Analysis of an SEIR Epidemic Model with a General Feedback Vaccination Law

M. De la Sen , S. Alonso-Quesada, A.Ibeas and R. Nistal

Abstract-**This paper discusses and formulates a continuous-time SEIR -type epidemic model of pseudo-mass action type with finitely distributed delays under a very general, and in general time-varying, vaccination control rule which eventually generates feedback actions on the susceptible, infectious and recovered subpopulations. A lot of particular vaccination laws can be got from the proposed general one. The equilibrium points are characterized and their local stability properties discussed depending on the limits of the vaccination control gains provided that they converge asymptotically.**

Keywords- **Epidemic models; distributed delays; SEIR model; feedback vaccination controls; equilibrium points.**

I. INTRODUCTION

THIS research is concerned with a SEIR epidemic model, subject to finitely distributed delays and eventual vaccination, which is of pseudo-mass action type in the sense that the infective transmission rate does not depend directly on the total population, [1-3]. The continuous-time model has the following characteristics and properties: a) The vaccination controls have eventual feedback actions of the susceptible, infected and recovered subpopulations and also an independent term which have, in general, time-varying gains with a constant term plus an incremental one. The independent term selection guarantees the non-negativity of the state-trajectory solution for all time so as to reflect real situations. The structure of the vaccination control law is very general and it can be also implemented in the case when the subpopulation numbers are not precisely known b) The disease-free and the endemic equilibrium points are characterized as well as their local asymptotic stability properties in the case that the vaccination controller gains converge asymptotically to limits. It is proved that the infection is non-permanent and the state-trajectory solution converges asymptotically to the disease-free equilibrium point if the disease infective rate is under a certain maximum threshold.

Manuscript received February 1, 2015; revised February 13, 2015. This work was supported in part by the Spanish Government for Grants DPI2012-30651and DPI2013-47825-C3-1-R, and to the Basque Government and UPV/EHU for Grant IT378-10.
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There are new characteristics in the scheme concerning the generality of the vaccination law related to the existing previous background .

II.FEEDBACK VACCINATION EPIDEMIC MODEL

Consider the following SEIR epidemic model with a delayed-distributed transmission effect:

$$\dot{S}(t) = b(1-V(t)-S(t)) - \beta S(t)\int_0^h f(\tau) I(t-\tau)\, d\tau \quad (1)$$

$$\dot{E}(t) = \beta S(t)\int_0^h f(\tau) I(t-\tau)\, d\tau - (b+\kappa)E(t) \quad (2)$$

$$\dot{I}(t) = \kappa E(t) - (b+\gamma) I(t) \quad (3)$$

$$\dot{R}(t) = bV(t) + \gamma I(t) - bR(t) \quad (4)$$

$$V(t) = \left(sat_{[0,1]}\overline{V}\right)(t) = \begin{cases} \overline{V}(t) & if \quad \overline{V}(t) \in [0,1] \\ 1 & if \quad \overline{V}(t) > 1 \end{cases} \quad (5)$$

$$\overline{V}(t) = k_1(t)\hat{S}(t) + k_2(t)\hat{I}(t) + k_3(t)\hat{S}(t)\int_0^h f(\tau)\hat{I}(t-\tau)\, d\tau + k_4(t) \quad (6)$$

