Proceedings of the World Congress on Engineering 2015 Vol I
WCE 2015, July 1 - 3, 2015, London, U.K.

Numerical Solutions of a Class of Nonlinear Volterra Integral Equation

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***Abstract*—We consider numerical solutions of a class of nonlinear (nonstandard) Volterra integral equations. We first prove the existence and uniqueness of the solution of the Volterra integral equation in the context of the space of continuous funtions over a closed interval. We then use one point collocation methods and quadrature methods with a uniform mesh to construct solutions of the nonlinear VIE. We conclude that the repeated Simpson's rule gives better solutions when a reasonably large value of the stepsize is used.**

***Index Terms*—Nonstandard Volterra integral equations, collocation methods, quadrature methods.**

I. Introduction

In this paper we study the nonlinear (nonstandard) Volterra integral equation of the second kind of the form

$$u(t) = \sum_{j=1}^{r} b_j \left(g_j(t) + \int_0^t k_j(t,s)u(s)\,ds \right)^j, t \in [0,T], \quad (1)$$

where ($r \in \mathbb{N}, r \geq 2$), with $b_j \in \mathbb{R}$ and g_j, k_j are continuous functions.

In its closed form $u = Vu$, (1) is nonstandard in that the defining operator V has the structure

$$Vu = \sum_{j=1}^{r} b_j (g_j + V_j u)^j,$$

where $V_j u$ is the standard linear Volterra operator.

Our primary interest is in the numerical solutions of (1). These are Collocation methods and quadrature methods. To this end, we make thefollowing fundamaental considerations:

1) Well-posedness of the problem;
2) Construction of the algorithms;
3) Study of accuracy and convergence.

Volterra integral equations play an important part in scientific and engineering problems such as population dynamics, spread of epidemics, semi-conductor devices, wave prograt-ion, superfluidity and travelling wave analysis, Saveljeva [1]. In cases where the kernel is of convolution type ($K(t,s) = K(t-s)$) the solutions to (1) include elliptic functions and natural generalizations of these functions which also have wide applications in the fields of science and engineering [2]. This class of Volterra integral equations was considered by Sloss and Blyth [2] who proved the existence and uniqueness of the solution in the Banach space L^2 and applied the Corrington's Walsh function method to (1).

Manuscript received April 20, 2015. This work was supported in part by the National Research Foundation (NRF) of South Africa.

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Much work has been done in the study of numerical solutions to Volterra integral equations using collocation methods [1], [3], [4], [5], [6], [7]. Benitez and Bolos [8] pointed out that collocation methods have proven to be a very suitable technique for approximating solutions to nonlinear integral equations because of their stability and accuracy. Other authors such as [9], [11], [12], [10] used quadrature rules like repeated trapezoidal and repeated Simpson's rule to solve linear Volterra integral equations. However, collocation methods and quadrature rules have not been used to approximate solutions to (1).

II. Well-posedness of the problem

The following theorem shows that when $r = 2$ and $b_1 = 0$ the integral equation (1) has a unique solution in the space $C[0,d]$. Theorem 2 gives sufficient conditions for the solution to (1) for general r to exist.

Theorem 1 The integral equation

$$z(t) = b\left(g(t) + \int_0^t k(t,s)z(s)\,ds \right)^2 \quad (2)$$

with $g \in C[0,1], b \in \mathbb{R}$, and $k(t,s) \in C([0,1]\times[0,1])$, has a unique solution u and the solution belongs to $I_d = [0,d], 0 < d \leq 1$, with

$$0 < d < \frac{1}{K}\left[\frac{1}{2K \mid b \mid} - \parallel g \parallel_\infty \right] - \parallel u \parallel_\infty \quad (3)$$

where

$$K = \sup_{[0,1]\times[0,1]} \mid k(t,s) \mid .$$

Theorem 2 There exists a solution u of (1), where $u \in C[0,d]$ provided that

$$N_b \sum_{j=1}^{r} j \mid b_j \mid (\parallel g_j \parallel_\infty + K_j d)^{j-1} K_j < 1$$

and

$$\sum_{j=1}^{r} \mid b_j \mid (\parallel g_j \parallel_\infty + K_j d) < d$$

where N_b is the number of nonzero b_j.

