equations, fluid motion, swirling turbulent flow, potential field, gradient of pressure**

I. Introduction

THIS paper is presented to ensure effective dissemination of mathematical theory for the Navier-Stokes problem and deals with common nature phenomena of turbulence. Turbulent fluid flow is a nonlinear multiscale phenomenon which poses some difficult and fundamental problems in theoretical and mathematical physics. The Navier-Stokes equations describe interactions between fluctuations and their directions for different wavelengths and have a great interest in mathematical modeling of turbulent process. Mathematical solution for a practical complex problem requires a perspective using some alternative approach which different from that is needed for studying the fundamental issues. It is worth stressing that turbulence is fundamentally interesting and practical importance for engineering models of turbulent effects. This importance provides motivation for this research, therefore we have presented a new analytic method of solution for the Navier-Stoke problem. This result is the first step to mathematical understanding of the elusive phenomena of turbulence.

The Navier-Stokes equations as nonlinear partial differential equations in real natural situation were formulated in 1821 and appeared to give an accurate description of fluid flow including laminar and turbulent features. Concerning the large literature on the Navier-Stoke problem we mention only some papers which consider particularly relevant for our purpose. We have focused on the global existence, uniqueness and smoothness. Examples of weak solution were given by Caffarelli [1], Sheffer[2]. A critical analysis for many analytic and numerical solutions of Navier-Stokes equations was given by Fefferman [3]. We will follow this idea of existence of weak solution given in [3] by using the energy conservation low for the external force and gradient of pressure. Theoretical prediction and analysis of turbulence has been the fundamental problem of fluid dynamics. Turbulence is a continuous phenomenon that exists on a large range of length and time scales. As there exists different scales which energy is transferred from the larger scales to the smaller scales where energy is dissipated into heat by molecular viscosity. This paper has focused on conservative force fields by using the energy conversion process. Without here going into details of the flow field as producing potential and swirling turbulent flows were defined stability conditions for fluid motion. There we may get some fundamental information about behavior of potential, kinetic and static energies which are required for the description their mechanics of the turbulent fluid motion.

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II. Mathematical Formulation of the Problem

Suppose that infinite spaces $\Omega = R^3$ and $\Omega_T = R^3 \times (0 < t < \infty)$, $\vec{u}(x,t) = u_1(x,t)\vec{i} + u_2(x,t)\vec{j} + u_2(x,t)\vec{k}$ is the velocity vector and $p(x,t)$ is the fluid pressure field. We consider the Navier -Stokes equations in the following form

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = -\frac{1}{\rho}\nabla p + \nu\Delta\vec{u} + \vec{f}(x,t) \quad \text{in} \quad \Omega_T \qquad (1)$$

$$div\ \vec{u} = 0 \qquad \text{in} \qquad \Omega_T \qquad (2)$$

with the initial conditions

$$\vec{u}\ \Big|_{t=0} = \vec{u}_0(x) \quad \text{on} \quad \Omega \qquad (3)$$

Here, a vector function

$$\vec{f}(x,t) = f_1(x,t)\vec{i} + f_2(x,t)\vec{j} + f_2(x,t)\vec{k}$$

denotes an external force, ν is a kinematic viscosity, ρ is a fluid density, the symbol ∇ denotes the gradient with respect to the function , the symbol Δ denotes the three dimensional Laplace operator.

We will construct a weak solution for the Navier-Stokes initial value problem (1)-(3). The weak formulation for the problem (1)-(3) is based on the introduced technique for an incompressible potential and swirling turbulent flow. There we assume that

$$|\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2} \to 0 \quad for\ |x| = \sqrt{x_1^2 + x_2^2 + x_3^2} \to \infty$$

The initial value problem (1)-(3) is concerned with the fundamental solution for Poisson and heat conduction equations. Particular attention is paid to the integral representation of solutions with their initial values for the viscous Newtonian fluid which is the basis of hydrodynamics. Note, that many problems formally exist for any Reynolds numbers and it can have an exact solution, but not all partial differential equation can describe real-nature phenomenon, therefore we will consider the basic equations of hydrodynamics that correctly can be solved (existence, uniqueness and stability). The requirement of stability is caused by the fact that physical evidence is usually determined from experiment and approximately, therefore we must be sure that the determined solution is the stability solution. This requirement of stability seems to be important, therefore we must construct Lyapunov theory for the Navier-Stokes problem which will be a powerful determining method for defining the stability or instability domains for nonlinear selected systems.