where $S(t)$, $E(t)$, $I(t)$ and $R(t)$ are, respectively, the susceptible, exposed, infectious and recovered subpopulations at time t and $V:[0,\infty)\to[0,1]$ is a non-negative real feedback vaccination control defined through the real control gains $k_i:[0,\infty)\to[k_{min},k_{max}]$; $i=1,2,3,4$ with $k_{max}\geq k_{min}\geq 0$. The functions $\hat{S}(t)=(1+\alpha_S(t))S(t)$ and $\hat{I}(t)=(1+\alpha_I(t))I(t)$, $\forall t\geq 0$ with $\overline{\alpha}_{Sm}\leq\alpha_S(t)\leq\overline{\alpha}_{SM}$ and $\overline{\alpha}_{Im}\leq\alpha_I(t)\leq\overline{\alpha}_{IM}$, $\forall t\geq 0$ are estimates of the susceptible and infectious subpopulations used for the vaccination law implementation with $\overline{\alpha}_{Sm}, 1\geq\overline{\alpha}_{SM}\left(\geq\overline{\alpha}_{Sm}\geq -1\right)$ and $\overline{\alpha}_{Im}, 1\geq\overline{\alpha}_{IM}\left(\geq\overline{\alpha}_{Im}\geq -1\right)$ being known real constants which are minimum and maximum relative per-unit errors of the estimates. The following positive parameters parameterize the system (1)-(6):

b is the birth-rate of the population,
β is the infectivity disease rate,
κ is the transition rate from the exposed subpopulation to the infectious one,

ISBN: 978-988-19253-4-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

γ is the transition rate from the infectious subpopulation to the recovered one. Note that γ^{-1} is the average time that an infectious individual stays at this stage before recovering completely.
It is assumed that $f:[0,\infty)\to[0,\infty)$ and that

$$\int_0^h f(\tau)\,d\tau = 1 \;,\; \int_0^h \tau\, f(\tau)\,d\tau < \infty \qquad (7)$$

One gets the following dynamics for the total population $N(t)=S(t)+E(t)+I(t)+R(t)$, $\forall t\geq 0$:

$$\dot{N}(t) = b\left(1-N(t)\right),\ \forall t\geq 0 \qquad (8)$$

which implies that at any existing equilibrium point, $N^* = \lim_{t\to\infty} N(t) = 1$. A result which guarantees the linear structure of the feedback vaccination law is the following:

Theorem 1. The vaccination law is given by

$$V(t) = \left(k_1(t)+k_3(t)\int_0^h f(\tau)\hat{I}(t-\tau)\,d\tau\right)\hat{S}(t)+k_2(t)\hat{I}(t)+k_4(t)$$
$$,\forall t\geq 0 \qquad (9)$$

, so that it does not enter the saturation zone at any time, if

$$k_{max} = \max_{1\leq i\leq 4}\left(k_i(t)\right) \leq \frac{1}{\hat{S}(t)\left[1+\int_0^h f(s)\hat{I}(t-s)ds\right]+1}$$

$$= \frac{1}{\left(1+\alpha_S(t)\right)S(t)\left[1+\int_0^h f(s)\left(1+\alpha_I(s)\right)I(t-s)ds\right]+1},$$
$$,\forall t\geq 0 \qquad (10)$$

A sufficient conditions for the above condition to hold is:

$$k_{max} \leq \frac{1}{\left(1+\bar{\alpha}_{SM}\right)S(t)\left[1+\left(1+\bar{\alpha}_{IM}\right)\int_0^h f(s)I(t-s)ds\right]+1}$$
$$,\ \forall t\geq 0 \qquad (11)$$

which holds under the stronger constraint:

$$k_{max} \leq \frac{1}{2S(t)\left[1+2\int_0^h f(s)I(t-s)ds\right]+1},\ \forall t\geq 0$$

$$(12)$$

III. EQUILIBRIUM POINTS

The disease-free and endemic equilibrium points are characterized and discussed in the following:

Theorem 2. The disease-free equilibrium point of (1)-(6) is

$$x_1^* = \left(S_1^*, E_1^*, I_1^*, R_1^*\right)^T = \left(\frac{1-k_4^*}{1+k_1^*\left(1+\alpha_S^*\right)}, 0, 0, \frac{k_4^*+k_1^*\left(1+\alpha_S^*\right)}{1+k_1^*\left(1+\alpha_S^*\right)}\right)^T \qquad (13)$$

and the corresponding equilibrium vaccination value and total population are, respectively, $V^* = R^* = \frac{k_4^*+k_1^*\left(1+\alpha_S^*\right)}{1+k_1^*\left(1+\alpha_S^*\right)}$ and $N^* = S^* + R^* = 1$ if

