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Dropped Ball Travel Time using Variation and Integration

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Abstract — **Spherical iron balls are used every day in the drilling and completion of oil wells. They can be used to operate downhole tools, set well packers, perform acid diversion and fracture wells. This is often achieved by dropping one or several balls into the well tubular from surface. Under gravity, the dropped ball travels kilometers down hole until it reaches the expected position or "seat". It is important to know the duration of the descent because time is money; the daily cost of the drillship must be monitored. Another reason is that one must know the time the ball reaches its final depth so the next operational step can be carried out. Modern wells are becoming more complex, horizontal, deeper and multi-lateral. In many cases, the empirical experience alone will not be reliable. Therefore, the following calculation methods could be helpful.**

Index Terms — **Abel's Integral Equation, Calculus of Variation, Lagrange Equation with Constraints, Wellbore.**

I. Introduction

THIS paper presents a rigorous approach to calculate the time a ball spends travelling through a well tubular, from surface to the expected depth.

The paper is divided in three main parts.

The first part is an overview of calculus of variations, summarized from reference [1], chap 2. The part recalls Lagrange and Euler equations with constraints.

The second part presents some types of integral equations, namely Fredholm, Volterra and singular; details can be found in reference [1], chap 3.

In the third and last part, we show how to use concepts of part I and II to evaluate the travel time of the dropped ball.

To that end, three methods are compared. The first one is an intuitive approach that uses speed and distance in classical mechanics. The second method is based on singular integral equation, namely Abel's integral equation, presented in part II. In the last one, we use Lagrange equations with constraints, reviewed in part I.

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II. Calculus of Variation

A. *Maxima and Minima, Lagrange Multipliers*

Let $y(x)$ be a function of x. A necessary condition for the existence of a maximum or minimum at a point x_0 inside (a,b) is that $\frac{dy}{dx}=0$ at x_0; in addition, a sufficient condition that y be a maximum (resp. minimum) is that ([1], p119-122)

$$\frac{d^2y}{dx^2}<0 \text{ (resp. } \frac{d^2y}{dx^2}>0)$$

Let $f(x_1,x_2,\dots,x_n)$ be a continuously differentiable function of n variables. Generally, a necessary condition for f to have a relative maximum or minimum value at an interior point of a region is that

$$df=\frac{\partial y}{\partial x_1}dx_1+\frac{\partial y}{\partial x_2}dx_2+\cdots+\frac{\partial y}{\partial x_n}dx_n=0$$

for all permissible values of $dx_1,dx_2,\dots,dx_n$. At such an interior point, the function f is said to be stationary.
If the n variables are independent, a sufficient condition that y be a maximum or minimum is:

$$\frac{\partial y}{\partial x_1}=\frac{\partial y}{\partial x_2}=\cdots=\frac{\partial y}{\partial x_n}=0$$

Let's assume that the n variables are not independent but related by N conditions $\emptyset_k(x_1,x_2,\dots,x_n)$=0 so that N variables are expressed in terms of the $n-N$ remaining variables. It can be shown that the values of $f(x_1,x_2,\dots,x_n)$ with N constraints (N<n) $\emptyset_1(x_1,x_2,\dots,x_n)$=0, $\emptyset_2(x_1,x_2,\dots,x_n)$=0,…, $\emptyset_N(x_1,x_2,\dots,x_n)$=0 can be expressed by

$$\left.\begin{array}{c}\frac{\partial f}{\partial x_1}+\lambda_1\frac{\partial\emptyset_1}{\partial x_1}+\lambda_2\frac{\partial\emptyset_2}{\partial x_1}+\cdots+\lambda_N\frac{\partial\emptyset_N}{\partial x_1}=0\\ \frac{\partial f}{\partial x_2}+\lambda_1\frac{\partial\emptyset_1}{\partial x_2}+\lambda_2\frac{\partial\emptyset_2}{\partial x_2}\dots+\lambda_N\frac{\partial\emptyset_N}{\partial x_2}=0\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \frac{\partial f}{\partial x_n}+\lambda_1\frac{\partial\emptyset_1}{\partial x_n}+\lambda_2\frac{\partial\emptyset_2}{\partial x_n}\dots+\lambda_N\frac{\partial\emptyset_N}{\partial x_n}=0\end{array}\right\}$$

Where $\lambda_1,\lambda_2,\dots,\lambda_N$ are arbitrary values known as Lagrange multipliers. These conditions are the conditions that $f+\lambda_1\emptyset_1+\lambda_2\emptyset_2+\cdots+\lambda_N\emptyset_N$ be stationary when no constraints are present.

B. *Euler Equation, Extremals, Stationary Functions*

It can be shown that ([1], p.123-125) the continuous differentiable function $y(x)$, satisfying $y(x_1)=y_1$ and $y(x_2)=y_2$ for which the integral $I=\int_{x_1}^{x_2}F(x,y,y')dx$ takes on a maximum or minimum value, satisfies the *Euler equation*:

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}=0$$

Solutions to Euler's equation are called *extremals*. An extremal that satisfies end conditions is called *stationary function* of the variational problem.