Let us describe the used method in the proofs of existence and uniqueness for the Navier-Stokes problem (1)-(3). The key idea is to exclude the pressure function from the equation (1) by using rotor or divergence operators. According to these transformations we can give the integral representative for the velocity vector and the energy conservation condition for the determining pressure distribution. We involve this method to show that the velocity vector with respect to the pressure function exists and satisfies the energy conservation low. We split the construction of the solution for Navier-Stokes problem (1)-(3) into two steps. In the first step we claim that we may assume

$$rot \ \vec{f} = 0 \ , \ rot \ \vec{u}_0 = 0$$

Then we will get

$$grad \ \left(\frac{u^2}{2} + \frac{p}{\rho} - \Phi \right) = 0$$

where

$$grad \ \Phi(x,t) = -\vec{f}(x,t)$$

Due to this assertion we can find a weak solution for the problem (1)-(3). It is proved that under the energy conservation low there exists a unique velocity vector given by the integral representation. There we get a stable solution of the Navier-Stokes problem (1)-(3). In the second step we assume that

$$rot \ \vec{f} \neq 0$$

Due to this assertion we obtain the second kind nonlinear matrix Volterra-Fredholm integral equation which is solved by using the method of successive approximation in Hilbert space. Under above assumption there exists a unique unstable solution with the appropriate properties.

III. Velocity Components and Function of Pressure for the Potential Field

Turbulent motion is supported by the subjected power from some external forces and initial velocity. The shape of turbulent region is determined by the property which has shown stability or instability of the velocity motion and the pressure distribution. Stabilizing mechanisms have been advocated to explain features observed in numerical simulations of turbulence.

Using well-known formula of vector analysis

$$\frac{1}{2} grad \ \vec{u}^2 = [\vec{u} \times rot \ \vec{u}] + (\vec{u}\nabla)\vec{u} \tag{4}$$

and operator $rot \ \vec{u} = \nabla \times \vec{u}$ which is the determinant of the third order

$$rot \ \vec{u} = \nabla \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \\ u_1 & u_2 & u_3 \end{vmatrix}$$

we have got

$$\frac{\partial \vec{u}}{\partial t} + grad \ \left(\frac{p}{\rho} + \frac{1}{2}\vec{u}^2\right) = [\vec{u} \times rot \, \vec{u}] + \nu\Delta\vec{u} + \vec{f}(x,t) \tag{5}$$

Considering the function which represents potential energy

$$grad \ \Phi(x,t) = -\vec{f}(x,t) \tag{6}$$

and using the *divergence* operator we can get an important expression for potential energy

$$\Phi(x,t) = div \ \vec{f} * \frac{1}{4\pi |x - \xi|}$$

Here symbol $*$ is a convolution between two functions: $div \ \vec{f}$ and $\frac{1}{4\pi |x-\xi|}$

Function

$$|x-\xi| = \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + (x_3-\xi_3)^2}$$

is a distance between the point $\xi = (\xi_1, \xi_2, \xi_3)$ and the point $x = (x_1, x_2, x_3)$.

Assume that

$$rot \ \vec{u} = 0 \tag{7}$$

we have got

$$\frac{\partial \vec{u}}{\partial t} + grad \ \left(\frac{p}{\rho} + \frac{1}{2}\vec{u}^2\right) = \nu\Delta\vec{u} + \vec{f}(x,t) \tag{8}$$

Using the *divergence* operator and condition (2) for the expression (8) we obtain the equation

$$\frac{p}{\rho} + \frac{u^2}{2} - div \ \vec{f} * \frac{1}{4\pi |x-\xi|} = 0 \tag{9}$$

This expression (9) represents the conservation of energy.

Applying the expression

$$grad \ \left(\frac{u^2}{2} + \frac{p}{\rho}\right) + \vec{f} = 0$$

to the Navier-Stokes equations (1)-(3) we obtain the mathematical problem

$$\frac{\partial \vec{u}}{\partial t} - \nu\Delta\vec{u} = 2\vec{f} \tag{10}$$

$$div \ \vec{u} = 0 \tag{11}$$

with initial condition

$$\vec{u}\,\Big|_{t=0} = \vec{u}_0(x) \tag{13}$$

Following the classical procedure we can get solutions for the problem (10)-(13) in the integral sum of the parabolic potentials

$$\vec{u} = \int_{R^3} \vec{u}_0(\xi) G(x-\xi,t)d\xi + 2\int_0^t d\tau \int_{R^3} \vec{f}(\xi,\tau) G(x-\xi,t-\tau)d\xi \tag{14}$$

where

$$G(x,\xi,t) = \frac{e^{-\frac{(x-\xi)^2}{4\nu t}}}{\left(2\sqrt{\pi\nu t}\right)^3}$$

