Realization of Program Motion of a Gyrostat with Variable Inertia Moments

Sergey Bezglasnyi

***Abstract*—In paper the task about stabilization of program motions of a balanced gyrostat with time-dependent inertia moments is considered. The active stabilizing control attached to the gyrostat by the principle of feedback is constructed. Conditions under which the desirable program motions property of asymptotic stability is possible are received. The task is solved on the base of a method of Lyapunov functions and a method of the limit equations and the limit systems.**

***Index Terms*—Gyrostat, active control, Lyapunov function, feedback, stabilization.**

I. Introduction

PROBLEMS about spatial orientation of satellites and aircraft in an orbit have important applied value and are widely considered by authors in many notes. Spatial motions of aircraft concerning the center of masses are modeled by spherical motions of solid bodies or systems of bodies, in particular, gyrostats. The basic methods and principles of control of rotational motions of bodies and systems were studied, for example, in notes [1-3]. Modern domestic and foreign writers investigate tasks about stability of equilibrium positions and stationary motions of gyrostats in orbits [4, 5], about resonant and chaotic modes of motions [6, 7], about stabilization of the set program motions of gyrostats of various structure [8, 9].

This paper is devoted to research of opportunities of realization of program motions concerning the center of masses of two coaxial bodies system (gyrostat) of variable structure. As a gyrostat of variable structure two bodies – the carrier and the rotor allowing relative rotation round the general axis are considered. The rotor has time-dependent inertia moments. In paper the problem of stabilization is solved by active external control by the principle of feedback. The presented results are received on the base of a method of Lyapunov functions of the classical stability theory and of a method of the limit equations and limit systems [10].

II. Problem Definition and Motion Equations

Let $O'\xi\eta\zeta$ is an inertial coordinate system; $O\alpha\beta\gamma$ is coordinate system randomly moving generally in relation to inertial system of coordinates. The gyrostat represents system of two solid bodies. Body T_1 is the carrier and body T_2 is rotor which rotates around with the angular speed $\bar{\omega}^{02} = (0, 0, \dot{\delta})^{\mathrm{T}}$ directed on an axes Oz of not inertial coordinate system $Oxyz$, rigidly connected with the carrier (the symbol $(\ )^{\mathrm{T}}$ denotes transposing). The angle $\delta = \delta(t)$ of rotor rotation is a continuous function of time. $Ox_2y_2z_2$ is rigidly connected with the rotor coordinate system. Axis Oz and Oz_2 are coincide.

We will assume that the centers of mass O_1 and O_2 both bodies are on the general axis of rotation and don't change the positions ($O_1O_2 = const$), the general center of mass of a gyrostat is in a point O (Fig. 1). We will consider that the rotor have the time-dependent inertia moments, and not changing positions of the centers of mass of both bodies. As a result the center of mass of all system have constant position in the point O. The condition about dependence of a tensor of inertia of a body on time allows to characterize existence of mobile parts of a design, redistribution or motion of masses in a gyrostat.

Let are set two mutually perpendicular basis vectors $\bar{s}_{01}$

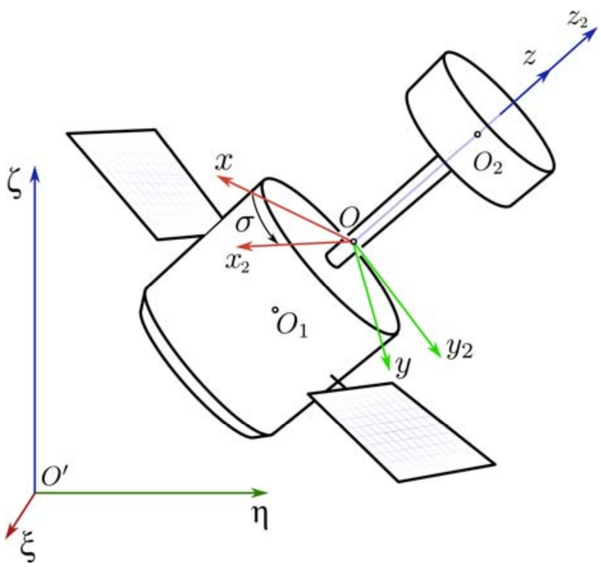

Fig. 1. Gyrostat.

and $\bar{s}_{02}$, holding invariable position in system of coordinates $O\alpha\beta\gamma$, and let are set two mutually perpendicular basis vectors $\bar{r}_{01}$ and $\bar{r}_{02}$, permanently connected with system of coordinates $Oxyz$. We will put: $\bar{s}_{03} = \bar{s}_{01} \times \bar{s}_{02}$, $\bar{r}_{03} = \bar{r}_{01} \times \bar{r}_{02}$.