More generally, for n independent variables x,y,… and m dependent variable u,v,… there are m Euler equations of n+1 terms of first order and various orders of the form:

$$F_u - \left(\frac{\partial}{\partial x}F_{u_x} + \frac{\partial}{\partial x}F_{u_y} + \cdots\right) + \left(\frac{\partial^2}{\partial x^2}F_{u_{xx}} + \frac{\partial^2}{\partial x\partial y}F_{u_{xy}} + \frac{\partial^2}{\partial y^2}F_{u_{yy}} + \cdots\right) - \left(\frac{\partial^3}{\partial x^3}F_{u_{xxx}} + \cdots\right) + \left(\frac{\partial^4}{\partial x^4}F_{u_{xxxx}} + \cdots\right) - \cdots = 0$$

C. Constraints and Lagrange Multipliers

Assuming that we have two dependent variables u and v, one independent variables x and the *Lagrange multiplier* λ. Then, for $\delta\int_{x_1}^{x_2} F(x,u,v,u_x,v_x)dx = 0$ with the constraint $\phi(u,v) = 0$, we must have (refer to [1], p139-143):

$$\left.\begin{aligned}\frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial F}{\partial u} - \lambda\phi_u = 0\\ \frac{d}{dx}\left(\frac{\partial F}{\partial v_x}\right) - \frac{\partial F}{\partial v} - \lambda\phi_v = 0\end{aligned}\right\}\text{with } \lambda$$

or

$$\phi_v\left[\frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial F}{\partial u}\right] - \phi_u\left[\frac{d}{dx}\left(\frac{\partial F}{\partial v_x}\right) - \frac{\partial F}{\partial v}\right] = 0 \text{ after } \lambda \text{ is eliminated.}$$

Let $y(x)$ be prescribed at end points $y(x_1) = y_1$ and $y(x_2) = y_2$ and determined such that

$$\int_{x_1}^{x_2} F(x,y,y')dx = \min \textit{ or max}$$

Let assume a single constraint of the form of a definite *integral*

$$\int_{x_1}^{x_2} G(x,y,y')dx = k$$

where k is a constant. Given the above two assumptions, the Euler equation becomes

$$\frac{d}{dx}\left[\frac{\partial}{\partial y'}(F+\lambda G)\right] - \frac{\partial}{\partial y}(F+\lambda G) = 0$$

This result will be used in part IV, paragraph E.

D. Hamilton's Principal

The *Hamilton's principle*, detailed in reference ([1], p.148-150), is one of the most basic and important principles of mathematical physics. It is of the general form

$$\int_{t_1}^{t_2}(\delta T + f.\delta r)dt = 0$$

where T is the kinetic energy, f the force acting on a particle and r is the vector from a fixed origin at time t. When a potential function exits, that is, when the forces acting are conservative, the Hamilton's principle takes the form $\delta\int_{t_1}^{t_2}(T-V)dt = 0$, with V being the *potential energy*. The form can be rewritten as $\delta\int_{t_1}^{t_2} L dt = 0$, where the energy difference $L = T - V$ is referred to as the *kinetic potential* or *Lagrangian function*. For non-conservative forces, recourse must be to use the element of *work* "$f.\delta r$" done by the force f in a small *displacement* δr. The derivation is extended to a *summation* for a system of *N* particles and to an *integration* for a *continuous* system.

E. Lagrange Equations

For a dynamical system with n degrees of freedom it is usually possible to choose n independent geometrical position quantities $q_1, q_2, \ldots, q_n$ of all components of the system known as *generalized coordinates*. The total kinetic energy T may depend upon the $q's$ and their time rates of change $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$ called *generalized velocities*. For a *conservative* system the total potential energy V is a function only of position and does not depend upon the generalized velocities.

The work done by the force system with small displacements is

$$-\delta V = \delta\Phi = \sum_1^n f_k\delta r_k = Q_1\delta q_1 + Q_2\delta q_2 + \cdots + Q_n\delta q_n$$

with

$$Q_1 = -\frac{\partial V}{\partial q_1} = \frac{\partial\Phi}{\partial q_1};\ Q_2 = -\frac{\partial V}{\partial q_2} = \frac{\partial\Phi}{\partial q_2};\ldots;\ Q_n = -\frac{\partial V}{\partial q_n} = \frac{\partial\Phi}{\partial q_n}$$

Where Q's are referred to as *generalized forces*.
The Hamilton's principle then leads to n Euler equations, known as Lagrangian equations of the following form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

whenever the variations of the n q's are independent.
Since $\frac{\partial V}{\partial\dot{q}_i} \equiv 0$, we obtain:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i,\ (i=1,2,\ldots,n)$$

More details can be found in reference ([1], p.151-164).

F. Constraints in Dynamical Systems. Nonholomorphic Constraints

Let a system of n coordinates have k restrictive and auxiliary independent *equations of constraint* of the form $\emptyset_j(q_1,q_2,\ldots,q_k) = 0$, j=1,2…,k. We deduce n equations, each having the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i + \lambda_1\frac{\partial\emptyset_1}{\partial q_i} + \lambda_2\frac{\partial\emptyset_2}{\partial q_i} + \cdots + \lambda_k\frac{\partial\emptyset_k}{\partial q_i} \quad (\text{i}=1, 2,\ldots, \text{n})$$

It is apparent that any term $\lambda_k\frac{\partial\emptyset_k}{\partial q_i}$ is of the nature of a generalized force. However, the work done in any set of displacements by the force due to the k[th] constraint is given by

$$\lambda_k\frac{\partial\emptyset_k}{\partial q_1}\delta q_1 + \lambda_k\frac{\partial\emptyset_k}{\partial q_2}\delta q_2 + \cdots + \lambda_k\frac{\partial\emptyset_k}{\partial q_n}\delta q_n$$

and therefore vanishes if the displacements satisfy the constraint conditions as $\frac{\partial\emptyset_k}{\partial q_1}\delta q_1 + \frac{\partial\emptyset_k}{\partial q_2}\delta q_2 + \cdots + \frac{\partial\emptyset}{\partial q_n}\delta q_n = 0$.
In certain cases, the constraint conditions are only expressible in the form $C_1\delta q_1 + C_2\delta q_2 + \cdots + C_n\delta q_n = 0$ where the left hand member is not proportional to the variation of any function. In such a case, the constraint is said to be *nonholonomic* and the $\frac{\partial\emptyset_k}{\partial q_1},\ldots,\frac{\partial\emptyset_k}{\partial q_n}$ are replaced by $C_{k1},\ldots,C_{kn}$ using the method of Lagrange multipliers.