III. Numerical Methods

A. Collocation methods

In our work we focus on one point collocation methods (see [13]).

Let $t_n := nh$ $(n = 0,1,...,N-1)$ define a uniform partition for $I = [0,T]$ and set $Z_N := t_0,...,t_N$, $I_0 := [t_0,t_1]$ $I_n := (t_n,t_{n+1}]$ $(1 \leq n \leq N-1)$. The solution to (1) will be approximated by using collocation in the piecewise constant polynomial space $S_0^{-1}(Z_N)$

ISBN: 978-988-19253-4-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

For a given real number c_1, define the set $X_N := t_{n,1}$ of *collocation points* by

$$t_{n,1} = t_n + c_1 h \ (0 \leq c_1 \leq 1, \quad n = 0, ..., N-1). \quad (4)$$

The collocation solution $u_n \in S_0^{-1}(Z_N)$ is defined by the collocation equation

$$u_n(t) = \sum_{j=1}^{r} b_j \left(g_j(t) + \int_0^t k_j(t,s) u(s)\, ds \right)^j, \quad t \in X_N. \quad (5)$$

Since

$$u_n(t) = u_n(t_n + \nu h) = L_1(\nu) U_{n,1}, \quad \nu \in (0,1] \quad (6)$$

where $L_1(\nu) = 1$ and is a *Lagrange fundamental polynomial*.

Thus for $t = t_{n,1} := t_n + c_1 h$ and $0 < c_1 \leq 1$ the collocation equation (5) assumes the form

$$\begin{aligned} u_n(t) &= \sum_{j=1}^{r} b_j \left(g_j(t) + \int_0^{t_n} k_j(t,s) u_i(s) \right. \\ &+ \left. h \int_0^{c_1} k_j(t, t_n + sh) u_n(t_n + sh)\, ds \right)^j \end{aligned}$$

Expressing the collocation equation in terms of the stage values we get

$$\begin{aligned} U_{n,1} &= \sum_{j=1}^{r} b_j \left(g_j(t_{n,1}) + F_{jn}(t_{n,1}) \right. \\ &+ \left. h \left(\int_0^{c_1} k_j(t_{n,1}, t_n + sh)\, ds \right) U_{n,1} \right)^j. \quad (7) \end{aligned}$$

Let $t \in I_n$ and define

$$F_{jn}(t) := \int_0^{t_n} k_j(t,s) u_i(s)\, ds \quad (8)$$

Then

$$F_{jn}(t_{n,1}) = h \sum_{i=0}^{n-1} \left(\int_0^1 k_j(t_{n,1}, t_i + sh)\, ds \right) U_{i,1} \quad (9)$$

The term $F_{jn}(t_{n,1})$ is called the *lag term* corresponding to the collocation solution, [13].

The iterated approximation u^I corresponding to u is defined by

$$u^I(t) = \sum_{j=1}^{r} b_j \left(g_j(t) + \int_0^t k_j(t,s) u(s)\, ds \right)^j \quad t \in I \quad (10)$$

(see [14], [4], [5])

Set $t = t_n \in \bar{Z_N}$ and use (6) we may write (10) in the form

$$\begin{aligned} u^I(t_n) &= \sum_{j=1}^{r} b_j \left(g_j(t_n) \right. \\ &+ \left. h \sum_{i-0}^{n-1} \int_0^1 k_j(t_n, t_i + sh)\, ds U_{i,1} \right)^j \quad (11) \end{aligned}$$

B. Repeated trapezoidal rule

Using the trapezoidal rule we construct the solution to the integral equation (1), (see [12]). Let

$$\begin{cases} t_0 = a, t_n = b \\ t_i = t_0 + ih \quad i = 0, 1, 2, ..., n \\ u(t_i) = \sum_{j=1}^{r} b_j \left(g_j(t_i) + \sum_{l=1}^{i} \int_{t_{l-1}}^{t_i} k_j(t_i, s) u(s)\, ds \right)^j, \\ \quad i = 1, 2, ..., n \end{cases} \quad (12)$$

The approximation of the integral in (12) by repeated trapezoidal rule will result in a discretized system.