$$\lim_{t\to\infty} k_i(t) = k_i^* \;;\quad i=1,2,3,4 \;,\quad \lim_{t\to\infty}\alpha_S(t) = \alpha_S^* \;,$$
$$\lim_{t\to\infty}\alpha_I(t) = \alpha_I^* \qquad (14)$$

and that for the disease-free equilibrium point (13), one has

$$\max_{1\leq i\leq 4} k_i^* \leq \frac{1}{1+\left(1+\alpha_S^*\right)S_1^*} = \frac{1+k_1^*\left(1+\alpha_S^*\right)}{1+\left(1+\alpha_S^*\right)\left(1+k_1^*-k_4^*\right)}$$
$$,\quad \forall t\geq 0 \qquad (15)$$

If $k_4^* = (1-\alpha)k_1^*$ for $\alpha\in[0,1]$ then k_1^* and k_4^* satisfy (15) if $k_1^* \in \left[0, \frac{\sqrt{1+4\alpha\left(1+\alpha_S^*\right)}-1}{2\left(1+\alpha_S^*\right)}\right)$.

Theorem 3. The following properties hold:

(i) A necessary condition for an equilibrium point $x_2^* = \left(S_2^*, E_2^*, I_2^*, R_2^*\right)^T$ to exist being an endemic equilibrium (that is, $E_2^* I_2^* \neq 0$) of (1)-(6) is that the infectivity disease rate is large enough satisfying $\beta > \frac{(b+\kappa)(b+\gamma)}{\kappa}$.

(ii) A necessary condition for such an endemic equilibrium point to exist under a saturation-free equilibrium vaccination V_2^* is that $V_2^* \in \left[0, 1-\frac{(b+\kappa)(b+\gamma)}{\beta\kappa}\right)$.

ISBN: 978-988-19253-4-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

(iii) The exposed and infectious subpopulations of the endemic equilibrium point satisfy the constraints:

$$E_2^* \in \left(0 , \frac{b+\gamma}{b+\gamma+\kappa} \frac{\beta\kappa-(b+\kappa)(b+\gamma)}{\beta\kappa} \right)$$

$$I_2^* \in \left(0 , \frac{\beta\kappa-(b+\kappa)(b+\gamma)}{\beta(b+\gamma+\kappa)} \right) \quad (16)$$

(iv) If $k_1^* < \frac{\beta\kappa-(b+k)(b+\gamma)}{1+\alpha_S^*}$ then there is a unique feasible,in the sense that all its components are positive, endemic equilibrium point.

The local stability of the disease-free equilibrium point is now characterized.

Theorem 4. The following properties hold:

(i) The delay-free disease-free equilibrium point of the deterministic SEIR model (1)-(6) under a linear limiting control satisfying the conditions of Theorem 1 is locally uniformly asymptotically stable. In the presence of distributed delay, the system is still locally uniformly asymptotically stable if the transfer matrix $\left(sI-A_1^*\right)^{-1}\hat{H}(s)$ is in $\boldsymbol{RH}_\infty^{4\times 4}$ with H_∞ norm

$$\left\| \left(sI-A_1^*\right)^{-1}\hat{H}(s) \right\|_\infty < 1,$$

where $\hat{H}(s)=\beta S_1^*\left(1-e^{hs}\right)e_{12}e_3^T\hat{f}(s)$, where s denotes the Laplace transform argument, $\hat{f}(s)=L(f(t))$, $e_{12}=(-1,0,0,1)^T$ $e_3=(0,0,1,0)^T$, and

$$A_1^* = \begin{bmatrix} -b\left(1+k_3^*\left(1+\alpha_S^*\right)S_1^*\right) & 0 & 0 & 0 \\ 0 & -(b+\kappa) & 0 & 0 \\ 0 & \kappa & -(b+\gamma) & 0 \\ bk_3^*\left(1+\alpha_S^*\right)S_1^* & 0 & \gamma & -b \end{bmatrix}$$