III. Integral Equations

A. Introduction

An *integral equation* is an equation in which a function to be determined appears under an integral sign. It is said to be *linear* when no nonlinear functions of the unknown function are involved. The most frequent form is the *Fredholm equation*:

$$\alpha(x)y(x) = F(x) + \lambda\int_a^b K(x,\varepsilon)y(\varepsilon)d\varepsilon$$

If $b = x$ is identified with the current variable, the equation is known as the *Volterra equation*. $\lambda, a \textit{ and } b$ are constant. $K(x,\varepsilon)$ is known as the *kernel*. The integral equation is said to be of the *first kind* if $\alpha \equiv 0$, *second kind* if $\alpha \equiv 1$ and of the *third kind* if α is a function. If α is positive, the Fredholm equation can take the form

$$\sqrt{\alpha(x)}y(x) = \frac{F(x)}{\sqrt{\alpha(x)}} + \lambda\int_a^b \frac{K(x,\varepsilon)}{\sqrt{\alpha(x)\alpha(\varepsilon)}}\sqrt{\alpha(\varepsilon)}y(\varepsilon)d\varepsilon$$

and be considered of the second kind in the unknown function $\sqrt{\alpha(x)}y(x)$ with a modified kernel. For a twodimensional variable $w(x,y)$, the Fredholm equation is of the form

$$\alpha(x,y)w(x,y) = F(x,y) + \lambda \iint_{\Re} K(x,y;\varepsilon,\eta)w(\varepsilon,\eta)d\varepsilon d\eta$$

Certain integral equations can be deduced from or reduced to differential equations making use of the formulae:

$$\frac{d}{dx}\int_{A(x)}^{B(x)} F(x,\varepsilon)d\varepsilon = \int_{A(x)}^{B(x)} \frac{\partial F(x,\varepsilon)}{\partial x} d\varepsilon + F[x,B(x)]\frac{dB}{dx} - F[x,A(x)]\frac{dA}{dx}$$

For instance the function $I_n(x) = \int_a^x (x-\varepsilon)^{n-1} f(\varepsilon)d\varepsilon$ will lead to $\frac{dI_n}{dx} = (n-1)I_{n-1}$; $\frac{d^n I_n}{dx^n} = (n-1)!\, f(x)$; and consequently $\int_a^x \ldots \int_a^x f(x)dx = \frac{1}{(n-1)!}\int_a^x (x-\varepsilon)^{n-1} f(\varepsilon)d\varepsilon$.

B. Singular Integral Equation

An integral equation in which the range of integration is infinite, or which the kernel $K(x,\varepsilon)$ is discontinuous, is called a *singular* integral equation (refer to [1], p.271-273). For instance:

-The *Fourier sine transform* $F(x) = \int_0^\infty \sin(x\varepsilon)\, y(\varepsilon)d\varepsilon$, associated with $y(x) = \lambda \int_0^\infty \sin(x\varepsilon)\, y(\varepsilon)d\varepsilon$. It can be inverted uniquely in the form

$$y(x) = \frac{2}{\pi}\int_0^\infty \sin(x\varepsilon)\, F(\varepsilon)d\varepsilon$$

The equation possesses $\lambda = \pm\sqrt{\frac{2}{\pi}}$ as characteristic numbers of *infinite multiplicity*, that is, each value of λ corresponds to infinitely many independent characteristic functions:

$y_1(x) = \sqrt{\frac{2}{\pi}} e^{-ax} + \frac{x}{a^2+x^2}$ $(x>0)$ and $y_2(x) = \sqrt{\frac{2}{\pi}} e^{-ax} - \frac{x}{a^2+x^2}$ $(x>0)$

This is due to the parameter a that can take on any positive value. This is in contrast with the fact that the characteristic numbers of a *nonsingular* Fredholm equation corresponds only to a *finite* number of independent characteristic functions.

-The *Laplace transform* $F(x) = \int_0^\infty e^{-x\varepsilon} y(\varepsilon)d\varepsilon$ is associated with the homogenous integral equation of the second kind:

$$y(x) = \lambda \int_0^\infty e^{-x\varepsilon} y(\varepsilon)d\varepsilon \quad (x>0)$$

Let us consider the *Gamma function* and the relations $\int_0^\infty e^{-x\varepsilon}\varepsilon^{a-1}d\varepsilon = \Gamma(a)x^{-a}$ $(a>0)$ and $\int_0^\infty e^{-x\varepsilon}\varepsilon^{-a}d\varepsilon = \Gamma(1-a)x^{a-1}$ $(a<1)$.

Then let us divide the first one by $\sqrt{\Gamma(a)}$ and the second one by $\sqrt{\Gamma(1-a)}$, and adding the resultant equations and taking $\lambda = \frac{1}{\sqrt{\Gamma(a)}\sqrt{\Gamma(1-a)}}$ $(0<a<1)$, $y(x) = \lambda \int_0^\infty e^{-x\varepsilon} y(\varepsilon)d\varepsilon$ we obtain:

$y(x) = \sqrt{\Gamma(1-a)}x^{a-1} + \sqrt{\Gamma(a)}x^{-a}$ (x>0).

