Mathematical Modeling of Additive Manufacturing Technologies

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***Abstract*—Mathematical modeling of additive manufacturing technologies is aimed at improving the performance of device, machine, and mechanism parts. These technologies include stereolithography, electrolytic deposition, thermal and laser-based 3D printing, 3D-IC fabrication technologies, etc. They are booming nowadays, because they can provide rapid low-cost high-accuracy production of 3D items of arbitrarily complex shape (in theory, from any material). However, deformation and strength problems for products manufactured with these technologies yet remain to be solved. The fundamentally new mathematical models considered in the paper describe the evolution of the end product stress-strain state in additive manufacturing and are of general interest for modern technologies in engineering, medicine, electronics industry, aerospace industry, and other fields.**

***Index Terms*—Additive manufacturing technology, mathematical modeling, mechanics of growing solids, stress-strain state, deformation, strength.**

I. Introduction

MATHEMATICAL modeling of a variety of natural phenomena and technological processes requires taking into account the *material evolution* and *remodeling* of a solid, which can be associated with creation and annihilation of material points or with internal constraint redistribution in the bulk of the solid. If such changes are accompanied with deformation of the entire solid, then what we deal with is a *growing solid*, whose properties are highly unusual. Models of winding and welding, vapor deposition, photopolymerization, and ion implantation processes can serve as examples [1], [2]. The solid material composition in such processes is changed either by adding macroscopic volumes, whose locally thermostatic states can be described by statistical parameters such as temperature, distortion, and tension, or by implanting individual atoms or molecules (referred to as *extra substance* in [3]), which from the macroscopic viewpoint leads to distributed defect evolution in the boundary layer. Winding and welding are examples of the former, and ion implantation is an example of the latter. Sometimes both mechanisms should be taken into account, as is the case with the vapor deposition process, which involves the adherence of atomic clusters consisting of large numbers of coupled atoms as well as the adherence of individual atoms of ions bombarding the growth surface.

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We point out that growth is often closely associated with defects formation processes. In particular, vapor phase deposition causes continuous defect formation in growing structures, which can readily be shown by estimating the crystal growth rate. Indeed, the atoms condensed from vapor to a crystal surface with regular atomic structure are only weakly coupled with the surface and evaporate back with high probability. But if there is an unfinished atomic plane on the growth surface, then the atoms that hit the plane edge become strongly coupled. This forces the unfinished atomic plane to be completed rapidly, and the crystal growth stops until there is formed a sufficiently large nucleus of a new atomic plane. One can estimate the probability of such a nucleus to appear as well as the resulting crystal growth rate, which proves to be many orders of magnitude smaller than observed in experiments. This apparent paradox can be explained by assuming that there are a large number of defects continuously formed on the crystal surface, which play the role of nuclei for independently growing islands of atomic planes [4]. With this growth mechanism, the force interactions arising between these islands result in a residual stress field.

Kröner showed in his pioneering paper [3] that the residual stresses in *simple materials* [5] can be represented in terms of the incompatibility of the local distortion field defined in the reference description by methods of non-Euclidean geometry. Thus, the geometric language of the theory of smooth manifolds can be used to describe not only solids with distributed defects but also growing solids.

Stress-strain state analysis for growing solids has been carried out in numerous papers [6]–[11], where a number of trends in generalizing classical continuum mechanics have been used. One of these is developed in the framework of the theory of *inhomogeneity* (structural heterogeneity) arising from a special connection of parts of a body rather than from distinctions in the physical properties of the materials of these parts [12], [13]. This kind of structural inhomogeneity also arises in bodies made of a single material, which are homogeneous in the classical sense. To distinguish between these two kinds of inhomogeneity, we use the term *material uniformity* [5] for the latter one.

The growth of a solid is usually viewed as a process where additional material is joined to the solid, which is deformed in the process. It is assumed that the additional material may have the form of material surfaces, threads, or drops and be deposited to the main body in some stressed state. Moreover, the growing body, together with additional material, can be viewed as a single body represented by multiple solid components, and the growth process can be treated as the generation of constraints providing that the number of connected components of the solid decreases. This leads to a change in the topology of the body. In particular, boundary points become interior points. Here the process of gluing a bundle of paper sheets can be used as an example. Prior to gluing, we have a set with many (possibly, very many) connected components, but after gluing we have a connected set. If each sheet were subjected to some deformation from the standard (uniform) state in the gluing process, no smooth deformation of the final body after gluing would be able to bring all the sheets to the standard state simultaneously. The response of a local part of the body to external loadings for the case in which it is defined by the elasticity tensor would vary from point to point of the body in any configuration defining an immersion in Euclidean space. Thus, the body is inhomogeneous even if it is made from a single material (i.e., is materially uniform). This example illustrates the existence of a special type of inhomogeneity, which is studied in [12], [14], [15].

Thus, inhomogeneity results from the growth process and is related to varying physical and mechanical properties of the material. One can say that it arises in special assembling scenarios. To describe the response of the solid to external inputs, one can either treat it as a nonuniform solid, which results in a complicated description of the constitutive equations, or somehow reconstruct the natural global configuration of the solid and use it as a reference configuration. For simple materials, the latter can be done if one allows embedding the reference shapes in a space with a more flexible definition of geometric properties (e.g., in an affine connection space) and defining a global natural shape with additional geometric parameters such as the torsion, curvature, and nonmetricity of the connection.

