Valuing Multi-asset Spread Options by the Lie-Trotter Operator Splitting Method

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Abstract — **In this paper, by means of the Lie-Trotter operator splitting method, we have presented a new unified approach not only to rigorously derive Kirk's approximation but also to obtain a generalisation for multi-asset spread options straightforwardly. The derived price formula for the multi-asset spread option bears a great resemblance to Kirk's approximation in the two-asset case. More importantly, our approach is able to provide a new perspective on Kirk's approximation and the generalisation; that is, they are simply equivalent to the Lie-Trotter operator splitting approximation to the Black-Scholes equation.**

Keywords: Spread options; Kirk's approximation; Lie-Trotter operator splitting method

1. Introduction

Despite the rapid development of spread options whose payoff is contingent upon the price difference of two underlying assets, pricing spread options is a very challenging task and receives much attention in the literature.[1] The major difficulty lies in the lack of knowledge about the distribution of the spread of two correlated lognormal random variables. The simplest approach is to evaluate the expectation of the final payoff over the joint probability distribution of the two correlated lognormal underlyings by means of numerical integration. However, practitioners often prefer to use analytical approximations rather than numerical methods because of their computational ease. Among various analytical approximations, *e.g.* Carmona and Durrleman (2003), Deng et al. (2008), Bjerksund and Stensland (2011), Venkatramana and Alexander (2011),[1-4] Kirk's approximation seems to be the most widely used and is the current market standard.[5] It is well known that Kirk's approximation extends from Margrabe's exchange option formula with no rigorous derivation.[6] Recently, Lo (2013) applied the idea of WKB method to provide a derivation of Kirk's approximation and discuss its validity.[7] Nevertheless, it is not straightforward to provide a generalisation of Kirk's approximation for the case of multi-asset spread option via this approach.

Accordingly, it is the aim of this paper to present a simple unified approach, namely the Lie-Trotter operator splitting method,[8] not only to rigorously derive Kirk's approximation but also to obtain a generalisation for the case of multi-asset spread option in a straightforward manner. The derived price formula for the multi-asset spread option bears a great resemblance to Kirk's approximation in the two-asset case. More importantly, the proposed approach is able to provide a new perspective on Kirk's approximation and the generalisation; that is, they are simply equivalent to the Lie-Trotter operator splitting approximation to the Black-Scholes equation. Illustrative numerical examples for the three-asset spread options are also shown to demonstrate both the accuracy and efficiency of our generalised Kirk approximation.

2. Multi-asset spread options

To price a European N-asset call spread option, we need to solve the N-dimensional Black-Scholes equation

$$\left\{\hat{L}_{BS}-\frac{\partial}{\partial\tau}\right\}P\left(\{F_i\},\tau\right)=0 \qquad (1)$$

with

$$\hat{L}_{BS}=\sum_{i,j=1}^{N}\frac{1}{2}\rho_{ij}\sigma_i\sigma_jF_iF_j\frac{\partial^2}{\partial F_i\partial F_j}-r\ , \qquad (2)$$

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subject to the final payoff condition

$$P\left(\{F_i\},0\right)=\max\left(F_N-\sum_{j=1}^{N-1}F_j-K,0\right)\ ,\qquad(3)$$

where F_i is the future price of the lognormal underlying asset i with the volatility σ_i, ρ_{ij} is the correlation between the assets i and j, K is the strike price, r is the risk-free interest rate, and τ is the time-to-maturity. Unfortunately, this is a very formidable task. In the following we apply the Lie-Trotter operator splitting method to derive a closed-form approximate price formula for the multi-asset spread option, which bears a great resemblance to the price formula of Kirk's approximation in the two-asset case.

***Proposition* :**

The price of the N-asset spread option can be approximated by

$$\begin{aligned}&P\left(\{F_i\},\tau\right)\\ \approx\ &\left\{F_N N\left(\theta_1\right)-\left(F_++K\right)N\left(\theta_2\right)\right\}e^{-r\tau}\end{aligned}\qquad(4)$$

where

$$\begin{aligned}F_+&=\sum_{j=1}^{N-1}F_j&(5)\\ \theta_1&=\frac{1}{\bar\sigma_-\sqrt{\tau}}\left\{\ln\left(\frac{F_N}{F_++K}\right)+\frac{1}{2}\bar\sigma_-^2\tau\right\}&(6)\\ \theta_2&=\theta_1-\bar\sigma_-\sqrt{\tau}&(7)\\ \bar\sigma_-&=\sqrt{\sigma_N^2-2\tilde\rho\sigma_N\bar\sigma_++\bar\sigma_+^2}&(8)\\ \bar\sigma_+&=\tilde\sigma_+\left(\frac{F_+}{F_++K}\right)&(9)\\ \tilde\sigma_+&=\frac{\sqrt{\sum_{j,k=1}^{N-1}\rho_{jk}\sigma_j\sigma_kF_jF_k}}{F_+}&(10)\\ \tilde\rho&=\frac{1}{\tilde\sigma_+}\left(\frac{\sum_{j=1}^{N-1}\rho_{jN}\sigma_jF_j}{F_+}\right)\ .&(11)\end{aligned}$$