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is the Green's function for the three dimensional infinite space R^3. Green's function $G(x,\xi,t)$ for their derivations has estimations

$$\left|\frac{\partial}{\partial x_i} G(x,\xi,t)\right| \le \frac{e^{-\frac{(x-\xi)^2}{8\nu t}}}{\left(\sqrt{\pi}\right)^3 \nu^2 t^2} \qquad (i=1,2,3)$$

Notice that assumption (7) is closely related with the following conditions

$$rot\ \vec{f} = 0\ ,\ rot\ \vec{u}_0 = 0$$

Using properties of the Green's function $G(x,\xi,t)$ and its derivative evaluations we have got a uniqueness and stable solution (14) satisfying following estimation

$$\|\vec{u}\|_{H^{(2,1)}_{\Omega_T}} \le M_0(\|\vec{u}_0\|_{H^{(1)}_{\Omega}} + 2t\|\vec{f}\|_{L_2})$$

Condition for the scalar pressure function $p(x,t)$

$$\frac{P}{\rho} + \frac{u^2}{2} - div\ \vec{f} * \frac{1}{4\pi|x-\xi|} = 0 \qquad (9)$$

predicts a steady feature which introduces a stable turbulent motion. This equation links with the energy conservation low and characterizes steady behavior for the turbulent motion that can be the main property of stability turbulent flows. After using the Navier-Stokes equation (1) has been obtained following estimation for pressure function $p(x,t)$ with norm on Hilbert space $\|p\|_{H^{(1,0)}_{\Omega_T}} \le M_0(\|\vec{u}_0\|_{H^{(1)}_{\Omega}} + \|\vec{f}\|_{L_2})$

Consequently, we see that a stability of the turbulent flow depends on the condition (9).

IV. VELOCITY COMPONENTS AND FUNCTION OF PRESSURE FOR TURBULENT SWIRLING MOTION

Fundamental interest in the study of unsteady features is an instable swirling motion which characterizes high Reynolds numbers. and new obtained condition of the turbulent motion admits solution that can be predicted in terms of the rotation function which is concerned instable fluid flow.

Assume that

$$grad\left(\frac{u^2}{2} + \frac{p}{\rho} - \Phi\right) \ne 0\ , \qquad (14)$$

then the Navier-Stokes equations (1)-(3) can be written as follows:

$$\frac{\partial\vec{u}}{\partial t} - \nu\Delta\vec{u} - \left[\vec{u}\times rot\ \vec{u}\right] + grad\left(\frac{u^2}{2} + \frac{p}{\rho} - \Phi\right) = \vec{f}^* + 2f \qquad (15)$$

There vector function $\vec{f}^*$ is a convolution between vector and matrix

$$\vec{f}^8 = \begin{pmatrix} -\frac{\partial^2 f_1}{\partial x_2^2} - \frac{\partial^2 f_1}{\partial x_3^2} + \frac{\partial^2 f_2}{\partial x_1 \partial x_2} + \frac{\partial^2 f_3}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f_1}{\partial x_1 \partial x_2} - \frac{\partial^2 f_2}{\partial x_1^2} - \frac{\partial^2 f_2}{\partial x_3^2} + \frac{\partial^2 f_3}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f_1}{\partial x_1 \partial x_3} + \frac{\partial^2 f_3}{\partial x_1 \partial x_3} - \frac{\partial^2 f_3}{\partial x_1^2} - \frac{\partial^2 f_3}{\partial x_2^2} \end{pmatrix} * \begin{pmatrix} \frac{1}{4\pi|x-\xi|} & 0 & 0 \\ 0 & \frac{1}{4\pi|x-\xi|} & 0 \\ 0 & 0 & \frac{1}{4\pi|x-\xi|} \end{pmatrix}$$

Considering condition $rot\ \vec{f}^* \ne 0$ and using rotor operator we obtain equation

$$rot\left[\frac{\partial}{\partial t}\vec{u} - \left[\vec{u}\times rot\ \vec{u}\right] - \nu\Delta\vec{u}\right] = rot\ \vec{f}^* \qquad (16)$$

Donate that

$$\vec{g} = \frac{\partial}{\partial t}\vec{u} - \left[\vec{u}\times rot\ \vec{u}\right] - \nu\Delta\vec{u} \qquad (17)$$

$$\vec{z} = rot\ \vec{f}^*$$

With respect to (16) we have got vector equation

$$rot\ \vec{g} = \vec{z} \qquad (18)$$

Expressing the function $\vec{g}$ in terms of function $\vec{z}$ we can consider system of equations

$$\begin{aligned} \frac{\partial g_3}{\partial x_2} - \frac{\partial g_2}{\partial x_3} &= z_1 \\ \frac{\partial g_1}{\partial x_3} - \frac{\partial g_3}{\partial x_1} &= z_2 \\ \frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_2} + \frac{\partial g_3}{\partial x_3} &= 0 \end{aligned} \qquad (19)$$