C. Repeated Simpson's rule

We use repeated Simpson's rule to construct the solution to the integral equation (1), (see [9]).

If n is even, Simpson's rule rule may be applied to each subinterval $[t_{2i}, t_{2i+1}, t_{2i+2}]$. The approximation of (1) in the even nodes t_{2m} is given by

$$u_{2m} = \sum_{j=1}^{r} b_j \left[g_j(t_{2m}) + \int_a^{t_{2m}} k_j(t_{2m}, s) u(s)\, ds \right]^j, \quad (13)$$

from which a discretized system is obtained.

IV. Numerical Computations

In our work we consider examples of (1) when $r = 2$. We use (7) to approximate the solutions considering two special cases: $c_1 = 1/2$, (implicit midpoint method) and $c_1 = 1$ (implicit Euler method). We also use the repeated trapezoidal and repeated Simpson's rule. Since the methods are implicit we perform an iterative procedure at each step implementing a tolerence of 10^{-4}. For each method we used three different values of h: $h = 0.01$, $h = 0.005$ and $h = 0.0025$.

A. Example 1

Consider the nonlinear VIE

$$u(t) = 2 \left(1 + \int_0^t (t-s) u(s)\, ds \right)^2 \qquad 0 \leq t \leq 1 \quad (14)$$

which arises from a nonlinear differential equation in [15] where $b_1 = 0$ and $b_2 = 2$.

1) Using implicit Euler method: When $c_1 = 1$, and $t_{n,1} = t_n + h$, the collocation solution of (14) is given by

$$U_{n,1} = 2 \left(1 + F_n(t_{n,1}) + U_{n,1} \frac{h^2}{2} \right)^2 \quad (15)$$

where

$$F_n(t_{n,1}) = h \sum_{i=0}^{n-1} \left(t_n - t_i + \frac{h}{2} \right) U_{i,1}$$

Figure 1 shows the solution to (14) at $h = 0.01$, $h = 0.005$, and $h = 0.0025$.

ISBN: 978-988-19253-4-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

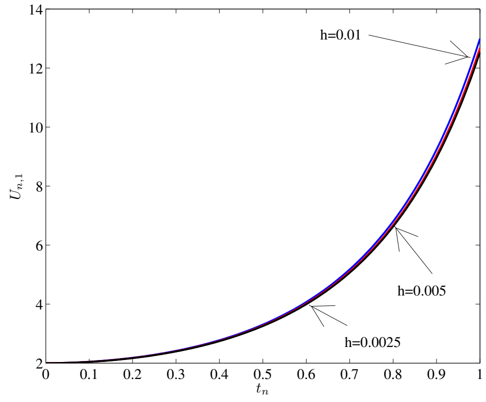

Fig. 1. The collocation solution of (14) when $c_1 = 1$

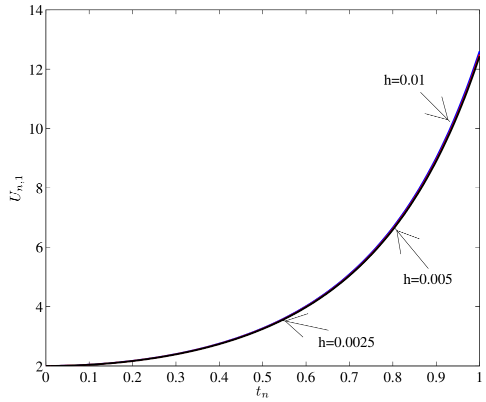

Fig. 2. The collocation solution of (14) when $c_1 = \frac{1}{2}$

2) Using implicit midpoint method: When $c_1 = \frac{1}{2}$, and $t_{n,1} = t_n + \frac{h}{2}$, the collocation solution of (14) is given by

$$U_{n,1} = 2\left(1 + F_n(t_{n,1}) + U_{n,1}\frac{h^2}{8}\right)^2 \qquad (16)$$

where

$$F_n(t_{n,1}) = h\sum_{i=0}^{n-1}(t_n - t_i)\,U_{i,1}$$

Figure 2 shows the solution to (14) at $h = 0.01$, $h = 0.005$, and $h = 0.0025$.