***Proof* :**

Introducing the new variables: $f_i\equiv F_i-F_+$ for $i=1,2,\ldots,N-2$ and $F_+\equiv\sum_{j=1}^{N-1}F_j$, Eq.(1) can be cast in the form

$$\begin{aligned}&\left\{\hat L_N+\hat L_++\hat L_{N+}+\hat L_R-r\right\}P\left(\{F_i\},\tau\right)\\ =\ &\frac{\partial P\left(\{F_i\},\tau\right)}{\partial\tau}\ ,\end{aligned}\qquad(12)$$

where

$$\begin{aligned}\hat L_N&=\frac{1}{2}\sigma_N^2F_N^2\frac{\partial^2}{\partial F_N^2}&(13)\\ \hat L_+&=\frac{1}{2}\tilde\sigma_+^2F_+^2\frac{\partial^2}{\partial F_+^2}&(14)\\ \hat L_{N+}&=\tilde\rho\sigma_N\tilde\sigma_+F_+F_N\frac{\partial^2}{\partial F_+\partial F_N}&(15)\end{aligned}$$

and

$$\begin{aligned}&\hat L_R\\ =\ &\frac{1}{2}\sum_{k=1}^{N-2}\sum_{n=1}^{N-2}\left\{\left(\rho_{kn}\sigma_kF_k-2\sum_{m=1}^{N-1}\rho_{mn}\sigma_mF_m\right)\times\right.\\ &\left.\sigma_nF_n+\sum_{p=1}^{N-1}\sum_{q=1}^{N-1}\rho_{pq}\sigma_p\sigma_qF_pF_q\right\}\frac{\partial^2}{\partial f_n\partial f_k}+\\ &\frac{1}{2}\sum_{k=1}^{N-2}\sum_{n=1}^{N-1}\left(\rho_{kn}\sigma_kF_k-\sum_{m=1}^{N-1}\rho_{mn}\sigma_mF_m\right)\times\\ &\sigma_nF_n\frac{\partial^2}{\partial F_+\partial f_k}\ .\end{aligned}\qquad(16)$$

The formal solution of Eq.(12) is given by

$$\begin{aligned}&P\left(\{F_i\},\tau\right)\\ =\ &\exp\left\{\tau\left(\hat L_N+\hat L_++\hat L_{N+}+\hat L_R-r\right)\right\}\times\\ &\max\left(F_N-F_+-K,0\right)\ .\end{aligned}\qquad(17)$$

Then we apply the Lie-Trotter operator splitting method to obtain an approximation to the formal solution $P\left(\{F_i\},\tau\right)$,[8] namely (see the Appendix)

$$\begin{aligned}&P^{LT}\left(\{F_i\},\tau\right)\\ =\ &\exp\left\{\tau\left(\hat L_N+\hat L_++\hat L_{N+}-r\right)\right\}\times\\ &\exp\left\{\tau\hat L_R\right\}\max\left(F_N-F_+-K,0\right)\\ =\ &e^{-r\tau}\exp\left\{\tau\left(\hat L_N+\hat L_++\hat L_{N+}\right)\right\}\times\\ &\max\left(F_N-F_+-K,0\right)\ ,\end{aligned}\qquad(18)$$

where the relation

$$\hat L_R\max\left(F_N-F_+-K,0\right)=0\ .\qquad(19)$$

is utilized.

Next, in terms of the two new variables

$$R_1=\frac{F_N}{F_++K}\quad\text{and}\quad R_2=F_++K\ ,\qquad(20)$$

we rewrite $P^{LT}\left(\{F_i\},\tau\right)$ as follows:

$$\begin{aligned}P^{LT}\left(\{F_i\},\tau\right)&=e^{-r\tau}\exp\left\{\tau\left(\hat L_A+\hat L_B\right)\right\}\times\\ &\quad R_2\max\left(R_1-1,0\right)\ ,\end{aligned}\qquad(21)$$

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where

$$\hat{L}_A = \frac{1}{2}\bar{\sigma}_-^2 R_1^2 \frac{\partial^2}{\partial R_1^2} \quad (22)$$

$$\hat{L}_B = \frac{1}{2}\bar{\sigma}_+^2 R_2^2 \frac{\partial^2}{\partial R_2^2} - (\tilde{\rho}\sigma_N - \bar{\sigma}_+)\bar{\sigma}_+ R_1 \frac{\partial}{\partial R_1} + (\tilde{\rho}\sigma_N - \bar{\sigma}_+)\bar{\sigma}_+ R_1 R_2 \frac{\partial^2}{\partial R_1 \partial R_2} \quad (23)$$

$$\bar{\sigma}_- = \sqrt{\sigma_N^2 - 2\tilde{\rho}\sigma_N\bar{\sigma}_+ + \bar{\sigma}_+^2} \quad (24)$$

$$\bar{\sigma}_+ = \tilde{\sigma}_+ \left(\frac{F_+}{F_+ + K}\right) . \quad (25)$$