We point out that the inhomogeneity in solids can be described without using the ideas of non-Euclidean geometry. Clearly, if a configuration is an embedding of a body in a space that is necessarily Euclidean, then an inhomogeneous solid does not have a global natural configuration; i.e., any configuration is not free of residual stresses. At the same time, we have to use a stressed configuration as a reference, which complicates the statement of constitutive relations. In particular, they have one more tensor argument known as *implant* [9], [10], [14] which characterizes the initial local deformation. However, the geometric meaning of the implant becomes clear if we treat it as the initial ("assembly") local deformation of the element in the natural state, which directly leads to the notion of local transformation of the natural frame used in the geometry of a space with absolute parallelism and thus introduces the concept of non-Euclidean geometry. Therefore, we prefer to use the geometric language from the very beginning.

II. Body as a smooth manifold

In what follows, we use the concept of a body as an abstract smooth manifold, that is, an open subset of some topological space equipped with a special *material connection*. This concept allows one to describe the inhomogeneity phenomenon in materially uniform bodies in a rather elegant geometric way. The foundations of the theory of inhomogeneity have been laid down in the milestone work by Noll [12] and developed by Wang, Epstein, and Maugin [13]–[15]. Since inhomogeneity results from an accretion process, we can hope that this geometric approach will be effective for the problems considered.

We treat a body $\mathfrak{B}$ as a smooth manifold without boundary. This means that $\mathfrak{B}$ is a set equipped with a topology satisfying the separation axiom and can be covered by finitely many overlapping open sets $U_k \subset \mathfrak{B}$ homeomorphic to open subsets in $\mathbb{R}^n$. The homeomorphisms are established by coordinate mappings (charts) $\chi_k : U_k \to \mathbb{R}^n$ such that for every intersection $U_k \cap U_p$ the corresponding composition $\chi_k \circ \chi_p^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and has sufficiently many derivatives. Note that n can be 1, 2, or 3 depending on whether the body is a fiber, a membrane, or a solid, respectively. We refer to n as the *dimension* of the body. The material points are elements of the set $\mathfrak{B}$ and can be identified by their coordinates provided by the charts χ_k. The collection of charts $\{\chi_k\}_{k=1}^{l}$ defines an atlas (of order l) of the manifold. If a manifold can be covered by an atlas of the first order, then this manifold is *trivial*. The need for nontrivial atlases is clear for one- and two- dimensional solids. (A sphere is a simple example.) At first glance, it seems that the three-dimensional case is different, and only trivial atlases are needed. Indeed, a three-dimensional solid embedded in Euclidean space can be modeled by a trivial manifold covered by a single chart whose values are just the *Cartesian coordinates* of the points that constitute the body. But this impression is wrong! In fact, the structure of the atlas should be consistent with the *material connection* (see below), and this consistency may require nontrivial atlases; Wang [13] showed this by examples (one of which is the famous "Möbius crystal") Of course, all such bodies have nontrivial inhomogeneity structure. We point out that these bodies can be created by appropriate growing processes that "sew" these three-dimensional bodies from two-dimensional surfaces. Thus, the notion of atlas plays a significant role in the theory of growing bodies.

Connection in general is a rule that determines the transformation of a vector as it moves along a path (curve) on $\mathcal{B}$ that carries the vector from one fiber to another. A linear (affine) connection determines the linear transformation under infinitesimal transport, i.e., a mapping $\nabla : T_{\mathfrak{X}}(\mathfrak{B}) \to \mathcal{L}(T_{\mathfrak{X}}(\mathfrak{B}), T_{\mathfrak{X}}(\mathfrak{B}))$. In a local chart, one has $\nabla_{\partial_\alpha}\partial_\beta = \Gamma^\gamma_{\alpha\beta}\partial_\gamma$, where the $\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols of the connection. A linear connection ∇ is said to be *compatible* with a metric $\boldsymbol{g}$ on the manifold if the inner product of two arbitrary vectors remains the same after the parallel transport of these vectors along an arbitrary curve. It can be shown that ∇ is compatible with $\boldsymbol{g}$ if and only if $\forall \boldsymbol{u} \nabla_{\boldsymbol{u}} \boldsymbol{g} = \boldsymbol{0}$. Consider an n-dimensional manifold $\mathfrak{B}$ with a metric $\boldsymbol{g}$ and a connection ∇. The triple $(\mathfrak{B}, \nabla, \boldsymbol{g})$ is called a *Riemann–Cartan manifold*.

The *torsion* of a connection is the map $\boldsymbol{\mathfrak{T}} : T_{\mathfrak{X}}(\mathfrak{B}) \times T_{\mathfrak{X}}(\mathfrak{B}) \to T_{\mathfrak{X}}(\mathfrak{B})$ defined by

$$\boldsymbol{\mathfrak{T}}(\boldsymbol{u}, \boldsymbol{v}) = \nabla_{\boldsymbol{u}}\boldsymbol{v} - \nabla_{\boldsymbol{v}}\boldsymbol{u} - [\boldsymbol{u}, \boldsymbol{v}].$$

In the components in a local chart, one has $\mathfrak{T}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta}$. A connection is said to be *symmetric* if it is torsion free, that is, if $\nabla_{\boldsymbol{X}}\boldsymbol{Y} - \nabla_{\boldsymbol{Y}}\boldsymbol{X} = [\boldsymbol{X}, \boldsymbol{Y}]$.[1] The *Riemannian curvature* is the map $\boldsymbol{\mathfrak{R}} : T_{\mathfrak{X}}(\mathfrak{B}) \times T_{\mathfrak{X}}(\mathfrak{B}) \to \mathcal{L}(T_{\mathfrak{X}}(\mathfrak{B}), T_{\mathfrak{X}}(\mathfrak{B}))$

[1] One can show that on every Riemannian manifold $(\mathfrak{B}, \boldsymbol{g})$ there exists a unique torsion-free linear connection ∇ compatible with $\boldsymbol{G}$, namely, the Levi-Civita connection.