$$= \begin{bmatrix} -b\left(1+k_3^*\left(1+\alpha_S^*\right)\frac{1-k_4^*}{1+k_1^*\left(1+\alpha_S^*\right)}\right) & 0 & 0 & 0 \\ 0 & -(b+\kappa) & 0 & 0 \\ 0 & \kappa & -(b+\gamma) & 0 \\ bk_3^*\left(1+\alpha_S^*\right)\frac{1-k_4^*}{1+k_1^*\left(1+\alpha_S^*\right)} & 0 & \gamma & -b \end{bmatrix} \quad (17)$$

(ii) A sufficient condition for Property (i) to hold is that the infective disease rate be small enough to satisfy $\beta < 2/\left(S_1^* \left\| \hat{f}(s) \right\|_\infty \left\| \left(sI-A_1^*\right)^{-1} \right\|_\infty \right)$.

Remark 2. Note that, since all the eigenvalues of A_1^* are negative then $\left(sI-A_1^*\right)^{-1} \in \boldsymbol{RH}_\infty$. In the delayed case, the reproduction number R_p is defined as:

$$R_p = \left\| e_3^T\left(s\boldsymbol{I}-A_1^*\right)^{-1}\hat{H}(s) \right\|_\infty = \sup_{\omega\in\boldsymbol{R}_{0+}} \left\| \left(\boldsymbol{i}\omega\boldsymbol{I}-A_1^*\right)^{-1}\hat{H}(\boldsymbol{i}\omega) \right\|_\infty$$

with $\boldsymbol{R}_{0+}=\boldsymbol{R}\cup\{0\}$, e_3 being the third unity vector in the canonical basis of $\boldsymbol{R}^4$ and $\boldsymbol{i}=\sqrt{-1}$. If $R_p<1$, equivalently, if $\beta < 2/\left(S_1^* \left\| \hat{f}(s) \right\|_\infty \left\| \left(sI-A_1^*\right)^{-1} \right\|_\infty \right)$, then the disease-free equilibrium point is locally asymptotically stable. If $R_p>1$, the infection propagates. Note that $R_p<1$ is equivalent to $\hat{B}(s)=e_3^T\left(s\boldsymbol{I}-A_1^*\right)^{-1}\hat{H}(s)$ being bounded real (i.e. Schur with real coefficients) and to the transfer function $\hat{S}(s)=\frac{1+\hat{B}(s)}{1-\hat{B}(s)}$ to be strictly positive real, [5-6].

IV. NUMERICAL SIMULATION WITH FURTHER ANALYSIS

This section contains some numerical examples illustrating the theoretical results introduced in the

ISBN: 978-988-19253-4-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

previous sections. The subsequent extended stochastic version of the deterministic model (1)-(6) is stated by modifying (1)-(4) as follows:

$$\dot{S}(t) = b(1-V(t)-S(t)) - \beta S(t)\int_0^h f(\tau)I(t-\tau)d\tau + \sigma_1\left(S(t)-S^*\right)\dot{w}_1(t) \quad (17)$$

$$\dot{E}(t) = \beta S(t)\int_0^h f(\tau)I(t-\tau)d\tau - (b+\kappa)E(t) + \sigma_2\left(E(t)-E^*\right)\dot{w}_2(t) \quad (18)$$

$$\dot{I}(t) = \kappa E(t) - (b+\gamma)I(t) + \sigma_3\left(I(t)-I^*\right)\dot{w}_3(t) \quad (19)$$

$$\dot{R}(t) = bV(t) + \gamma I(t) - bR(t) + \sigma_4\left(R(t)-R^*\right)\dot{w}_4(t) \quad (20)$$