With the identity $\Gamma(a).\Gamma(1-a) = \frac{\pi}{\sin \pi a}$, we obtain $\lambda = \sqrt{\frac{\sin \pi a}{\pi}}$ $(0<a<1 \Rightarrow 0<\lambda<1/\sqrt{\pi})$ which is in contrast with the fact that the characteristic values of λ for a *nonsingular* equation are *discretely* distributed and cannot constitute a *continuous "spectrum"*.

-*The Abel's integral equation* $F(x) = \int_0^x \frac{y(\varepsilon)}{\sqrt{x-\varepsilon}}d\varepsilon$ is treated in the following section.

C. Transforms, Convolution, Laplace Transform of Special Volterra Equation

([1], p.274-277).

-*Transforms*: If the double integral of the relationship

$$\int_a^b \int_a^b \Gamma(x,\varepsilon_1)\, K(\varepsilon_1,\varepsilon)y(\varepsilon)d\varepsilon d\varepsilon_1$$

can be evaluated as an iterated integral, then it follows that for $F(x) = \int_a^b K(x,\varepsilon)y(\varepsilon)d\varepsilon$ we have also $y(x) = \int_a^b \Gamma(x,\varepsilon)F(\varepsilon)d\varepsilon$. We refer to one of the function as the *transform* and the other as the *inverse transform*. The correspondence may or may not be unique.

-*The convolution*: The convolution of u(x) and v(x) is given by

$$\int_a^x \mathrm{u}(x-\varepsilon)v(\varepsilon)d\varepsilon$$

The fact that *the Laplace transform of the convolution is equal to the product of the Laplace transforms* permits the reduction of the Volterra equation

$$y(x) = F(x) + \int_a^x K(x-\varepsilon)y(\varepsilon)d\varepsilon$$

to

$$\mathscr{L}\mathrm{y(x)} = \mathscr{L}\mathrm{F(x)} + \mathscr{L}\mathrm{K(x)}\mathscr{L}\mathrm{y(x)}$$

Henceforth, $\mathscr{L}\, y(x) = \mathscr{L}F(x)/[1-\mathscr{L}\, K(x)]$, which right-hand member is calculable and it remains only to determine its inverse transform by the use of tables or otherwise.

-*Volterra equations of the first kind:*

It is often possible to reduce $F(x) = \int_a^x K(x,\varepsilon)y(\varepsilon)d\varepsilon$ to $F'(x) = K(x,x)y(x)\int_a^x \frac{\partial K(x,\varepsilon)}{\partial x}y(\varepsilon)d\varepsilon$ by differentiating its equal members; one must assume that $K(x,\varepsilon)$ (for $\varepsilon \le x$) is continuously differentiable, $K(x,x)\neq 0$ and $F(x)$ is differentiable.

By setting $\tilde{F}(x) = \frac{F'(x)}{K(x,x)}$ and $\tilde{K}(x,\varepsilon) = -\frac{1}{K(x,x)}\frac{\partial K(x,\varepsilon)}{\partial x}$ (or alternatively by considering $Y(x) = \int_a^x y(\varepsilon)d\varepsilon$), we obtain the equation of the *second kind* $y(x) = \tilde{F}(x) + \int_a^x \tilde{K}(x,\varepsilon)y(\varepsilon)d\varepsilon$, suitable to the method of successive substitutions.

-*Abel's integral equation*:

The Volterra equation $F(x) = \int_0^x \frac{y(\varepsilon)}{\sqrt{x-\varepsilon}}d\varepsilon$ is known as the *Abel's integral equation*. By diving both sides by $\sqrt{s-x}$ and integrating the result with respect to x over (0,s), s parameter, following by an inversion of the order of integration of the right-hand member, we obtain

$$\int_0^s \frac{F(x)}{\sqrt{s-x}}dx = \int_0^s \{\int_\varepsilon^s \frac{dx}{\sqrt{(x-\varepsilon)(s-x)}}\} y(\varepsilon)\, d\varepsilon$$

The change in variable $x = (s-\varepsilon)t + \varepsilon$ lead to: $x = \varepsilon \Rightarrow t = 0$ and $x = s \Rightarrow t = 1$; $dx = (s-\varepsilon)dt$ and $s - x = -(s-\varepsilon)t + s - \varepsilon \Rightarrow s - x = (s-\varepsilon)(1-t)$.

It follows that

$\int_\varepsilon^s \frac{dx}{\sqrt{(x-\varepsilon)(s-x)}} = \int_0^1 \frac{dt}{\sqrt{t(1-t)}} = \int_0^1 \frac{dt}{\sqrt{\frac{1}{4}-(t-\frac{1}{2})^2}} = [\arcsin 2(t-\frac{1}{2})]_0^1 = \pi$

which leads to the solution

$$y(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{F(\varepsilon)}{\sqrt{x-\varepsilon}}d\varepsilon$$

A generalized Abel's equation can also be found using a method similar to the preceding. One can also use proprieties of the *Laplace transform* of convolution, combined with the propriety $\Gamma(a).\Gamma(1-a) = \frac{\pi}{\sin \pi a}$ of the *Gamma function*.

Then, it follows that the general solution of

$$F(x) = \int_0^x \frac{y(\varepsilon)}{(x-\varepsilon)^\alpha}d\varepsilon$$

is of the form:

$$y(x) = \frac{\sin \alpha\pi}{\pi}\frac{d}{dx}\int_0^x (x-\varepsilon)^{\alpha-1}F(\varepsilon)d\varepsilon$$

$$(0 < \alpha < 1)$$

This result will be used in part IV, paragraph D.