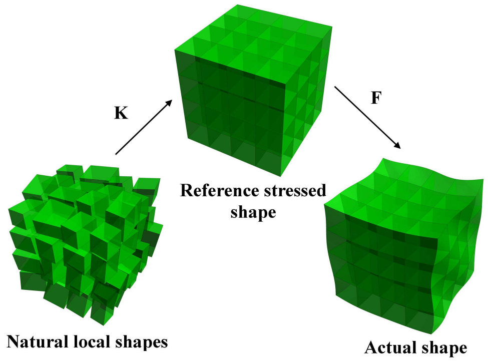

Fig. 1. Configurations and Deformations

defined by

$$\mathfrak{R}(\boldsymbol{u},\boldsymbol{v}) = \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}} - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}} - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}.$$

A metric-affine manifold is a manifold equipped with both a connection and a metric, $(\mathfrak{B}, \nabla, \boldsymbol{g})$. If the connection is metrically compatible, then the manifold is called a *Riemann–Cartan manifold*. If the connection is torsion free but has nontrivial curvature, then $\mathfrak{B}$ is called a *Riemannian manifold*. If the curvature of the connection vanishes but the torsion is nontrivial, then $\mathfrak{B}$ is called a Weitzenböck manifold. If both the torsion and the curvature vanish, then $\mathfrak{B}$ is a flat (*Euclidean*) manifold.

Let us explain the concept of *material connection* in a few words. It implements the idea of a local uniform reference configuration that carries an infinitesimal neighborhood of a material point to some uniform (typically natural) strain state. In the simplest but frequently studied cases, one can carry the whole body to a uniform state by some global configuration. In this case, the connection turns out to be Euclidean, and the theory becomes trivial. In general, there does not exist a smooth mapping transforming the infinitesimal neighborhoods of all material points to a uniform state simultaneously. That is why one equips a materially uniform body, i.e., a body all of whose material points are of the same kind, with some *artificial* (or *structural*) inhomogeneity. From the mechanical viewpoint, such bodies do not have shapes free from residual stresses. The only way to return the neighborhood of each material point to a natural state and hence relax the residual stresses is to cut the body into infinitely many parts and allow them to deform independently (Fig.1). This fictitious process in some sense is reciprocal to the growing process. One can find a detailed statement of the theory in [8], [11].

III. Growing Body

Now let us give a precise definition of growing body. We consider growing bodies that can be represented as continuous families of nested bodies. Recall that, according to the definitions given above, the manifolds that represent bodies have no boundaries. Certainly, physical bodies have boundaries. The boundary points are not included in the open set $\mathfrak{B}$, but their union is the set $\partial\mathfrak{B} = \overline{\mathfrak{B}} \setminus \mathfrak{B}$, which represents the boundary of the body $\mathfrak{B}$. We can assume that $\partial\mathfrak{B}$ is a smooth manifold whose dimension is the dimension of $\mathfrak{B}$ minus one. Finally, we believe that the inclusion of material can be represented as a continuous adjunction of *material surfaces* (in the sense of [16]) to the boundary $\partial\mathfrak{B}$. These considerations can be summarized as follows:

A *layerwise* growing body is a continuous monotone (with respect to inclusion) one-parameter family of manifolds

$$\mathfrak{C} = \{\mathfrak{B}_\alpha\}_{\alpha\in I}\,, \quad \forall\alpha < \beta \in I \quad \mathfrak{B}_\alpha \subset \mathfrak{B}_\beta, \tag{1}$$

where $I = (a, b) \subset \mathbb{R}$ is an open interval, such that the following property holds:

$$\forall\alpha \in I \quad \forall\boldsymbol{\mathfrak{X}} \in \mathfrak{B}_\alpha \setminus \overline{\mathfrak{B}}_a \quad \exists\gamma \in I \quad \boldsymbol{\mathfrak{X}} \in \partial\mathfrak{B}_\gamma. \tag{2}$$

We refer ro $\mathfrak{B}_a$ as to initial body and to $\mathfrak{B}_b$ as to final body. Since the family (1) represents some evolution process, we refer to α as the *evolution parameter*. Property (2) states that any interior point that belongs to the adjoined part of the body has been an element of the boundary manifold at some stage of growth, and so the topological dimension of its neighborhood has changed. This is specific for growing bodies.

We single out a particular class of growth, namely, *complete surface growth*, by the following condition:

$$\forall\alpha < \beta \in I \quad \partial\mathfrak{B}_\beta \notin \mathfrak{B}_\alpha,$$

or, equivalently, $\forall\alpha < \beta \in I \; \partial\mathfrak{B}_\beta \cap \partial\mathfrak{B}_\alpha = \emptyset$. This property ensures that for each $\mathfrak{B}_\alpha \in \mathfrak{C}$ there exists a continuous projection of $\mathfrak{B}_\alpha \setminus \overline{\mathfrak{B}}_a$ onto an interval $I' \subset I$ exists, so that one can interpret the manifold $\mathfrak{B}_\alpha \setminus \overline{\mathfrak{B}}_a$ as a trivial bundle over I' whose fibers are the manifolds $\partial\mathfrak{B}_\gamma$, $\gamma \in I'$.