where $\dot{w}_i(t)$ are mutually independent standard Wiener processes $(0,1)$, i.e. mutually independent definite integrals from zero to time t of a zero mean, unit variance white Gaussian stochastic processes, that is $w_i(0)=0$, $E[dw_i(t)]=0$, $E\left[w_i^2(t)\right]=t$; $i=1,2,3,4$ for $t\geq 0$ with E denoting expectation, the functions $t\to w_i(t)$; $i=1,2,3,4$ are almost everywhere surely continuous and σ_i; $i=1,2,3,4$ are real parameters. The parameters of the model are given by $b^{-1}=25.5$ days, $\kappa^{-1}=2.2$ days $\beta=1.66$ days^{-1}, $h=3.5$ days and $\gamma=\kappa$. The function $f(t)$ is defined by $f(t)=0$ for $t\in[0,1.5)$, $f(t)=\Lambda/N(t)$ for $t\in[1.5,3.5]$ in similarity with the standard incidence rate for delay-free models. Λ is the normalization constant guaranteeing $\int_0^h f(\tau)d\tau=1$. The initial conditions are $S(0)=0.5$, $E(0)=0.1$, $I(0)=0.2$ and $R(0)=0.2$ so that the initial total population is $N(0)=1$. The constant values $\alpha_S(t)=\alpha_S^*=-0.05$, $\alpha_I(t)=\alpha_I^*=0.1$ and $\alpha_R(t)=\alpha_R^*=-0.1$ are used for simulation purposes. The Wiener processes parameters are $\sigma_1=0.1$, $\sigma_2=0.2$, $\sigma_3=0.3$ and $\sigma_4=0.2$. Figure 1 shows the final values achieved by the trajectory of the system in the absence of vaccination for the deterministic case. The system gets the endemic equilibrium values $S_2^*=0.32$, $E_2^*=0.05$, $I_2^*=0.05$ and $R_2^*=0.58$ By comparing Figures 1 and 2 we can see that both, the deterministic and stochastic systems possess the same equilibrium points.

The vaccination control law given by (5)-(6) is now applied to the system in order to eradicate the illness from the population. The control parameters $k_1(t)=k_1^*=0.6>0$ and $k_4(t)=k_4^*=-0.7<0$ provide an stability abscissa for A_{10}^* of $\rho_0=0.0016$. Moreover, $k_3(t)=k_3^*$ is selected to be $k_3(t)=k_3^*=-0.001$.