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We now state the task about realization and stabilization of program motion of gyrostat (or about three-axis orientation [8]). Namely, we have to find the attached to carrier control moment $\bar{M}_C$ which would stabilize basis vector $\bar{r}_{01}$ in the direction $\bar{s}_{01}$ and basis vector $\bar{r}_{02}$ in the direction $\bar{s}_{02}$. Then according to basis vector $\bar{r}_{03}$ will be focused in the direction $\bar{s}_{03}$.

Let the coordinate system $O\alpha\beta\gamma$ rotates with the angular speed $\bar{\omega}_0 = \bar{\omega}_0(t)$ rather motionless inertial coordinate system $O'\xi\eta\zeta$. Here the function of time $\bar{\omega}_0 = \bar{\omega}_0(t)$ is the program motion of the gyrostat.

According to the theorem of change of the kinetic moment of system the motion equation of the first body we will take in the form

$$\frac{\tilde{d}\bar{K}_1}{dt} + \bar{\omega}\times\bar{K}_1 = \bar{M}_1' + \bar{M}_C + \bar{M}_2 \tag{1}$$

where the sign "tilde" over a sign of differential d denotes a capture of a local derivative, that is in mobile system $Oxyz$, $\bar{\omega}$ is the angular speed of rotation of the carrier in coordinate system $Oxyz$; $\bar{K}_1 = I_1\bar{\omega}$ is the kinetic moment of the carrier, $I_1 = const$ is its tensor of inertia, $\bar{M}_1'$ is the torque of the external forces attached to the carrier, $\bar{M}_C$ is the control torque, $\bar{M}_2$ is the torque operating on the carrier from the rotor. Thus, the motion equation (1) can be written in a look:

$$I_1\dot{\bar{\omega}} + \bar{\omega}\times I_1\bar{\omega} = \bar{M}_1' + \bar{M}_C + \bar{M}_2, \tag{2}$$

where the point denotes a time-derivative. The equation of motion of the rotor in rigidly connected with the carrier coordinate system $Oxyz$:

$$\hat{\delta}\left[\frac{\tilde{d}\bar{K}_2}{dt} + \bar{\omega}_2\times\bar{K}_2\right] = \bar{M}_2' - \bar{M}_2,\ \hat{\delta} = \begin{pmatrix}\cos\delta & -\sin\delta & 0\\ \sin\delta & \cos\delta & 0\\ 0 & 0 & 1\end{pmatrix} \tag{3}$$

Here $\bar{\omega}_2 = \hat{\delta}^{-1}(\bar{\omega} + \bar{\omega}^{02})$ is the absolute angular speed of the rotor in coordinate system $Oxyz$, $\bar{K}_2 = I_2\hat{\delta}^{-1}(\bar{\omega} + \bar{\omega}^{02})$ is its kinetic moment and $I_2 = I_2(t)$ is its tensor of inertia, $\bar{\omega}^{02} = (0,0,\dot{\delta})^{\mathrm{T}}$ is the angular speed of rotation of the rotor concerning the carrier, $\bar{M}_2'$ is the torque of the external forces attached to the rotor, and $-\bar{M}_2$ is the torque twisting the rotor concerning the carrier. Matrix $\hat{\delta}$ is a transition tensor from coordinate system $Ox_2y_2z_2$ to coordinate system $Oxyz$. We will write the equation (3) in a look:

$$\begin{aligned}&\hat{\delta}I_2\hat{\delta}^{-1}(\dot{\bar{\omega}} + \dot{\bar{\omega}}^{02}) + \hat{\delta}\dot{I}_2\hat{\delta}^{-1}(\bar{\omega} + \bar{\omega}^{02}) + \hat{\delta}I_2\dot{\hat{\delta}}^{-1}(\bar{\omega} + \bar{\omega}^{02}) +\\ &+(\bar{\omega} + \bar{\omega}^{02})\times I_2\hat{\delta}^{-1}(\bar{\omega} + \bar{\omega}^{02}) = \bar{M}_2' - \bar{M}_2.\end{aligned} \tag{4}$$