The idea of continuity can be formalized by means of some measure (such as volume or mass) on the manifolds $\mathfrak{B}_\alpha$. If a certain measure $\mathrm{mes}(\ldots)$ has been introduced, then the *continuity* of the family (1) is equivalent to the following property:

$$\forall\varepsilon \quad \exists\delta \quad \forall\alpha < \beta \in I \quad \beta - \alpha < \delta \Rightarrow \mathrm{mes}(\mathfrak{B}_\beta \setminus \mathfrak{B}_\alpha) < \varepsilon.$$

It is also possible to interpret the layerwise character of growth in terms of measure as follows:

$$\lim_{\beta\to\alpha} \frac{\mathrm{mes}(\mathfrak{B}_\beta \setminus \mathfrak{B}_\alpha)}{\beta - \alpha} = k,$$
$$0 < k < \infty, \quad \exists\Omega \subset \mathfrak{B}_\beta \setminus \mathfrak{B}_\alpha \quad \dim\Omega = \dim\mathfrak{B}_\alpha - 1.$$

The latter condition states that the infinitesimal increment, which is the set difference between two nearby instances of the family $\mathfrak{C}$ representing the growing body, is asymptotically equivalent to a body of dimension less by one. For example, if the $\mathfrak{B}_\alpha$ are three-dimensional manifolds, then Ω is two-dimensional, and its mechanical response can be described by relations suitable for membranes, shells, *etc.*

Clearly, the definition given above does not cover all possible ways of surface growth that can be implemented as a continuous process of joining surfaces, fibers, or drops. An appropriate classification can be obtained on the geometric basis. In this framework, we treat a growing body as a bundle of smooth manifolds; the topological structure of the bundle, in particular, the dimensions of the base and a typical fiber, depends on the accretion process. See [8], [11] for details.

It is conventional in rational mechanics [5, p. 35] that bodies are presented in the physical space $\mathcal{E}$ as shapes $\mathcal{B}_\alpha \subset \mathcal{E}$. On the one hand, shapes $\mathcal{B}_\alpha$ are connected subsets of the physical space, and on the other hand, every shape is the image of a configuration $\varkappa_\alpha : \mathfrak{B}_\alpha \to \mathcal{B}_\alpha$ and belongs to the class of admissible configurations that equip the body $\mathfrak{B}_\alpha$ with the structure of a smooth manifold. We associate two shapes with each element of the family (1), the reference shape $\mathcal{B}_\alpha^R = \varkappa_\alpha^R \mathfrak{B}_\alpha$ and the actual shape $\mathcal{B}_\alpha = \varkappa_\alpha \mathfrak{B}_\alpha$. Thus, the family (1) induces the corresponding families of reference and actual configurations as well as families of reference and actual shapes. From now on, we use exactly these definitions of configurations and shapes. Note that the reference shape is not stress free in general.

Note that the conventional notation for the position of a material point $\boldsymbol{\mathfrak{X}}_\alpha$ in the reference configuration is $T\mathcal{E} \ni \boldsymbol{X}_\alpha = \varkappa_\alpha^R(\boldsymbol{\mathfrak{X}})$, and $\boldsymbol{x}_\alpha$ is used for the position in the actual configuration; i.e., $T\mathcal{E} \ni \boldsymbol{x}_\alpha = \varkappa_\alpha(\boldsymbol{\mathfrak{X}})$. The composition $\varphi_\alpha = \varkappa_\alpha \circ (\varkappa_\alpha^R)^{-1}$ of these configurations, i.e., $\varphi_\alpha : \boldsymbol{X}_\alpha \mapsto \boldsymbol{x}_\alpha$, is the *deformation*. The derivative in the mapping φ, which exists owing to the differentiability of the configurations and their inverses, is known as the *deformation gradient* (which is actually not a gradient at all) $\boldsymbol{F} = \partial x^m / \partial X^n \boldsymbol{e}_m \otimes \boldsymbol{e}^n \equiv \partial x^m / \partial X^n \partial_m \otimes dx^n$. We always assume that $J = \det \boldsymbol{F} > 0$.

In general, the configuration of a growing body is a smooth mapping of the material manifold equipped with a material connection onto the physical manifold, whose connection is substantially different.

We assume that the bodies $\mathfrak{B}_\alpha$ are materially uniform, simple and elastic [5], so their response can be defined by the response functional,

$$\boldsymbol{T}_\alpha = \mathfrak{S}(\boldsymbol{H}_\alpha). \tag{3}$$

Here $\mathfrak{S}(\ldots)$ is the response functional, which is nonlinear in the general case. We assume that $\mathfrak{S}(\ldots)$ does not explicitly depend on the evolution parameter α. The tensor $\boldsymbol{T}_\alpha$ is some kind of stress tensor field. (To be definite, we use the term *Cauchy stresses*). Assume that $\mathfrak{S}$ satisfy the principle of frame indifference and has been calibrated,

$$\mathfrak{S}(\boldsymbol{0}) = \boldsymbol{0}, \qquad \lim_{\det H \to 0} |\mathfrak{S}(\boldsymbol{H})| = \infty.$$

The tensor $\boldsymbol{H}_\alpha$ is a smooth tensor field representing the local distortion. It can be written in the form of multiplicative decomposition

$$\boldsymbol{H}_\alpha = \boldsymbol{F}_\alpha \circ \boldsymbol{K}_\alpha, \tag{4}$$

where $\boldsymbol{F}_\alpha$ is the conventional deformation gradient, i.e., the linearization of the mapping $\gamma_\alpha : \mathcal{B}_\alpha^R \to \mathcal{B}_\alpha$, which can be represented by the relative gradient $\boldsymbol{\nabla}_{\varkappa_\alpha^R}$ as follows [5]:

$$\boldsymbol{F}_\alpha = \boldsymbol{\nabla}_{\varkappa_\alpha^R} \gamma_\alpha. \tag{5}$$

It is important to note that the tensor field $\boldsymbol{F}_\alpha$ is compatible in the following sense: there exists a vector field whose gradient gives $\boldsymbol{F}_\alpha$. Note that this property does not hold in general for the second factor on the right-hand side in (4), namely, for the smooth tensor field $\boldsymbol{K}_\alpha$. This field was dubbed the *implant field* in [14]. Indeed, $\boldsymbol{K}_\alpha$ is a field of linear transformations acting on the undistorted incompatible infinitesimal parts and joining them without gaps into a global reference configuration.