Apply to system (19) three-dimensional Fourier transform

$$-is_3\tilde{g}_2 + is_2\vec{g}_3 = \tilde{z}_1$$

$$is_3\vec{g}_1 - is_1 g_3 = \tilde{z}_2$$

$$is_1\vec{g}_1 + is_2\vec{g}_2 + is_3\vec{g}_3 = 0$$

we can define functions

$$\vec{g}_1 = -\frac{(s_1 s_2\tilde{z}_1 + (s_2^2 + s_3^2)\tilde{z}_2)i}{s_3(s_1^2 + s_2^2 + s_3^2)}$$

$$\vec{g}_2 = \frac{((s_1^2 + s_2 s_3)\tilde{z}_1 + s_1 s_2\tilde{z}_2)i}{s_3(s_1^2 + s_2^2 + s_3^2)}$$

$$\vec{g}_3 = \frac{(-s_2 s_3\vec{z}_1 + s_1 s_3\vec{z}_2)i}{s_3(s_1^2 + s_2^2 + s_3^2)}$$

Based on the formula for integration

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos\ \beta t}{t^2 + \alpha^2} = \frac{\pi}{2\alpha}e^{-|\beta|\alpha}\ ,\alpha > 0$$

and formulas for the Fourier transformation

$$\frac{\pi}{2\sqrt{s^2+a^2}}\exp(-b\sqrt{s^2+a^2} \to K_0(a\sqrt{x^2+b^2}),\quad a,b>0$$

$$K_0(s\sqrt{a^2+b^2}) \to \frac{1}{\sqrt{a^2+b^2+c^2}}$$

we obtain the inverse

$$\frac{1}{s_1^2+s_2^2+s_3^2} \to \frac{1}{|x-\xi|}$$

There symbol $\to$ indicates transitions from the representation to the original.

Using representation

$$\tilde{Z}_0(f_1,f_2,f_3) = \begin{cases} \tilde{z}_1 = is_2\tilde{f}_3^* - is_3\tilde{f}_2^* \\ \tilde{z}_2 = is_3\tilde{f}_1^* - is_1\tilde{f}_3^* \\ 0 \end{cases}$$

we obtain following vector equation

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$$\frac{\partial \vec{u}}{\partial t} - \nu \Delta \vec{u} - \left[\vec{u} \times rot \quad \vec{u}\right] = \vec{b}(x,t)$$

where vector

$$\vec{b}(x,t) = \frac{1}{4\pi}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{20}$$

has components

$$b_1 = \left(\frac{\partial^3}{\partial x_3 \partial^2 x_2} + \frac{\partial^3}{\partial^3 x_3}\right)\int_{R^3}\frac{1}{|x-\xi|}\int_{-\infty}^{\infty}\theta(\xi_3 - \varsigma_3) f_1^*(\xi_1,\xi_2,\varsigma_3,t)d\varsigma_3 d\xi -$$

$$-\left(\frac{\partial^3}{\partial x_3 \partial x_2 \partial x_1} + \frac{\partial^3}{\partial x_1 \partial^2 x_2} + \frac{\partial^3}{\partial x_1 \partial^2 x_3}\right)\int_{R^3}\frac{1}{|x-\xi|}\int_{-\infty}^{\infty}\theta(\xi_3 - \varsigma_3) f_2^*(\xi_1,\xi_2,\varsigma_3,t)d\varsigma_3 d\xi +$$

$$+\frac{\partial^3}{\partial x_3 \partial x_2 \partial x_1}\int_{R^3}\frac{1}{|x-\xi|}\int_{-\infty}^{\infty}\theta(\xi_3 - \varsigma_3) f_3^*(\xi_1,\xi_2,\varsigma_3,t)d\varsigma_3 d\xi$$

$$b_2 = -\Bigg[\frac{\partial^3}{\partial x_3 \partial x_2 \partial x_1}\int_{R^3}\frac{1}{|x-\xi|}\int_{-\infty}^{\infty}\theta(\xi_3 - \varsigma_3) f_1^*(\xi_1,\xi_2,\varsigma_3,t)d\varsigma_3 d\xi +$$

$$+\left(\frac{\partial^3}{\partial x_3 \partial^2 x_1} + \frac{\partial^3}{\partial x_2 \partial^2 x_3} + \frac{\partial^3}{\partial x_2 \partial^2 x_1}\right)\int_{R^3}\frac{1}{|x-\xi|}\int_{-\infty}^{\infty}\theta(\xi_3 - \varsigma_3) f_2^*(\xi_1,\xi_2,\varsigma_3,t)d\varsigma_3 d\xi +$$