From the equations (2) and (4) we will receive the motion equations of all system concerning the center of mass:

$$\begin{aligned}&(I_1 + \hat{\delta}I_2\hat{\delta}^{-1})\dot{\bar{\omega}} + (\hat{\delta}\dot{I}_2\hat{\delta}^{-1} + \hat{\delta}I_2\dot{\hat{\delta}}^{-1})\left(\bar{\omega} + \bar{\omega}^{02}\right) + \bar{\omega}\times I_1\bar{\omega} +\\ &+\hat{\delta}I_2\hat{\delta}^{-1}\dot{\bar{\omega}}^{02} + (\bar{\omega} + \bar{\omega}^{02})\times I_2\hat{\delta}^{-1}\left(\bar{\omega} + \bar{\omega}^{02}\right) = \bar{M}' + \bar{M}_C.\end{aligned} \tag{5}$$

Here torque $\bar{M}' = \bar{M}_1' + \bar{M}_2'$ characterizes impact of external forces on all gyrostat. We will assume further that external forces are absent, that is $\bar{M}' = 0$.

III. Program Control and Stabilizing Control

We will designate the control torque $\bar{M}_C = \bar{M}_p + \bar{M}_s$. Here torque $\bar{M}_p$ is the program control, torque $\bar{M}_s$ is the stabilizing control. If in an initial time point at $t = 0$ a reference point $\bar{r}_{01}$, $\bar{r}_{02}$, $\bar{r}_{03}$ coincides with a reference point $\bar{s}_{01}$, $\bar{s}_{02}$, $\bar{s}_{03}$, then direct substitution $\bar{\omega}_0(t)$ in the motion equations (5) we will calculate the program control torque:

$$\begin{aligned}\bar{M}_p &= (I_1 + \hat{\delta}I_2\hat{\delta}^{-1})\dot{\bar{\omega}}_0 + (\hat{\delta}\dot{I}_2\hat{\delta}^{-1} + \hat{\delta}I_2\dot{\hat{\delta}}^{-1})\left(\bar{\omega}_0 + \bar{\omega}^{02}\right) +\\ &+\hat{\delta}I_2\hat{\delta}^{-1}\dot{\bar{\omega}}^{02} + \bar{\omega}_0\times I_1\bar{\omega}_0 + (\bar{\omega}_0 + \bar{\omega}^{02})\times I_2\hat{\delta}^{-1}(\bar{\omega}_0 + \bar{\omega}^{02})\end{aligned} \tag{6}$$

The program control torque (6) realizes the program motion with an angular speed $\bar{\omega}_0(t)$ of the gyrostat. But in the presence of initial deviations or actions of small perturbations we will constructed the additional stabilizing torque $\bar{M}_s$ which would provide asymptotic stability of this program motion.

Let us introduce the new generalized coordinates (deflections) $\bar{x}$ according to equality

$$\bar{\omega} = \bar{\omega}_0(t) + \bar{x}\,. \tag{7}$$

We substitute formulas (6) and deflections in equation (5), than we have the control equation in deflections:

$$\begin{aligned}&(I_1 + \hat{\delta}I_2\hat{\delta}^{-1})\dot{\bar{x}} + (\hat{\delta}\dot{I}_2\hat{\delta}^{-1} + \hat{\delta}I_2\dot{\hat{\delta}}^{-1})\bar{x} +\\ &+\bar{x}\times I_1\bar{\omega}_0 + \left(\bar{x} + \bar{\omega}_0\right)\times I_1\bar{x} + \bar{x}\times I_2\hat{\delta}^{-1}(\bar{\omega}_0 + \bar{\omega}^{02} + \bar{x}) +\\ &+(\bar{\omega}_0 + \bar{\omega}^{02})\times I_2\hat{\delta}^{-1}\bar{x} = \bar{M}_s.\end{aligned} \tag{8}$$