By virtue of its incompatibility, the implant field $\boldsymbol{K}_\alpha$ induces an inhomogeneity that can be represented by a non-Euclidean material connection, which is a certain type of affine connection admissible on the manifold $\mathfrak{B}_\alpha$. In abstract terms, this connection can be defined as a field of

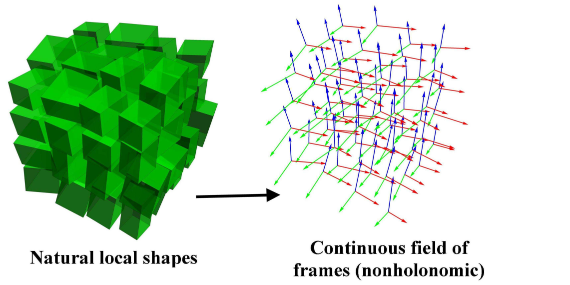

Fig. 2. Correspondence Between Local Shapes and Space with Absolute Parallelism

operators Γ_α that map some tangent vector $\boldsymbol{h}$ onto a linear operator [12],

$$\Gamma_\alpha \boldsymbol{h} = \boldsymbol{K}_\alpha^{-1} \boldsymbol{\nabla}_{\boldsymbol{K}_\alpha}(\boldsymbol{K}_\alpha \boldsymbol{h}) \boldsymbol{K}_\alpha,$$

It is clear that the material connection is actually a geometric representation of the implant field $\boldsymbol{K}_\alpha$. The non-Euclidean features of such a connection are represented by the torsion tensor field $\mathfrak{T}_\alpha$,

$$\mathfrak{T}_\alpha(\boldsymbol{h}, \boldsymbol{p}) = (\Gamma_\alpha \boldsymbol{p})\boldsymbol{h} - (\Gamma_\alpha \boldsymbol{h})\boldsymbol{p} - [\boldsymbol{h}, \boldsymbol{p}].$$

Here $[\cdot, \cdot]$ denotes the Lie bracket, which represents the commutator of tangent vector fields [12], [17]. One can express these relations in terms of the natural frame ∂_ν induced by a certain coordinate map, say $\varkappa_\alpha^R$, on the manifolds $\mathcal{B}_\alpha$ in the form [14]

$$\boldsymbol{e}_\beta = (\boldsymbol{K}_\alpha)^{\nu}_{\cdot\beta} \partial_\nu, \quad (\Gamma_\alpha)^\beta_{\gamma\nu} = \left(\boldsymbol{K}_\alpha^{-1}\right)^{\rho}_{\cdot\gamma,\nu} (\boldsymbol{K}_\alpha)^\beta_{\cdot\rho},$$
$$(\mathfrak{T}_\alpha)^\beta_{\gamma\nu} = (\Gamma_\alpha)^\beta_{\gamma\nu} - (\Gamma_\alpha)^\beta_{\nu\gamma},$$

where the $\boldsymbol{e}_\beta$ combine to form a nonholonomic system of frames corresponding to the "implantation" by $\boldsymbol{K}_\alpha$.

The connection turns the manifold $\mathfrak{B}_\alpha$ into a Cartan space (space with absolute parallelism, teleparallel space [17]) (Fig.2). Indeed, using the operators Γ_α, we can define a rule of parallel transport on $\mathfrak{B}_\alpha$. The implant field $\boldsymbol{K}_\alpha$ defines arbitrary affine transformations of the natural frame ∂_ν at all points of $\mathfrak{B}$, and so the frame field $\boldsymbol{e}_\beta$ becomes nonholonomic. The rule of parallel transport can be stated as follows. A vector is transported parallelly if its projections onto the frames $\boldsymbol{e}_\beta$ remain invariant. Since the field $\boldsymbol{K}_\alpha$ defines the inhomogeneity, we see that this situation has a vivid physical interpretation; i.e., and observer traveling with the moving frame sees no inhomogeneities, just as a *geodesic observer* does not "feel" any gravitation field in general relativity; e.g., see [14]. Actually, one can represent the local shapes as a continuous family of stressed reference shapes (Fig.3).

We have already pointed out that the response functional $\mathfrak{S}$ does not depend on α explicitly. Formally, this means that, for every $\alpha < \beta \in I$, the response of the body $\mathfrak{B}_\alpha$ at a fixed interior material point $\mathfrak{x} \in \mathfrak{B}_\alpha$ is equal to that of its successor $\mathfrak{B}_\beta$. From the physical viewpoint, this means that the properties of already accreted material do not change as the growth process continues. The "implantation" parameters corresponding to a material point are completely determined at the accretion time and remain unchanged ever after. In other words, the inhomogeneity arises owing to the growth process on the boundary and does not develop further in the

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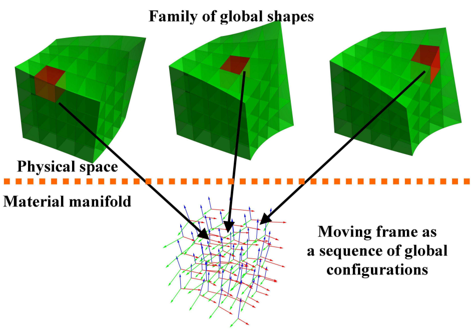

Fig. 3. Representation for Local Shapes as a Family of Reference Shapes

bulk. In terms of [15], this means that remodeling does not occur in the bulk. Thus, we exclude processes like plasticity or shrinkage from consideration. We shall refer to growing bodies under such condition as *nonrearranged*:

A layerwise growing body is *nonrearranged* if

$$\forall \alpha < \beta \in I \quad \forall \boldsymbol{x} \in \mathfrak{B}_\alpha \quad \mathfrak{T}_\beta(\boldsymbol{x}) = \mathfrak{T}_\alpha(\boldsymbol{x}). \tag{6}$$

Note that the property of nonrearrangement in the bulk is typical of surface growth models as opposed to so-called volume growth models [15].