3) Using the iterated collocation: For $c_1 = \frac{1}{2}$ the iterated collocation solution of (14) is given as

$$u^I(t_n) = 2\left(1 + h\sum_{i=0}^{n-1}\int_0^1 (t_n - t_i - sh)\,ds U_{i1}\right)^2$$

Integrate to obtain

$$u^I(t_n) = 2\left(1 + h\sum_{i=0}^{n-1}(t_n - t_i - \frac{h}{2})U_{i1}\right)^2 \qquad (17)$$

The iterated collocation solution of (14) with three different values of h is shown in Figure 3

4) Using repeated trapezoidal rule: For the VIE (14) $u(0) = 2$ and

$$u(t_n) = 2\left(1 + \frac{h}{2}t_n u(0) + h\sum_{i=1}^{n-1}(t_n - t_{n-1})u_{n-1}\right)^2$$

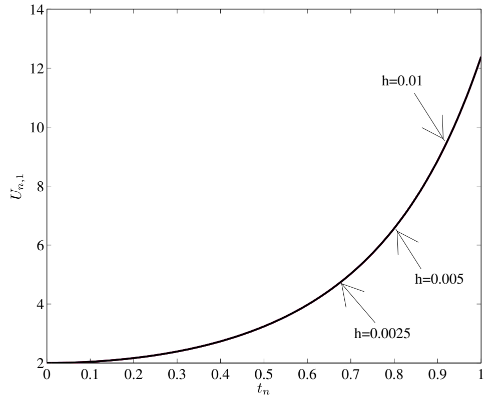

Fig. 3. The iterated collocation solution of (14) when $c_1 = \frac{1}{2}$

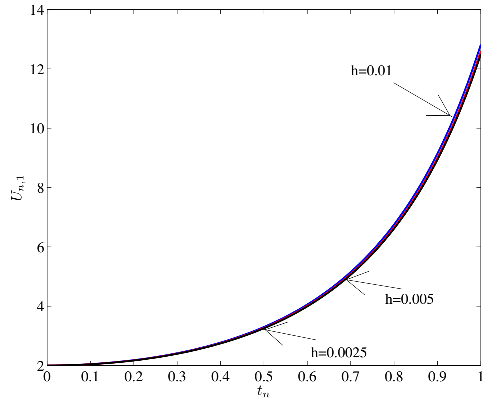

Fig. 4. The solution of (14) by the repeated trapezoidal rule

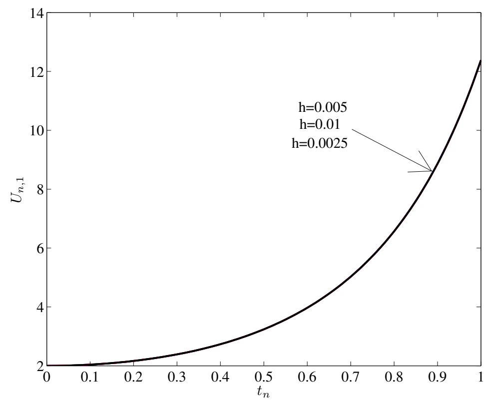

Fig. 5. The solution of (14) by the repeated Simpson's rule

Figure 4 shows the solution to the VIE (14) for the three values of h used.

5) Using repeated Simpson's rule: When $t = 0$, $u(0) = 2$ for equation (14) and

$$\begin{aligned} u(t_{2m}) &= 2\Big[1 + \frac{h}{3}((3t_{2m} - 2t_1)u(0) \\ &+ (2t_{2m} - 2t_{2m-1})u(t_{2m})) \\ &+ \frac{2h}{3}\sum_{l=0}^{m-1}(3t_{2m} - t_{2l+1} - t_l - t_{2l-1})u(t_{2l})\Big]^2 \end{aligned}$$

The solution to (14) using repeated Simpson's rule is shown in Figure 5.