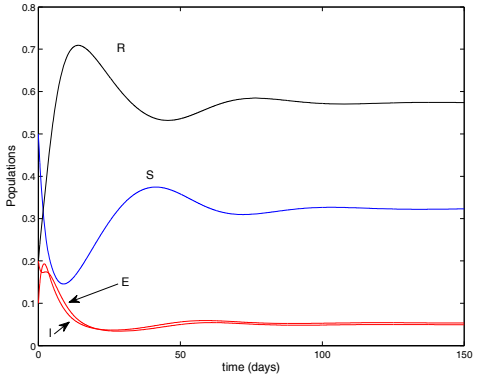

Figure 1. Deterministic endemic equilibrium point in the absence of vaccination

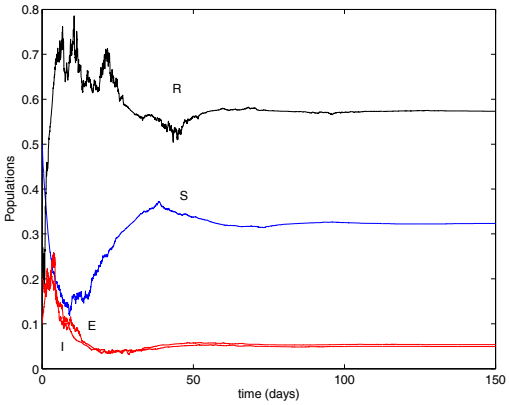

Figure 2. Stochastic endemic equilibrium point in the absence of vaccination

ISBN: 978-988-19253-4-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

The control parameter function $k_2(t)$ does not have to satisfy any special requirement that cannot be accomplished by using the others. Therefore, it will be fixed to zero, $k_2(t)=0$, for the sake of simplicity. The last control parameter function $k_5(t)$ is potentially time-varying since its purpose is to guarantee that the control law always lies within $\bar{V}(t)\in[0,1]$ (the linear feedback condition). It can be seen in Figure 3 that the disease is removed asymptotically from the population since the exposed and infectious subpopulations converge to zero. Figure 4 displays the time evolution of the vaccination. It is confined to the interval $[0,1]$ by the action of the control function $k_5(t)$, depicted in Figure 5. It can be seen in Figure 3 that $R_1^*=0.75$ which is exactly the equilibrium value of the vaccination as Figure 4 reveals. These results also hold in the stochastic case. Thus, Figure 6 displays the system's trajectory when a Wiener process is added to the system dynamics while Figure 7 shows the corresponding vaccination function. Therefore, we can see in Figures 6 and 7 that the disease is asymptotically removed, the percentages of susceptible and immune correspond to those selected beforehand and the vaccination function converges to the value of immune at equilibrium. The solution of the SEIR model under the standard independent Wiener processes (17)-(20) and the vaccination feedback law (5)-(6) of the given class is given by

$$x(t)=\Phi(t)x_0+\int_0^t \Phi(t-\tau)\left(\left(A(\tau)-A_{10}^*\right)x(\tau)+v(\tau)-\Omega\, x_1^*\,\dot{w}(\tau)\right)d\tau$$

$\forall t\geq 0$, where the evolution operator is given by:

$$\Phi(t)=\Psi_s(t)\Phi_d(t)=\Psi_w(t)\Psi_0(t)\Phi_d(t)=e^{\Omega w(t)+\left(A_{10}^*-\Omega^2/2\right)t} \quad , \quad \forall t\geq 0$$

with

$$\Psi_s(t)=e^{\Omega w(t)-\left(\Omega^2/2\right)t},$$

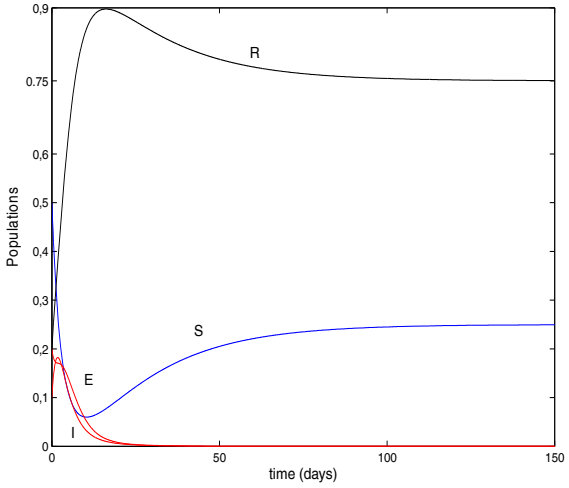

Figure 3. State trajectory when the feedback control law is applied. Deterministic case

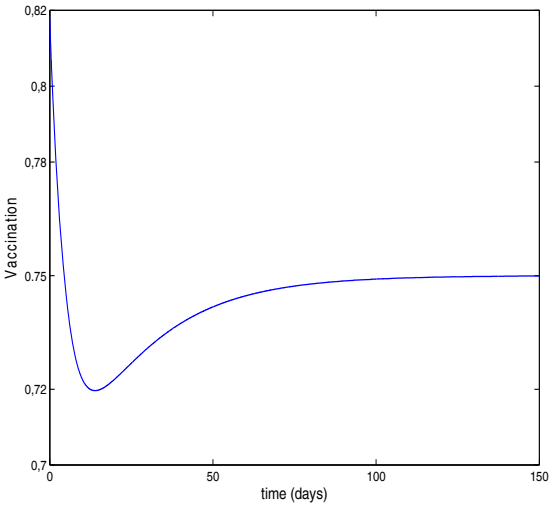

Figure 4. Vaccination law. Deterministic case

$$\Psi_w(t)=e^{\Omega w(t)},\ \Psi_0(t)=e^{-\left(\Omega^2/2\right)t} \text{ and } \Phi_d(t)=e^{A_{10}^* t}$$