It is not difficult to take into account the natural geometric definition of Cartan connection and its torsion $\mathfrak{T}_\alpha$, but this can require some additional calculations. Thus, in many cases it is preferable to deal directly with the implant field $\boldsymbol{K}_\alpha$ inducing certain type of connection. To this end, we can choose a *hereditary* family of reference shapes, i.e., a family that satisfies the following relation:

$$\forall \alpha < \beta \in I \quad \mathcal{B}_\alpha = \varkappa^R_{\beta|\alpha} \mathfrak{B}_\alpha,$$

where $\varkappa^R_{\beta|\alpha}$ is the restriction of the mapping $\varkappa^R_\beta$ to the domain $\mathfrak{B}_\alpha \subset \mathfrak{B}_\beta$. This means that the family of reference shapes can be treated as a sequence of continuations of the set $\mathcal{B}^R_a$ to the set $\mathcal{B}^R_b$. From the mechanical viewpoint, this means that the shape $\mathcal{B}^R_a$ continuously increases by the addition of a flux of material surfaces to its boundary, while the state of already adhered material particles remains unchanged. Obviously. these shapes are not stress free and can be in equilibrium only under special external fields of bulk forces and surface forces on the boundary (Eshelby forces). In this case, the condition (6) can be represented in terms of the implant $\boldsymbol{K}_\alpha$ as follows:

$$\forall \alpha < \beta \in I \quad \forall \boldsymbol{x} \in \mathfrak{B}_\alpha \quad \boldsymbol{H}_\alpha(\boldsymbol{x}) = \boldsymbol{H}_\beta(\boldsymbol{x}).$$

Furthermore, if the family $\mathfrak{C}$ (1) admits differentiation with respect to the parameter α, then condition (6) can be rewritten in terms of the field equation,

$$\dot{\boldsymbol{K}} = \boldsymbol{0}, \tag{7}$$

where the symbol $\boldsymbol{K}$ denotes the mapping $\boldsymbol{K} : I \ni \alpha \mapsto \boldsymbol{K}_\alpha$ and the dot stands for differentiation with respect to the parameter α.

IV. Boundary Value Problem in the Hyperelastic Case

Let us introduce the elastic potential $W^{\varkappa_R}_{\varkappa}$, that is, the elastic energy per unit volume in the reference state $\varkappa_R$, which can be interpreted as a function of three arguments $\boldsymbol{F}$, $\boldsymbol{K}$, and $\mathfrak{X}$ [14],

$$W^{\varkappa_R}_{\varkappa}(\boldsymbol{K}, \boldsymbol{F}, \mathfrak{X}) = J^{-1}_{\boldsymbol{K}} W^{\varkappa_C}_{\varkappa}(\boldsymbol{H}, \mathfrak{X}) = J^{-1}_{\boldsymbol{K}} W^{\varkappa_C}_{\varkappa}(\boldsymbol{K} \boldsymbol{F}, \mathfrak{X}).$$

One can express the Piola stress tensor $\boldsymbol{T}^{\varkappa_R}_{\varkappa}$ corresponding to $\varkappa_R$ by the formula

$$\boldsymbol{T}^{\varkappa_R}_{\varkappa} = \frac{\partial W^{\varkappa_R}_{\varkappa}}{\partial \boldsymbol{F}^*} = J^{-1}_{\boldsymbol{K}} \boldsymbol{K}^* \cdot \frac{\partial W^{\varkappa_C}_{\varkappa}}{\partial \boldsymbol{H}^*}.$$

The stress tensor $\boldsymbol{T}^{\varkappa_C}_{\varkappa}$ corresponding to $\varkappa_C$ can be defined fiberwise as follows:

$$\boldsymbol{T}^{\varkappa_C}_{\varkappa} = \frac{\partial W^{\varkappa_C}_{\varkappa}}{\partial \boldsymbol{H}^*}.$$

The boundary value problem for an accreted solid is determined by the equations of equilibrium in $V(t)$ with boundary $\Omega(t)$ whose parametrically depends on time,

$$\boldsymbol{\nabla}_{\varkappa_R} \cdot \left[J^{-1}_{\boldsymbol{K}} \boldsymbol{K}^* \cdot \frac{\partial W^{\varkappa_C}_{\varkappa}(\boldsymbol{H}, \mathfrak{X})}{\partial \boldsymbol{H}^*} \bigg|_{\boldsymbol{H}=\boldsymbol{K}\cdot\boldsymbol{F}} \right] + \boldsymbol{b} = \boldsymbol{0}, \tag{8}$$

and the boundary conditions on $\Omega(t)$,

$$\boldsymbol{n}_{\varkappa_R} \cdot \left[J^{-1}_{\boldsymbol{K}} \boldsymbol{K}^* \cdot \frac{\partial W^{\varkappa_C}_{\varkappa}(\boldsymbol{H}, \mathfrak{X})}{\partial \boldsymbol{H}^*} \bigg|_{\boldsymbol{H}=\boldsymbol{K}\cdot\boldsymbol{F}} \right] \Bigg|_{\Omega(t)} = \boldsymbol{p}.$$