ISBN: 978-988-19253-4-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

TABLE I
ABSOLUTE ERRORS IN THE SOLUTION OF (18) WHEN $h = 0.01$

t	Impl.Euler	Impl.Midpt	Rept. Trpz.	Rept Simps.
0.1	0.0028	0.0014	0.0027	-
0.2	0.0057	0.0028	0.0055	-
0.3	0.0089	0.0043	0.0086	-
0.4	0.0127	0.0061	0.0121	0.0001
0.5	0.0171	0.0081	0.0162	0.0001
0.6	0.0226	0.0105	0.0209	0.0001
0.7	0.0295	0.0135	0.0239	-
0.8	0.0385	0.0172	0.0342	0.0001
0.9	0.0501	0.0219	0.0436	0.0001
1.0	0.0659	0.0279	0.0553	0.0002

TABLE II
ABSOLUTE ERRORS IN THE SOLUTION OF (19) WHEN $h = 0.01$

t	Impl.Euler	Impl.Midpt	Rept. Trpz.	Rept Simps.
0.1	0.0116	0.0057	0.0110	-
0.2	0.0239	0.0116	0.0229	0.0001
0.3	0.0389	0.0187	0.0369	-
0.4	0.0586	0.0274	0.0543	0.0002
0.5	0.0857	0.0390	0.0774	0.0003
0.6	0.1247	0.0549	0.1089	0.0005
0.7	0.1829	0.0773	0.1537	0.0009
0.8	0.2727	0.1104	0.2196	0.0016
0.9	0.4163	0.1607	0.3196	0.0028
1.0	0.6541	0.2399	0.4771	0.0049

B. Example 2

$$\begin{aligned} u(t) &= \left(1 + \int_0^t (t-s)u(s)\,ds\right) \\ &+ \frac{1}{2}\left(1 + \int_0^t (t-s)u(s)\,ds\right)^2,\ 0 \le t \le 1 \quad (18) \end{aligned}$$

where $b_1 = 1$ and $b_2 = \frac{1}{2}$. The integral equation (18) arises from a nonlinear differential equations the represent conservative systems,(see [16]. We used the four methods to approximate the solution to this example and example 3, and we present tables for the absolute errors in the solution. Table I shows the errors in the solution of (18) when $h = 0.01$.

C. Example 3

Consider the integral equation

$$\begin{aligned} u(t) &= 2\left(1 + \int_0^t (t-s)u(s)\,ds\right) \\ &+ \left(1 + \int_0^t (t-s)u(s)\,ds\right)^2,\ 0 \le t \le 1 \quad (19) \end{aligned}$$

where $b_1 = 2$ and $b_2 = 1$. The nonlinear VIE arises from a nonlinear differential equation in [17]. Shown in Table II are the errors in the solution of (19) when $h = 0.01$.

V. DISCUSSION

We approximated the solutions to example 1-3 using the implicit euler method, implicit midpoint method, repeated trapezoidal and repeated Simpson's rule using various values of the stepsize. At $h = 0.001$ and below we obtained a similiar solution from all the methods used, hence we take that as our 'exact' solution. Therefore, for sufficiently small h we get a good accuracy of the numerical solutions. When the stepsize is greater that 0.001 we obtained different numerical solutions from each of the four methods. We use the 'exact' solution and absolute error to study the performance of each method when the stepsize is increased.

From examination of errors, we observe that the Repeated Simpson's rule performs better followed by the implicit midpoint method then the repeated trapezoidal rule. Among the four methods used the implicit Euler method gives a larger error as h is increased.We then found an iterated collocation solution for the implicit midpoint method and the accuracy of the method improved as shown in Figure 3. According to our numerical results, we conclude that the repeated Simpson's rule performs well since it gives better solutions when a reasonably large value of the stepsize is used. These observations are consistent for all three examples used.

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