$$+\left(\frac{\partial^3}{\partial x_2 \partial^2 x_1} + \frac{\partial^3}{\partial x_3 \partial^2 x_2}\right)\int_{R^3}\frac{1}{|x-\xi|}\int_{-\infty}^{\infty}\theta(\xi_3 - \varsigma_3) f_3^*(\xi_1,\xi_2,\varsigma_3,t)d\varsigma_3 d\xi$$

$$b_3 = -\Bigg[\frac{\partial^3}{\partial x_3 \partial x_2 \partial x_1}\int_{R^3}\frac{1}{|x-\xi|}\int_{-\infty}^{\infty}\theta(\xi_3 - \varsigma_3) f_1^*(\xi_1,\xi_2,\varsigma_3,t)d\varsigma_3) d\xi -$$

$$-\left(\frac{\partial^3}{\partial x_2 \partial^2 x_3} - \frac{\partial^3}{\partial x_3 \partial^2 x_1}\right)\int_{R^3}\frac{1}{|x-\xi|}\int_{-\infty}^{\infty}\theta(\xi_3 - \varsigma_3) f_2^*(\xi_1,\xi_2,\varsigma_3,t)d\varsigma_3 d\xi -$$

$$-\frac{\partial^3}{\partial x_3 \partial^2 x_2}\int_{R^3}\frac{1}{|x-\xi|}\int_{-\infty}^{\infty}\theta(\xi_3 - \varsigma_3) f_3^*(\xi_1,\xi_2,\varsigma_3,t)d\varsigma_3 d\xi \Bigg]$$

Expanding the brackets $\left[\vec{u} \times rot\vec{u}\right]$, we obtain the expression

$$\left[\vec{u} \times rot\vec{u}\right] = u_2\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\vec{i} + u_3\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)\vec{j} + u_1\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)\vec{k} -$$

$$-u_3\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)\vec{i} - u_1\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\vec{j} - u_2\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)\vec{k}$$

which can be written as

$$\left[\vec{u} \times rot\,\vec{u}\right] = \left[u_2\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) - u_3\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)\right]\vec{i} +$$

$$+\left[u_3\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) - u_1\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\right]\vec{j} +$$

$$+\left[u_1\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) - u_2\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)\right]\vec{k}$$

For convenience, let $\left[\vec{u} \times rot\vec{u}\right]$ denote the coordinates of this vectors

$$u_1^{**} = u_2\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) - u_3\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)$$

$$u_2^{**} = u_3\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) - u_1\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$$

$$u_3^{**} = u_1\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) - u_2\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)$$

We consider

$$\frac{\partial \vec{u}}{\partial t} - \nu \Delta \vec{u} - \left[\vec{u} \times rot \quad \vec{u}\right] = \vec{b}(x,t)$$

Problem (10)-(13) is closely related with the nonlinear integral equation satisfying a equation

$$\vec{u} = \vec{u}^{**} * G + \vec{F}$$

where

$$\vec{F} = \vec{u}_0 * G + \vec{b} * G$$

$$G(x,\xi,t) = \frac{e^{-\frac{(x-\xi)^2}{4\nu t}}}{\left(2\sqrt{\pi \nu t}\right)^3}$$

is the Green's function in three dimensional whole space R^3. Properties of the Green's function and its derivative evaluation allow to solve the nonlinear matrix Volterra - Fredholm integral equation by using successive approximations. Using Betta function

$$B\left(n + \frac{1}{2}, \frac{1}{2}\right) = \frac{(n-1)!\,(\sqrt{\pi})^2}{n!} = \frac{\pi}{n}$$

and well-known properties of Green's function we have got estimations

$$\left\|u^{(0)}\right\|^2 \le M_0\left(\|u_0\|^2 + t\|f\|^2\right)$$

$$\left\|u^{(1)}\right\|^2 \le M_0\left(\|u_0\|^2 + t\|f\|^2 + \frac{M_0^2\sqrt{t}}{2}\left(\|u_0\|^2 + t\|f\|^2\right)^2\right.$$

$$\left\|u^{(2)}\right\|^2 \le \frac{M_0\left(\|u_0\|^2 + t\|f\|^2\right)}{1!} + \frac{M_0^2\sqrt{t}\left(\|u_0\|^2 + t\|f\|^2\right)^2}{2!} +$$

$$+\frac{M_0^3 t^{3/2}\left(\|u_0\|^2 + t\|f\|^2\right)^4}{3!} \tag{21}$$

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$$\left\|u^{(n)}\right\|^2 \le \frac{M_0\left(\|u_0\|^2 + t\|f\|^2\right)}{1!} + \frac{M_0^2\sqrt{t}\left(\|u_0\|^2 + t\|f\|^2\right)^2}{2!} +$$

$$+\frac{M_0^3 t^{3/2}\left(\|u_0\|^2 + t\|f\|^2\right)^4}{3!} + \dots + \frac{M_0^n t^{n+1/2}\left(\|u_0\|^2 + t\|f\|^2\right)^{2n}}{n!}$$