At first glance, the formal statement of the boundary value problem differs from the classical one only in that the boundary of the domain depends parametrically on time. However, there is a more profound difference: the elastic potential depends on the distortion tensor field, whose determination requires additional conditions. The particular form of these conditions depends on the geometric structure of the adhering elements, that is, essentially, on the structure of the bundle of material manifolds. If the growth of a body is due to a continuous influx of prestressed material surfaces to this body, then this condition can be written in the form

$$\boldsymbol{P}_{\varkappa_R} \cdot \left[J^{-1}_{\boldsymbol{K}} \boldsymbol{K}^* \cdot \frac{\partial W^{\varkappa_C}_{\varkappa}(\boldsymbol{H}, \mathfrak{X})}{\partial \boldsymbol{H}^*} \bigg|_{\boldsymbol{H}=\boldsymbol{K}\cdot\boldsymbol{F}} \right] \cdot \boldsymbol{P}_{\varkappa_R} \Bigg|_{\Omega(t)} = \boldsymbol{\mathcal{T}}.$$

Here $\boldsymbol{P} = (\boldsymbol{E} - \boldsymbol{n} \otimes \boldsymbol{n})$ is the projection onto the tangent plane to $\Omega(t)$. This equation expresses the fact that the fibers align with the specified tension determined by the surface tensor $\boldsymbol{\mathcal{T}}$, i.e., two-dimensional tensor of second rank defined on the tangent space to the adhering material surface.

If the growth results from continuous adherence of prestressed surfaces, then the equation for the distortion tensor $\boldsymbol{K}$ can be obtained from the relations of the theory of material surfaces (theory of solids with material boundary [16]). The effect of material surface adhering leads to an infinitesimal change in the stress–strain state of the accreted solid, but since the elementary adhesion act occurs on an infinitesimal time interval, the stress rate proves to be finite. This rate can be found from the equations of contact interaction between the 3D body and the adhering material surface. The equilibrium equation for the physical boundary (which is a bounding surface of the body in its actual state from the geometric viewpoint and a thin film in a membrane stress state from the mechanical viewpoint) can be written as $\boldsymbol{\nabla}_s \cdot \boldsymbol{\mathcal{T}} + \boldsymbol{b}_s = \boldsymbol{n}_{\varkappa_R} \cdot \boldsymbol{T}^{\varkappa_R}_{\varkappa} \Big|_{\Omega(t)}$, where $\boldsymbol{\nabla}_s$ is the surface nabla operator and $\boldsymbol{b}_s$ is the surface density of external forces acting on $\Omega(t)$. To complete the statement of the boundary

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value problem, we should pose a condition on the curvilinear boundary $\partial\Omega(t)$ of the surface $\Omega(t)$. This condition can be of the form $\widetilde{\boldsymbol{n}}\cdot\boldsymbol{T}\Big|_{\partial\Omega(t)} = \widetilde{\boldsymbol{f}}$. Here $\widetilde{\boldsymbol{n}}$ is the external unit normal to the curve $\partial\Omega(t)$ in the tangent plane and $\widetilde{\boldsymbol{f}}$ is a linear density of forces distributed on $\partial\Omega(t)$.

V. Note on the representations of stresses

In classical linear algebra, associated with a vector space is the dual space, i.e., the space of linear functionals (covectors). The distinction between vectors and covectors is commonly ignored in continuum mechanics. However, a superstructure on a vector space in the form of a linear functional space seems more natural from the physical viewpoint. If all the variables of vector nature are defined in the same vector space, which is usually equipped with an inner product, then there is a quite justified wish to treat the velocity vector and the force vector as elements of the same vector space, whose inner product is a scalar, i.e., power, exerted by the force on the velocity. On the other hand, taking into account their equivalence as elements of the same vector space one can formally consider their sum which is deprived of any physical meaning. The situation is fixed by the argument that the force should be a covector and the velocity should be a vector. The covector representation of the force treat it as a linear functional whose action on the velocity vector gives a scalar, i.e., power. Of course, the sum of a vector and a co-vector is not defined.

Following [18], we use the statement of continuum mechanics in terms of vector- and covector-valued forms (exterior forms), which are taken as replacements for the standard stresses. This statement seems to be more natural from the geometric viewpoint. It permits one to present the concept of stresses on a manifold with non-Euclidean connection.

We take an approach to stresses that treats them as covector-valued two-forms and regard them as fundamental variables,

$$\mathcal{T} \,:\, \mathcal{B}_\alpha \to T\mathcal{B}\otimes\Omega^2, \quad \mathcal{T} = *_2\boldsymbol{\sigma} = \sigma_{i\cdot}^{\cdot j}\boldsymbol{e}^i\otimes(*\boldsymbol{e}_j),$$

$$\mathcal{P} \,:\, \mathcal{B}_\alpha^R \to T\mathcal{B}\otimes\Omega^2, \quad \mathcal{P} = *_2\boldsymbol{P} = p_{i\cdot}^{\cdot j}\boldsymbol{e}^i\otimes(*\boldsymbol{e}_j).$$

Here $*$ denotes the Hodge star operator.

Physically $\mathcal{T}$ and $\mathcal{P}$ can be interpreted as follows. The stress upon pairing with a velocity field provides an area-form that is ready to be integrated over a surface to give the rate of work done by the stress on that surface [18]. The above-mentioned considerations remain valid on a manifold with non-Euclidean connection [19]. All one needs to define stress covector-valued forms in such cases is a volume 3-form.