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IV. Travel Time of a Dropped Ball along a Curved Wellbore

A. Problem Formulation

TABLE I
SYMBOL AND UNITS USED

Symbol	Quantity	SI unit
V	Potential Energy	Joule
T	Kinetic Energy	Joule
T	Time, duration strength	second
m	mass	kg
g	gravity	m/s^2
$\mathscr{L}$	Laplace Transform Operator	
ε	Vertical Depth at s	m
s	Distance along wellbore from the end point(seat)	m
L	Lagrangian	
Γ	Gamma Function	
$\dot{q}_i$	Time first derivative	
MD	Measured Depth	m
$s'(x)$	First derivative of s over x	

Well Profile

Measured Depth (MD-s) (m)	Vertical Depth (X-ε) (m)
0.00	0.00
100.45	100.45
600.55	590.08
850.11	820.03
1020.97	950.44
1620.57	1200.01
1911.30	1380.82
1923.91	1387.22
2506.14	1449.95
2918.53	1508.67
3202.29	1532.46
3313.72	1544.48
3491.31	1544.04
3619.13	1544.45
3975.24	1544.29
4303.23	1544.16
4886.62	1544.41
5202.33	1544.42

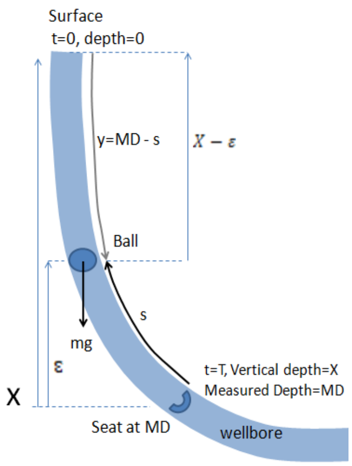

Fig. 1. Wellbore Profile

1- Constraint Equation: A horizontal well completion profile is defined by (s, ε) data points where s is the ***distance along the well from the terminal point***, the seat *measured depth* (MD=3200m), and ε is the vertical depth at s. Find a polynomial of second degree of the form $y(X,\varepsilon) = u(X-\varepsilon)^2 + v(X-\varepsilon) + w$ that best represents the ***well profile***. Use appropriate regression operations and verify the result with Excel "Trend line Options" and "Display Equation on Chart". Deduce the constraint equation of the form $s(\varepsilon) = a\varepsilon^2 + b\varepsilon + c$.

2- Equation of Speed:

a) Suppose that a ball of mass m starts from rest at time t=0 at the initial point at surface and slides downhole along the above wellbore curve under the action of gravity without friction, neglecting the buoyancy effect (assumed an unfilled well). If the seat depth is X and if the ball height from seat is ε at time t, first establish the equation of motion as a differential equation in s(t). Second, express the ***speed*** ds/dt of the ball as a function of t, X, g and ε. Notice that ds/dt =-dy/dt

b) Curved Wellbore, Time of Descent: Given the seat vertical depth at 1530m, deduce from the preceding result the ***time of descent*** T of the ball.

3- Time of Descent using Abel's Integral Equation:
First find a general solution to the Abel's Integral Equation using the Laplace transform of the convolution. Then, apply the result to the time equation found in the previous section and deduce the time of descent.

4- Time of Descent using Lagrange's Equations: write the generalized Lagrange's equations of the system; eliminate the parameter λ between the equations and find the relation between s and t at t=0, s=MD and t=T, s=0; solve for T, given X=1530m, MD=3200m.

B. Solution:

1) *Wellbore Profile Mathematical Model: Constraint Equation*

It is useful to obtain a well profile in equation (or set of equations) so it (they) can be used as a constraint (set of constraints) in Lagrange's equations of motion. We use the "line of best fit" or the least square regression technique for that effect ([2], [3]). This could be done by section but, since the wellbore looks like a curve rather than a line, and for the sake of simplicity, we choose to approximate the entire wellbore by one equation of the form:

$$s(\varepsilon) = a\varepsilon^2 + b\varepsilon + c$$

For the n data points, the equation can be rewritten in the form of matrix multiplication by

$$\begin{Bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{Bmatrix} = a \begin{Bmatrix} \varepsilon_1^2 \\ \varepsilon_2^2 \\ \vdots \\ \varepsilon_n^2 \end{Bmatrix} + b \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{Bmatrix} + c \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix} = \begin{bmatrix} \varepsilon_1^2 & \varepsilon_1 & 1 \\ \varepsilon_2^2 & \varepsilon_2 & 1 \\ \dots & \dots & \dots \\ \varepsilon_n^2 & \varepsilon_n & 1 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} \varepsilon_1^2 & \varepsilon_2^2 & \dots & \varepsilon_n^2 \\ \varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_n \\ 1 & 1 & \dots & 1 \end{bmatrix} \begin{Bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{Bmatrix}$$