Due to this fact we have the unique solution of the problem (10)-(13)

$$\vec{u} = \int_0^t d\tau \int_\Omega R\left[A(\vec{F}(\xi,\tau))\Gamma(x-\xi,t-\tau)\right]d\Omega + \vec{F}(x,t) \tag{22}$$

$$R[\cdot] = F(x,t) + \int_0^t d\tau \int_{R^3} A\left[F(\xi,\tau)\right]\Gamma(x-\xi,t-\tau)d\xi +$$

$$+\int_0^t d\tau \int_{R^3} A\left[\int_0^\tau d\tau_1 \int_{R^3} A\left[F(\xi,\tau)\right]\Gamma(\xi-\zeta,\tau-\tau_1)d\zeta\right]\Gamma(x-\xi,t-\tau)d\xi +$$

$$+\int_0^t d\tau \int_{R^3} A\left[\dots\left[\int_0^\tau d\tau_1 \int_{R^3} A\left[F(\xi,\tau)\right]\Gamma(\xi-\zeta,\tau-\tau_1)d\zeta\right]\dots\right]\Gamma(x-\xi,t-\tau)d\xi + \dots$$

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$$A(F)=\begin{pmatrix} \frac{1}{2}(F_1^2+F_2^2+F_3^2) & -F_1F_2 & -F_1F_3 \\ -F_1F_2 & \frac{1}{2}(F_1^2+F_2^2+F_3^2) & -F_2F_3 \\ -F_1F_3 & -F_2F_3 & \frac{1}{2}(F_1^2+F_2^2+F_3^2) \end{pmatrix}$$

where

$$\vec{F} = \vec{b} * G + u_0 * G$$

$$\Gamma(x-\xi,t) = \begin{pmatrix} \frac{\partial G(x-\xi,t)}{\partial x_1} \\ \frac{\partial G(x-\xi,t)}{\partial x_2} \\ \frac{\partial G(x-\xi,t)}{\partial x_3} \end{pmatrix}$$

Vector function $\vec{F}(x,t)$ satisfies following estimation

$$\left\|\vec{F}(x,t)\right\| \le C(\|\vec{u}_0\| + t\|\vec{b}\|)$$

Using the well-known properties of Green's functions we have got estimation for the vector velocity in the space $L_2(R^3 \times [0,T]$

$$\|\vec{u}\|_{L_2}^2 \le (\|\vec{u}_0\|_{L_2}^2 + t\|\vec{b}\|_{L_2)}^2)\Big[1 +$$

$$+ M_0(\|\vec{u}_0\|_{L_2}^2 + t\|\vec{b}\|_{L_2}^2)e^{(\|\vec{u}_0\|_{L_2}^2 + t\|\vec{b}\|_{L_2}^2)^2}\Big] \quad (22)$$

Following the classical procedure we get the uniqueness and stability of solution for the problem (1)-(3). Also we obtain equation for the pressure function

$$\frac{u^2}{2} + \frac{p}{\rho} - div\vec{f} * \frac{1}{4\pi|x-\xi|} - div f^{**} * \frac{1}{4\pi|x-\xi|} = 0 \quad (23)$$

where

$$\vec{f}^{**} = \vec{b} - \vec{f}^{*} - 2f$$

$$\|p\|_{L_2(\Omega_T)} \le C_0(\|\vec{u}_0\|_{L_2(\Omega)} + t\|\vec{\Psi}\|_{H^{(1,0)}_{\Omega_T}})\Big[1 +$$

$$+ C_1\sqrt{t}(\|\vec{u}_0\|_{L_2(\Omega)} + t\|\vec{\Psi}\|_{H^{(1,0)}_{\Omega_T}})e^{C_2 t(\|\vec{u}_0\|_{L_2(\Omega_T)} + t\|\vec{\Phi}\|_{H^{(1,0)}_{\Omega_T}})^2}\Big]$$

$$\|\Psi\|_{L_2} = \sqrt{(\Psi_1)^2 + (\Psi_2)^2 + (\Psi_3)^2}$$

$$\vec{\Psi}(x,t) = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} \equiv \begin{pmatrix} \int_{-\infty}^{\infty} \theta(x_1-\xi_1) f_1(\xi_1,x_2,x_3,t)d\xi_1 \\ \int_{-\infty}^{\infty} \theta(x_2-\xi_2) f_2(x_1,\xi_2,x_3,t)d\xi_2 \\ \int_{-\infty}^{\infty} \theta(x_3-\xi_3) f_3(x_1,x_2,\xi_3,t)d\xi_3 \end{pmatrix}$$

V. RESULTS AND DISCUSSION

Let us gather and formulate main results about properties of the vector velocity and the scalar function of pressure.