Equations (8) can also be transformed to local form by the Cartan calculus machinery. Consider the Cartan exterior derivative

$$\mathfrak{d} \,:\, T^*\mathcal{E}\otimes\Omega^{k-1} \to T^*\mathcal{E}\otimes\Omega^k; \mathcal{T}\mapsto\mathfrak{d}\mathcal{T}$$

$$\forall\boldsymbol{u} \quad \boldsymbol{u}\mathfrak{d}\mathcal{T} = d(\boldsymbol{u}\mathcal{T}) - \nabla\boldsymbol{u}\dot{\wedge}\mathcal{T},$$

where $\dot{\wedge}$ is, by definition, a pairing on the first elements of dyadic decomposition and a wedge product on the rest. Note that for $k = 0$, $\mathfrak{d}$ is reduced to the regular covariant derivative, while for $k = 3$, $\mathfrak{d}$ is identically zero.

Since the pullback of forms commutes with the exterior derivative, it is possible to use $\mathfrak{d}$ in the definition of a derivative $\mathfrak{D}$ on elements of the space $T^*\mathcal{E}\otimes\Omega^{k-1}$ such that the following diagram commutes [18]

$$\begin{array}{ccc} T^*\mathcal{E}\otimes\Omega^{k-1}(\mathfrak{B}^R) & \xleftarrow{\varphi^{*2}} & T^*\mathcal{E}\otimes\Omega^{k-1}(\mathfrak{B}) \\ \downarrow\mathfrak{D} & & \downarrow\mathfrak{d} \\ T^*\mathcal{E}\otimes\Omega^{k}(\mathfrak{B}^R) & \xleftarrow[\varphi^{*2}]{} & T^*\mathcal{E}\otimes\Omega^{k}(\mathfrak{B}) \end{array}$$

In terms of exterior calculus, the balance equations appear in the following form:

$$\mathfrak{d}\mathcal{T} + \boldsymbol{G}\otimes\varepsilon = \boldsymbol{0}, \quad \mathfrak{D}\mathcal{P} + \boldsymbol{g}\otimes\varepsilon = \boldsymbol{0}.$$

Here ε is the volume form corresponding to a certain parametrization of physical space. We again point out that the above-mention considerations remain valid on a manifold with a non-Euclidean connection. All one needs to define stress covector-valued forms in such cases is a volume 3-form.

References

[1] K. L. Choy, "Chemical vapour deposition of coatings," *Progress in Materials Science*, vol. 48, pp. 57–170, 2003.
[2] M. Nastasi and J. W. Mayer, *Ion Implantation and Synthesis of Materials*. Springer, 2010.
[3] E. Kröner, "Allgemeine kontinuumstheorie der versetzungen und eigenspannungen," *Arch. Rat. Mech. Anal.*, vol. 4, no. 1, pp. 273–334, 1967.
[4] W. A. Harrison, *Solid State Theory*. New York: Dover, 1980.
[5] C. Truesdell, *A First Course in Rational Continuum Mechanics: General concepts*. USA: Academic Press, 1991.
[6] N. K. Arutyunyan, A. D. Drozdov, and V. E. Naumov, *Mechanics of Growing Viscoelastoplastic Solids*. Moscow: Nauka, 1987, [in Russian].
[7] N. K. Aruyunyan, A. V. Manzhirov, and V. E. Naumov, *Contact Problems in Mechanics of Growing Solids*. Moscow: Nauka, 1991, [in Russian].
[8] A. V. Manzhirov and S. A. Lychev, "Mathematical theory of growing solids. Finite deformation," *Doklady Physics*, vol. 57, no. 4, pp. 160–163, 2012.
[9] M. Epstein and G. A. Maugin, "Thermomechanics of volumetric growth in uniform bodies," *Int. J. of Plast.*, vol. 16, no. 7, pp. 951–978, 2000.
[10] A. Yavari, "A geometric theory of growth mechanics," *J. of Nonlin. Sci.*, vol. 20, no. 6, pp. 781–830, 2010.
[11] A. V. Manzhirov and S. A. Lychev, "Residual stresses in growing bodies," in *Topical Problems in Solid and Fluid Mechanics*. Delhi: Elite Pub. House, 2011, pp. 66–79.
[12] W. Noll, "Materially uniform simple bodies with inhomogeneities," *1967*, vol. 27, no. 1, pp. 1–32, Arch. Rat. Mech. Anal.
[13] C. C. Wang, "On the geometric structure of simple bodies, or mathematical foundations for the theory of continuous distributions of dislocations," *Arch. Rat. Mech. Anal.*, vol. 27, no. 1, pp. 33–94, 1967.
[14] G. A. Maugin, *Material inhomogeneities in elasticity*. London: Chapman and Hall, 1993.
[15] M. Epstein, *The Geometrical Language of Continuum Mechanics*. Cambridge University Press, 2010.
[16] M. E. Gurtin and A. I. Murdoch, "A continuum theory of elastic material surfaces," *Arch. Rational Mech. Anal.*, vol. 27, pp. 291–323, 1975.
[17] Y. Choquet-Bruhat, C. Dewitt-Morette, and M. Dillard-Bleick, *Analysis, Manifolds and Physics. Part 1. Basics*. Amsterdam: North-Holland, 1982.
[18] E. Kanso, M. Arroyo, Y. Tong, A. Yavari, J. G. Marsden, and M. Desbrun, "On the geometric character of stress in continuum mechanics," *Z. Angew. Math. Phys.*, vol. 58, pp. 843–856, 2007.
[19] J. E. Marsden and T. J. R. Hughes, *Mathematical Foundations of Elasticity*. Dover Publications, 1994.

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