$$= \begin{bmatrix} \varepsilon_1^2 & \varepsilon_2^2 & \dots & \varepsilon_n^2 \\ \varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_n \\ 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_1^2 & \varepsilon_1 & 1 \\ \varepsilon_2^2 & \varepsilon_2 & 1 \\ \dots & \dots & \dots \\ \varepsilon_n^2 & \varepsilon_n & 1 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \sum \varepsilon_i^2 s_i \\ \sum \varepsilon_i s_i \\ \sum s_i \end{Bmatrix} = \begin{bmatrix} \sum \varepsilon_i^4 & \sum \varepsilon_i^3 & \sum \varepsilon_i^2 \\ \sum \varepsilon_i^3 & \sum \varepsilon_i^2 & \sum \varepsilon_i \\ \sum \varepsilon_i^2 & \sum \varepsilon_i & n \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{bmatrix} \sum \varepsilon_i^4 & \sum \varepsilon_i^3 & \sum \varepsilon_i^2 \\ \sum \varepsilon_i^3 & \sum \varepsilon_i^2 & \sum \varepsilon_i \\ \sum \varepsilon_i^2 & \sum \varepsilon_i & n \end{bmatrix}^{-1} \begin{Bmatrix} \sum \varepsilon_i^2 s_i \\ \sum \varepsilon_i s_i \\ \sum s_i \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{bmatrix} 2.26347E+17 & 44509532727 & 30626508.92 \\ 44509532727 & 30626508.92 & 21730.379 \\ 30626508.92 & 21730.379 & 18 \end{bmatrix}^{-1} * \begin{Bmatrix} 99482528978 \\ 66692545.75 \\ 45446.40 \end{Bmatrix} = \begin{Bmatrix} 0.0026876 \\ -1.918186 \\ 267.60127 \end{Bmatrix}$$

$$\Rightarrow y(X,\varepsilon) = 0.0026876\,(X-\varepsilon)^2 - 1.918186(X-\varepsilon) + 267.60127$$

$$\Rightarrow \mathrm{MD} - s(\varepsilon) = 0.0026876\,(X-\varepsilon)^2 - 1.918186(X-\varepsilon) + 267.60127$$

For X=1530m and MD=3200m we obtain

$$s(\varepsilon) = -0.0026876\,(1530-\varepsilon)^2 + 1.918186(1530-\varepsilon) - 267.60127 + 3200$$

Leading to the constraint equation:

$$s(\varepsilon) = -0.0026876\,\varepsilon^2 + 6.30587\varepsilon - 424.1795$$

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This equation is represented on the following graph, using the change in variable $x = 1530 - \varepsilon$ and $y = s(\varepsilon) - 3200$ for clarity purposes:

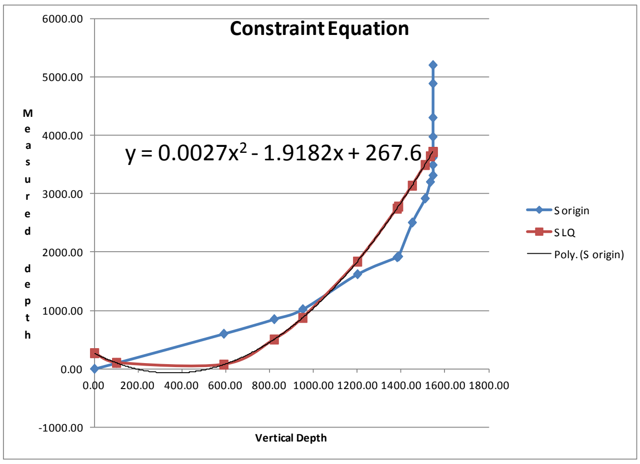

Fig. 2. Constraint Equation from Regression

2) *Equation of Speed - Solution Method 1*

The total energy is roughly approximated. The result is expected to be inaccurate.

1. Equation of Speed

If y=MD-s is the distance from surface (dy/dt=-ds/dt), neglecting the damping, the kinetic energy and the work done by gravity $W = mg(X-\varepsilon)$, hence the potential energy $V = -mg(X-\varepsilon)$, lead to:

$$T = \frac{1}{2}m(\dot{y})^2,\ V = -mg(X-\varepsilon) \Rightarrow \dot{y}^2 = 2g(X-\varepsilon) \Rightarrow \dot{y} = \sqrt{2g(X-\varepsilon)}$$

$$\text{but } \frac{dy}{dt} = -\frac{ds}{dt} \Rightarrow \frac{ds}{dt} = -\sqrt{2g(X-\varepsilon)}$$

2. Time of Descent to X=1530m

$$\frac{ds}{dt} = -\sqrt{2g(X-\varepsilon)} \Rightarrow \frac{dt}{ds} = \frac{-1}{\sqrt{2g(X-\varepsilon)}} \Rightarrow dt = \frac{-ds}{\sqrt{2g(X-\varepsilon)}},$$

note that at $\varepsilon = X$, $t = 0$ and at $\varepsilon = 0$, $t = T$

$$\Rightarrow T_2 - T_1 = T - 0 = T(X) = \int_{\varepsilon=X}^{\varepsilon=0} \frac{-s'(\varepsilon)}{\sqrt{2g(X-\varepsilon)}} d\varepsilon$$

$$= \int_{\varepsilon=0}^{\varepsilon=X} \frac{s'(\varepsilon)}{\sqrt{2g(X-\varepsilon)}} d\varepsilon$$

$$s(\varepsilon) = -0.0026876\,\varepsilon^2 + 6.30587\varepsilon - 424.1795$$

$$\Rightarrow s'(\varepsilon) = -0.0053752\varepsilon + 6.30587$$

$$\Rightarrow T(1530) = \int_0^{1530} \left(\frac{-0.0053752\varepsilon + 6.30587}{\sqrt{2 \text{ x } 9.81(1530-\varepsilon)}}\right) d\varepsilon$$

we evaluate the expression by integrating by part:

$$T(X) = \int_0^X \frac{A\varepsilon + B}{\sqrt{C + D\varepsilon}} d\varepsilon = \int_0^X (A\varepsilon + B)(C + D\varepsilon)^{-1/2} d\varepsilon =$$