Recall the notations $\Omega = R^3$ and $\Omega_T = R^3 \times (0 < t < \infty)$ we look for periodic solution for the problem (1)-(3).We assume that functions $f_i(x,t)$ and $u_{0i}(x,t)$ satisfy

$$u_{0i}(x) = u_{0i}(x + k_j), \quad f_i(x,t) = f_i(x+k_j,t)$$

for $1 \le j \le 3$, where $k_j = j^{th}$ is unit vector in R^3.

Theorem 1. Let $u_{0i}(x,t) \in H^{(2)}(\Omega)$ and $f_i(x,t) \in L_2(\Omega_T^{(i)})$ be periodic functions and $rot\ \vec{f} = 0$, $rot\ \vec{u}_0 = 0$. Then there exists a unique stable periodic solution

$$\bar{u} = \int_{R^3} \vec{u}_0(\xi)G(x-\xi,t)d\xi + 2\int_0^t d\tau \int_{R^3} \vec{f}(\xi,\tau)G(x-\xi,t-\tau)d\xi$$

for the Navier-Stokes problem (1) - (3) and a unique scalar function of pressure $p(x,t)$ which satisfies

$$\frac{u^2}{2} + \frac{p}{\rho} - div\ \vec{f} * \frac{1}{4\pi|x-\xi|} = 0 \quad (9)$$

Moreover, there exists positive constant M_0 such that for all functions $\vec{u}(x,t) \in H^{(2,1)}(\Omega_T)$ and $p(x,t) \in H^{(1.0)}(\Omega_T)$ satisfy the following estimates

$$\|\vec{u}\|_{H^{(2,1)}_{\Omega_T}} \le M_0(\|\vec{u}_0\|_{H^{(1)}_{\Omega}} + 2\|\vec{f}\|_{L_2})$$

$$\|p\|_{H^{(1,0)}_{\Omega_T}} \le M_0(\|\vec{u}_0\|_{H^{(1)}_{\Omega}} + 2\|\vec{f}\|_{L_2})$$

In processes dealing with **theorem 1** we notice that Bernoulli's equation as a consequence of the formula (9) which represents stability criteria. Due to this fact we can formulize this simple result.

Remark 1. Assume that $rot\ \vec{f} = 0$, $rot\ \vec{u}_0 = 0$ are satisfied. If $\vec{f} = C\vec{x} + \vec{d}$, where C matrix

$$C = \begin{pmatrix} \frac{c_1}{m} & 0 & 0 \\ 0 & \frac{c_2}{m} & 0 \\ 0 & 0 & -gh \end{pmatrix}$$

$\vec{d}$ - a numerical vector, m - a body's mass, c_1, c_2 are independent constants which satisfy the condition $c_1 + c_2 \ge 0$, g is the acceleration of gravity, h is the height. Then the fluid flow can be considered to be an incompressible flow which satisfies Bernoulli's equation

$$\frac{mp}{\rho} + \frac{mu^2}{2} + mgh = const \quad (24)$$

Here $\frac{mp}{\rho}$ is a binding energy of the mass elements, $\frac{mu^2}{2}$ - a kinetic energy , mgh - a potential energy .

The next theorem provides the result about unstable motion.

Theorem 2. Let $u_{0i}(x,t) \in H^{(2)}(\Omega)$ and $f_i(x,t) \in L_2(\Omega_T^{(i)})$ be periodic functions and $rot\ \vec{f} \ne 0$. Under this assumption there exists a unique unstable periodic solution of the Navier-Stokes problem (1) - (3)

$$\vec{u} = \int_0^t d\tau \int_\Omega R\left[A(\vec{F}(\xi,\tau))\Gamma(x-\xi,t-\tau)\right]d\Omega + \vec{F}(x,t)$$

$$\vec{F} = \vec{b} * G + \vec{u}_0 * G$$

and a unique scalar function of pressure $p(x,t)$ which satisfies

$$\frac{u^2}{2} + \frac{p}{\rho} - div\vec{f} * \frac{1}{4\pi|x-\xi|} - div\vec{f}^{**} * \frac{1}{4\pi|x-\xi|} = 0$$