It is clear that the exponential operator $\exp\left\{\tau\left(\hat{L}_A + \hat{L}_B\right)\right\}$ is difficult to evaluate, so we need to apply the Lie-Trotter operator splitting method again to approximate the operator by $\exp\left\{\tau\hat{L}_A\right\}\exp\left\{\tau\hat{L}_B\right\}$.[8] As a result, we obtain

$$\begin{aligned} & P^{LT}(\{F_i\},\tau) \\ \approx\ & e^{-r\tau}\exp\left\{\tau\hat{L}_A\right\}\exp\left\{\tau\hat{L}_B\right\} R_2 \max(R_1 - 1, 0) \\ \approx\ & e^{-r\tau} R_2 \exp\left\{\tau\hat{L}_A\right\}\max(R_1 - 1, 0) \quad (26) \end{aligned}$$

for

$$\hat{L}_B R_2 \max(R_1 - 1, 0) = 0 . \quad (27)$$

Since $\bar{\sigma}_-$ is independent of R_1, the standard workhorse of the Black-Scholes model can be used to evaluate $\exp\left\{\tau\hat{L}_A\right\}\max(R_1 - 1, 0)$ such that

$$P^{LT}(\{F_i\},\tau) \approx e^{-r\tau} R_2 \{R_1 N(\theta_1) - N(\theta_2)\} \quad (28)$$

where

$$\theta_1 = \frac{1}{\bar{\sigma}_-\sqrt{\tau}}\left\{\ln\left(\frac{F_N}{F_+ + K}\right) + \frac{1}{2}\bar{\sigma}_-^2\tau\right\} \quad (29)$$

$$\theta_2 = \theta_1 - \bar{\sigma}_-\sqrt{\tau} . \quad (30)$$

It is obvious that this approximate solution is identical to the approximate price formula given in Eq.(4). (Q.E.D.)

In terms of the spot asset prices, namely $S_i \equiv F_i \exp(-r\tau)$, the price formula given in Eq.(4) becomes

$$\begin{aligned} & P_{Kirk}(\{S_i\},\tau) \\ =\ & S_N N(d_1) - \left(\sum_{j=1}^{N-1} S_j + Ke^{-r\tau}\right) N(d_2) \quad (31) \end{aligned}$$

where

$$d_1 = \frac{\ln S_N - \ln\left(\sum_{j=1}^{N-1} S_j + Ke^{-r\tau}\right)}{\hat{\sigma}_-\sqrt{\tau}} + \frac{1}{2}\hat{\sigma}_-\sqrt{\tau} \quad (32)$$

$$d_2 = d_1 - \hat{\sigma}_-\sqrt{\tau} \quad (33)$$

$$\hat{\sigma}_- = \sqrt{\sigma_N^2 - 2\bar{\rho}\sigma_N\hat{\sigma}_+ + \hat{\sigma}_+^2} \quad (34)$$

$$\hat{\sigma}_+ = \frac{\sqrt{\sum_{j,k=1}^{N-1}\rho_{jk}\sigma_j\sigma_k S_j S_k}}{\sum_{j=1}^{N-1} S_j + Ke^{-r\tau}} \quad (35)$$

$$\bar{\rho} = \frac{1}{\hat{\sigma}_+}\left(\frac{\sum_{j=1}^{N-1}\rho_{jN}\sigma_j S_j}{\sum_{j=1}^{N-1} S_j}\right) . \quad (36)$$

Obviously, this approximate price formula resembles the price formula of Kirk's approximation in the two-asset case very closely. In fact, by setting $N = 2$ we can recover Kirk's approximation readily. Moreover, for the Lie-Trotter splitting approximation to be valid, we need to require $\hat{\sigma}_-^2\tau$ to be sufficiently small, namely $\bar{\sigma}_-^2\tau \ll 1$.