$$\left[(A\varepsilon + B)\frac{1}{(-\frac{1}{2}+1)D}(C + D\varepsilon)^{-\frac{1}{2}+1}\right]_0^X - \int_0^X A\frac{1}{(-\frac{1}{2}+1)D}(C + D\varepsilon)^{-\frac{1}{2}+1} d\varepsilon$$

$$T(X) = \left[\frac{2}{D}(A\varepsilon + B)(C + D\varepsilon)^{\frac{1}{2}}\right]_0^X - \int_0^X \frac{2A}{D}(C + D\varepsilon)^{\frac{1}{2}} d\varepsilon = \left[\frac{2}{D}(A\varepsilon + B)(C + D\varepsilon)^{\frac{1}{2}}\right]_0^X - \left[\frac{2A}{(\frac{1}{2}+1)D^2}(C + D\varepsilon)^{\frac{1}{2}+1}\right]_0^X = \left[\frac{2}{D}(A\varepsilon + B)(C + D\varepsilon)^{\frac{1}{2}}\right]_0^X - -\left[\frac{4A}{3D^2}(C + D\varepsilon)^{\frac{3}{2}}\right]_0^X$$

$$T(X) = \frac{2}{D}\left[(AX + B)(C + DX)^{\frac{1}{2}} - (B)(C)^{\frac{1}{2}}\right] - -\frac{4A}{3D^2}\left[(C + DX)^{\frac{3}{2}} - (C)^{\frac{3}{2}}\right]$$

$$T(1530) = \int_0^{1530} \left(\frac{-0.0053752\varepsilon + 6.30587}{\sqrt{30019 - 19.62\varepsilon}}\right) d\varepsilon$$

$$T(X) = \frac{2}{-19.62}\left[(-0.0053752 * 1530 + 6.30587)(30019 - 19.62 * 1530)^{\frac{1}{2}} - (6.30587)(30019)^{\frac{1}{2}}\right] - \frac{4(-0.0053752)}{3(-19.62)^2}\left[(30019 - 19.62 * 1530)^{\frac{3}{2}} - (30019)^{\frac{3}{2}}\right]$$

Curved Well, Time of descent (X = 1530m) = **14sec**

3) *Abel's Integral Equation - Solution Method 2*

This is achieved by solving the following Abel's integral equation:

$$T(x) = \int_0^X \frac{s'(\varepsilon)}{\sqrt{2g(X-\varepsilon)}} d\varepsilon$$

1. Solution to the Generalized Abel's Integral Equation

$$F(x) = \int_0^x \frac{y(\varepsilon)}{(x-\varepsilon)^\alpha} d\varepsilon \quad (0 < \alpha < 1)$$

To find the general solution, we use the propriety of the Laplace transform of a convolution ([1], p.274-277):

$$\mathscr{L}F(x) = \mathscr{L}x^{-\alpha}.\ \mathscr{L}y(x)$$

By virtue of

$$\mathscr{L}x^{-\alpha} = \Gamma(1-\alpha)s^{\alpha-1}$$

we obtain

$$\mathscr{L}y(x) = \frac{1}{\Gamma(1-\alpha)s^{\alpha-1}}\mathscr{L}F(x) = \frac{s^{1-\alpha}}{\Gamma(1-\alpha)}\mathscr{L}F(x)$$

It follows that

$$\frac{1}{s}\mathscr{L}y(x) = \frac{s^{1-\alpha}}{s\Gamma(1-\alpha)}\mathscr{L}F(x) = \frac{s^{-\alpha}}{\Gamma(1-\alpha)}\mathscr{L}F(x).$$

With the propriety of the Gamma function ([3], p.319)

$$\int_0^1 (1-t)^{\alpha-1}t^{-\alpha} dt = \Gamma(1-\alpha).\Gamma(\alpha) = \frac{\pi}{\sin(\alpha\pi)}$$

.Using

$$\Gamma(\alpha)s^{-\alpha} = \mathscr{L}x^{\alpha-1}$$

we obtain

$$\frac{1}{s}\mathscr{L}y(x) = \frac{\Gamma(\alpha)s^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)}\mathscr{L}F(x)$$

$$= \frac{\sin(\alpha\pi)}{\pi}\Gamma(\alpha)s^{-\alpha}\ \mathscr{L}F(x) = \frac{\sin(\alpha\pi)}{\pi}\mathscr{L}x^{\alpha-1}\ \mathscr{L}F(x)$$

$$\frac{1}{s}\mathscr{L}y(x) = \frac{\sin(\alpha\pi)}{\pi}\mathscr{L}x^{\alpha-1}\ \mathscr{L}F(x) \Rightarrow \int_0^x y(\varepsilon)\, d\varepsilon = \frac{\sin(\alpha\pi)}{\pi}\int_0^x (x-\varepsilon)^{\alpha-1}F(\varepsilon)d\varepsilon$$

After derivation of both sides of the equality, we deduce:

$$y(x) = \frac{\sin(\alpha\pi)}{\pi}\frac{d}{dx}\int_0^x (x-\varepsilon)^{\alpha-1}F(\varepsilon)d\varepsilon$$

2. Incorporating the Wellbore Profile

Using $\alpha = \frac{1}{2}$ in the above equation, the solution of

$$T(x) = \int_0^x \frac{s'(\varepsilon)}{\sqrt{2g(x-\varepsilon)}} d\varepsilon$$

is then given by

$$\frac{s'(x)}{\sqrt{2g}} = \frac{\sin(\frac{\pi}{2})}{\pi}\frac{d}{dx}\int_0^x (x-\varepsilon)^{-1/2}F(\varepsilon)d\varepsilon \Rightarrow y(x) = \frac{1}{\pi}\sqrt{2g}\int_0^x \frac{T(\varepsilon)}{\sqrt{(x-\varepsilon)}} d\varepsilon$$

Where $T(\varepsilon)$ is obtained by the method of point 1) from well data survey.