, so that

$$\Phi_d(0)=E[\Phi(0)]=E[\Psi_s(0)]=E[\Psi_0(0)]=E[\Psi_s(t)]=\boldsymbol{I},$$

where $\boldsymbol{I}$ is the identity matrix The evolution operator $\Phi(t)$ follows for a Wiener- type forced differential process of the form

ISBN: 978-988-19253-4-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$dx(t) = A_0 x(t)\,dt + \Omega x(t)\,dw(t) + F(t)$, $\forall t \geq 0$, [4], with homogeneous part $dx_0(t) = A_0 x(t)\,dt + \Omega x(t)\,dw(t)$.

We can get after some calculations the subsequent result:

Theorem 5: $\lim_{t\to\infty} E[x(t)] = x_1^*$ if Theorem 1, related to the deterministic version of the SEIR model.

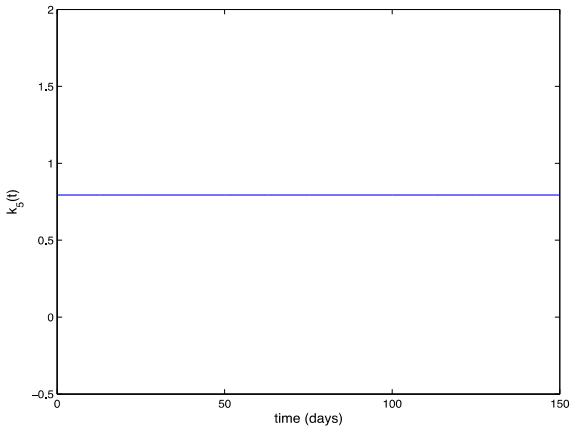

Figure 5. Evolution of $k_5(t)$

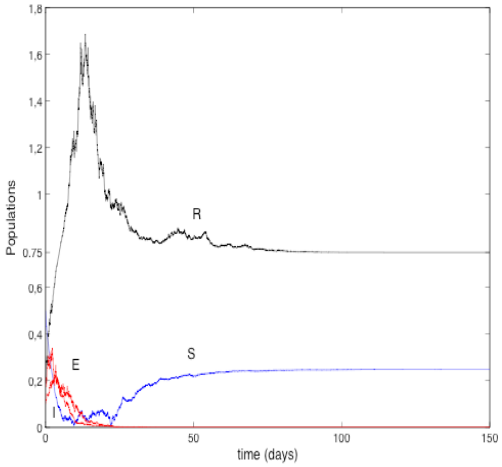

Figure 6. State trajectory when the feedback control law is applied. Stochastic case

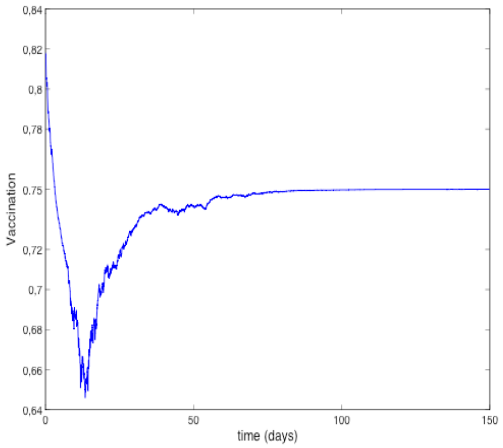

Figure 7. Vaccination law. Stochastic case free equilibrium point is lost

ACKNOWLEDGMENT

The authors are very grateful to the Spanish Government for Grants DPI2012-30651and DPI2013-47825-C3-1-R, and to the Basque Government and UPV/EHU for Grant IT378-10, SAIOTEK S-PE13UN039 and UFI 2011/07.

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