(23) where

$$\vec{f}^{**} = \vec{b} - \vec{f}^* - 2f$$

Moreover, there exist positive constants C_0, C_1, C_2 such that for all functions $\vec{u}(x,t) \in H^{(2,1)}(\Omega_T)$ and $p(x,t) \in H^{(1.0)}(\Omega_T)$ satisfy the following estimates

$$\|\vec{u}\|_{H^{(2,1)}_{\Omega_T}} \le C_0(\|\vec{u}_0\|_{H^{(1)}_\Omega} + \|\vec{\Psi}\|_{H^{(3,0)}_{\Omega_T}})\Big[1 + C_1(\|\vec{u}_0\|_{H^{(1)}_\Omega} + \|\vec{\Psi}\|_{H^{(3,0)}_{\Omega_T}})e^{C_2 t(\|\vec{u}_0\|_{H^{(1)}_\Omega} + \|\vec{\Psi}\|^2_{H^{(3,0)}_{\Omega_T}})^2}\Big]$$

$$\|p\|_{H^{(1,0)}_{\Omega_T}} \le C_0(\|\vec{u}_0\|_{H^{(1)}_\Omega} + \|\vec{\Psi}\|_{H^{(3,0)}_{\Omega_T}})\Big[1 + C_1(\|\vec{u}_0\|_{H^{(1)}_\Omega} + \|\vec{\Psi}\|_{H^{(3,0)}_{\Omega_T}})e^{C_2 t(\|\vec{u}_0\|_{H^{(1)}_\Omega} + \|\vec{\Psi}\|_{H^{(3,0)}_{\Omega_T}})^2}\Big]$$

where

$$\vec{\Psi}(x,t) = \begin{pmatrix} \int_{-\infty}^{\infty} \theta(x_3 - \varsigma_3) f_1(x_1, x_2, \varsigma_3, t) d\varsigma_3 \\ \int_{-\infty}^{\infty} \theta(x_3 - \varsigma_3) f_2(x_1, x_2, \varsigma_3, t) d\varsigma_3 \\ \int_{-\infty}^{\infty} \theta(x_3 - \varsigma_3) f_3(x_1, x_2, \varsigma_3, t) d\varsigma_3 \end{pmatrix}$$

$\theta(z)$ is Heaviside step function.

These mathematical endeavors can serve to enlarge our intuitive experiences respect to the nonlinear theories of partial differential equations. In this work we concentrate on those aspects of partial differential equations that can be represented in the terms of operators on a Hilbert spaces. There Navier-Stokes equations represent the evolution of the governing distribution functions, which depend on the velocity vector in the position of particles as a result of thermal excitation at any finite turbulent energy. In this way some difficulties arise in solving of the Navier-Stokes problem which was encountered in studying turbulent behavior for unstable motion. The fundamentals of our method have shown steady and unsteady behavior involving properties of the turbulent flows which demonstrate technological and principal importance at the forefront of classical approach. Expression of fluid energy

$$div\vec{f}^{**} * \frac{1}{4\pi|x-\xi|}$$

represents a departure from the average energy of the fluid known as eddy energy. Due to this fact we formalize the relation between components of the velocity vector and the pressure function given by the energy conservation low (9).

VI. CONCLUSION

The Navier-Stokes equations have been the basis for description and analysis of all turbulent phenomena and eexperimental selection of the regime turbulent fluctuation is costly and not always realizable process, therefore important argument for analytic research of the Navier-Stokes equation is to investigate an mathematical conception which is based on the Green's function and required a good deal with the parabolic and elliptic potential theory.We have presented the analytic method for the incompressible eddy turbulent problem which is expected to exist for all infinite domains. There are two unknown independent thermodynamic parameters (the velocity vector and the scalar function of pressure) which play a prominent role in the obtained integral representation of the velocity distribution and the description of the turbulent behavior of fluid motion. In processes dealing with governing equations the main point stressed that the velocity vector satisfy and the function of pressure satisfied their criteria of stability motion which is the energy conservation law. There the mathematical difficulty for determining the velocity vector associates with the nonlinear of Volterra-Fredholm matrix integral equation. There convenient procedure for the incompressible Navier-Stokes equations allows to use 'a priori' estimates and to prove existence and regularity of the weak solution. The weak formulation of the Navier-Stokes problem is based on the extension of idea to the case where the energy falls in the critical domain, due to the pressure transition. Moreover, basic concepts of the Navier-Stokes equations have been investigated in Hilbert space and weak formulation is based on the introduced technique for the turbulent flow. In this paper authors take the first step in developing analytic investigation which needs in constriction Green's function and the energy conservation law for the model initial and boundary tasks. Authors hope that this submitted analytic solution would be understood and would have been used for visualization of the turbulent processes and behavior of the pressure distribution in the considered areas.

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