3. Illustrative numerical examples

In this section illustrative numerical examples are presented to demonstrate the accuracy of our generalised Kirk approximation for the three-asset spread options. We examine a simple three-asset spread option with the final payoff $\max(S_3 - S_1 - S_2 - K, 0)$. Table I tabulates the approximate option prices estimated by our generalised Kirk approximation for different values of the strike price K and time-to-maturity T. Other input model parameters are set as follows: $r = 0.05$, $\sigma_1 = \sigma_2 = \sigma_3 = 0.3$, $\rho_{12} = 0.4$, $\rho_{23} = 0.2$, $\rho_{13} = 0.8$, $S_1 = 50$, $S_2 = 60$ and $S_3 = 150$. Monte Carlo estimates and the corresponding standard deviations are also presented for comparison. It is observed that the computed errors of the approximate option prices are capped at 1% (in magnitude). In fact, most of them are less than 0.5%. The effect of increasing the three volatilities (from 0.3 to 0.6) upon the approximate estimation of the option prices is also investigated. Only a slight increase occurs in the computed errors, and these errors are still less than 1% (in magnitude). As a result, it can be concluded that our generalised Kirk approximation for the three-asset spread option is found to be very accurate and efficient.

4. Conclusion

By means of the Lie-Trotter operator splitting method, we have presented a new unified approach

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not only to rigorously derive Kirk's approximation but also to obtain a generalisation for multi-asset spread options in a straightforward manner. The derived price formula for the multi-asset spread option bears a great resemblance to Kirk's approximation in the two-asset case. In fact, by setting the total number of assets to be two, we can recover Kirk's approximation readily. Thus, our generalisation possesses the same nice features as Kirk's approximation. Moreover, our approach is able to provide a new perspective on Kirk's approximation and the generalisation; that is, they are simply equivalent to the Lie-Trotter operator splitting approximation to the Black-Scholes equation.

Table I: Prices of a European three-asset call spread option. Other input parameters are: $r = 0.05$, $\sigma_1 = \sigma_2 = \sigma_3 = 0.3$, $\rho_{12} = 0.4$, $\rho_{23} = 0.2$, $\rho_{13} = 0.8$, $S_1 = 50$, $S_2 = 60$ and $S_3 = 150$. Here "GK" refers to our generalised Kirk approximation while "MC" denotes the Monte Carlo estimates with $900,000,000$ replications. The relative errors of the "GK" option prices with respect to the "MC" estimates are also presented.

$K \backslash T$	0.25	0.5	1	2	
30	13.5410	16.4210	20.7761	27.1974	EK
	−0.3%	−0.3%	−0.3%	−0.3%	error
	13.5763	16.4735	20.8471	27.2841	MC
	0.0089	0.0142	0.0185	0.0264	SD
35	10.3383	13.5024	18.1191	24.8196	EK
	−0.2%	−0.2%	−0.2%	−0.2%	error
	10.3576	13.5286	18.1541	24.8573	MC
	0.0086	0.0124	0.0176	0.0276	SD
40	7.6613	10.9586	15.7231	22.6176	EK
	0.0%	0.0%	0.0%	0.1%	error
	7.6608	10.9572	15.7197	22.6072	MC
	0.0068	0.0109	0.0161	0.0244	SD
45	5.5097	8.7824	13.5805	20.5856	EK
	0.4%	0.3%	0.3%	0.3%	error
	5.4903	8.7565	13.5421	20.5302	MC
	0.0051	0.0106	0.0141	0.0260	SD
50	3.8470	6.9540	11.6795	18.7162	EK
	0.9%	0.7%	0.6%	0.5%	error
	3.8140	6.9058	11.6109	18.6195	MC
	0.0040	0.0086	0.0136	0.0245	SD

Appendix:

Suppose that one needs to exponentiate an operator $\hat{C}$ which can be split into two different parts, namely $\hat{A}$ and $\hat{B}$. For simplicity, let us assume that $\hat{C} = \hat{A} + \hat{B}$, where the exponential operator $\exp\left(\hat{C}\right)$ is difficult to evaluate but $\exp\left(\hat{A}\right)$ and $\exp\left(\hat{B}\right)$ are either solvable or easy to deal with. Under such circumstances the exponential operator $\exp\left(\varepsilon\hat{C}\right)$, with ε being a small parameter, can be approximated by the Lie-Trotter operator splitting formula:

$$\exp\left(\varepsilon\hat{C}\right) = \exp\left(\varepsilon\hat{A}\right)\exp\left(\varepsilon\hat{B}\right) + \mathcal{O}\left(\varepsilon^2\right) \quad . \quad \text{(A.1)}$$

This can be seen as the approximation to the solution at $t = \varepsilon$ of the equation $d\hat{Y}/dt = \left(\hat{A} + \hat{B}\right)\hat{Y}$ by a composition of the exact solutions of the equations $d\hat{Y}/dt = \hat{A}\hat{Y}$ and $d\hat{Y}/dt = \hat{B}\hat{Y}$ at time $t = \varepsilon$. Details of the Lie-Trotter splitting approximation can be found in the references [8-13]. The Lie-Trotter splitting approximation is particularly useful for studying the short-time behaviour of the solutions of evolutionary partial differential equations of parabolic type because for this class of problems it is sensible to split the spatial differential operator into several parts each of which corresponds to a different physical contribution (e.g., reaction and diffusion).

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