3. Time of Descent

With

$$y = 1.3898(X - \varepsilon(t) \Rightarrow MD - s(t) = 1.3898$$

$$\varepsilon(t) \Rightarrow s = 1.3898\varepsilon - 1073.606$$

The time of descent becomes:

$$T(X) = \int_0^X \frac{B}{\sqrt{C + D\varepsilon}} d\varepsilon = \frac{2B}{D}\left[(C + DX)^{\frac{1}{2}} - BC^{\frac{1}{2}}\right] \Rightarrow$$

$$T(1530) = \int_0^{1530} \frac{1.3898}{\sqrt{30019 - 19.62\varepsilon}} d\varepsilon$$

$$T(X) = \frac{2 * 1.3898}{-19.62}\left[(30019 - 19.62 * 1530)^{\frac{1}{2}} - 30019^{\frac{1}{2}}\right]$$

Slant Well Time of descent (X = 1530m) = **25sec**

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4) *Lagrange's Equations - Solution Method 3*

Following the result of paragraph II.C or reference ([1], p.135-138, p.151-164), we use the generalized Lagrange equation with constraints in the form:

$$\frac{d}{dt}\left(\frac{\partial(F+\lambda G)}{\partial \dot{q}_i}\right)-\frac{\partial(F+\lambda G)}{\partial q_i}=0$$

where F=L is the Lagrangian and G=∅ the constraint equation.

$$y=1.3898(X-\varepsilon(t)) \Rightarrow MD-s(t)=1.3898\,\varepsilon(t) \Rightarrow s=1.3898\varepsilon-1073.606$$

$$\varepsilon(t) \Rightarrow s=1.3898\varepsilon-1073.606$$

$$T=\frac{1}{2}m\dot{s}^2, V=-mg(X-\varepsilon), \emptyset(s,\varepsilon)= s-1.3898\varepsilon+1073.606$$

L=T-V, $F=L+\lambda\emptyset=\frac{1}{2}m\dot{s}^2+mg(X-\varepsilon)+\lambda(s-1.3898\varepsilon+1073.606)$, which leads to

$$\frac{\partial(L+\lambda\emptyset)}{\partial\dot{\varepsilon}}=0;\ \frac{d}{dt}\left(\frac{\partial(L+\lambda\emptyset)}{\partial\dot{\varepsilon}}\right)=0;\ \frac{\partial(L+\lambda\emptyset)}{\partial\varepsilon}=-mg-1.3898\lambda$$

$$\frac{\partial(L+\lambda\emptyset)}{\partial\dot{s}}=m\dot{s};\ \frac{d}{dt}\left(\frac{\partial(L+\lambda\emptyset)}{\partial\dot{s}}\right)=m\ddot{s};\ \frac{\partial T}{\partial s}=\lambda;\ m\ddot{s}-\lambda=0$$

There follows:

$$\left.\begin{matrix}-mg-1.3898\lambda=0\\ m\ddot{s}=\lambda\end{matrix}\right\} \Rightarrow \frac{-mg}{1.3898}=m\ddot{s} \Rightarrow \frac{g}{-1.3898}=\ddot{s}$$

$$\frac{d^2s}{dt^2}=-\frac{g}{a} \Rightarrow \frac{ds}{dt}=-\frac{g}{a}t$$

$+B \Rightarrow$ at $t=0, \frac{ds}{dt}=0$ (starts free fall from rest), $B=0,\ s(t)=\frac{-g}{2a}t^2+C$

At t=0, y=0 and s=MD, we obtain $s(t)=MD-\frac{g}{2a}t^2$ and $T(s=0,y=MD)=\sqrt{\frac{2a}{g}(MD-s)}$ for $a=1.3898$, $g=9.81$ and $MD=3200$ hence,

Slant Lagrange $T(MD=3200)=\sqrt{\frac{2*1.3898}{9.81}3200}$=**30 sec**

V. Conclusion

Three methods were compared. One uses the speed approximation from which, a function of the speed is integrated and the time of travel is then deduced. This method gives 14 seconds in our example and appears a little bit short based on personal field experience. The method is inaccurate due to important assumptions that cannot be evaluated such as frictions. Frictions depend on the roughness of the tubular, the fluid, etc. It is hard to evaluate the actual friction because the inside of the tubular is different from the origin once in the wellbore. It is likely that it will have fines or scale deposits that are not predictable. Therefore this method is not the right way forward.

The other two methods, the Abel's equation and the calculus of variation give respectively 25 and 30 seconds. These are more realistic for the measured depth of our application example.

These methods of calculations are based on the mathematical model of the wellbore profile, meaning the wellbore is represented by an actual mathematical equation that we found from regression techniques. This mathematical equation is then considered as a physical constraint that imposes a path to the ball when applying Lagrange equations in calculus of variations. These methods are closer to actual experience and are better than the one from classical mechanic discussed earlier.

Acknowledgment

This paper is submitted as a continuation of my works started during my PhD in Applied Mathematics and as a solution for finding the time spent by a ball when descending a wellbore. Therefore, I would like to take this opportunity to express my gratitude to those that have been very supportive and of great help, many of my great colleagues, friends, former teachers, parents and close family, my mother Clementine Avome Ndong and my uncle, Paul Ovono Ndong for their moral encouragements throughout my life and studies.

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