We choose the stabilizing torque in the form

$$\bar{M}_s = -B\bar{x} + \sum_{i=1}^{3}\alpha_i(\bar{r}_{0i}\times\bar{s}_{0i}),\ \alpha_i = const > 0,\ i = 1,2,3. \tag{9}$$

Here $B = B(t)$ is the symmetric matrix of size three on three which is subject to definition. For vectors $\bar{s}_{0i}$ we have:

$$\dot{\bar{s}}_{0i} = -(\bar{\omega}_0 + \bar{x})\times\bar{s}_{0i},\quad i = 1,2,3. \tag{10}$$

The equation system (8), (10) with control (9) on a set

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$\{x = 0\}$ has only equilibrium positions:

$$\bar{x} = 0,\ \bar{s}_{01} = \pm\bar{r}_{01},\ \bar{s}_{02} = \pm\bar{r}_{02},\ \bar{s}_{03} = \bar{s}_{01} \times \bar{s}_{02}. \quad (11)$$

Derivative of Lyapunov function

$$V = \frac{1}{2}(\bar{x}^T (I_1 + \hat{\delta} I_2 \hat{\delta}^{-1})\bar{x} + \sum_{i=1}^{3} \alpha_i (\bar{r}_{0i} - \bar{s}_{0i})^2) \quad (12)$$

owing to system (8), (10) with (9) without composed above the second order of a small will have an appearance:

$$\begin{aligned} &\frac{dV}{dt} \cong -\bar{x}^T B\bar{x} - \bar{x}^T C\bar{x} - \\ &-\frac{1}{2}\bar{x}^T (\hat{\delta}\dot{I}_2\hat{\delta}^{-1} - \dot{\hat{\delta}} I_2 \hat{\delta}^{-1} + \hat{\delta} I_2 \dot{\hat{\delta}}^{-1})\bar{x}. \end{aligned} \quad (13)$$

Here elements of a symmetric matrix $C = \{c_{ij}\}$ are defined by formulas

$$\begin{aligned}
&c_{11} = (\omega_{02} + \omega_2^{02})(I_{2zx}\cos\delta - I_{2zy}\sin\delta) + \\
&+(\omega_{03} + \omega_3^{02})(I_{2yx}\sin\delta + I_{2zy}\cos\delta), \\
&2c_{12} = 2c_{21} = -(\omega_{01} + \omega_1^{02})(I_{2zx}\cos\delta - I_{2zy}\sin\delta) + \\
&+(\omega_{02} + \omega_2^{02})(I_{2zx}\sin\delta + I_{2zy}\cos\delta) + (\omega_{03} + \omega_3^{02})\cdot \\
&\cdot((I_{2xx} - I_{2yy})\cos\delta - 2I_{2xy}\sin\delta), \\
&c_{22} = (\omega_{03} + \omega_3^{02})(I_{2xx}\sin\delta + I_{2xy}\cos\delta) - \\
&-(\omega_{01} + \omega_1^{02})(I_{2zx}\sin\delta + I_{2zy}\cos\delta), \\
&2c_{13} = 2c_{31} = (\omega_{01} + \omega_1^{02})(I_{2yx}\cos\delta - I_{2yy}\sin\delta) + \\
&+(\omega_{02} + \omega_2^{02})(I_{2zz} - I_{2xx}\cos\delta + I_{2yx}\sin\delta) - \left(\omega_{03} + \omega_3^{02}\right) I_{2yz}, \\
&2c_{23} = 2c_{32} = (\omega_{01} + \omega_1^{02})(I_{2yx}\sin\delta + I_{2yy}\cos\delta - I_{2zz}) - \\
&-(\omega_{02} + \omega_2^{02})(I_{2xx}\sin\delta + I_{2yx}\cos\delta) + \left(\omega_{03} + \omega_3^{02}\right) I_{2xz}, \\
&c_{33} = (\omega_{01} + \omega_1^{02}) I_{2yz} - (\omega_{02} + \omega_2^{02}) I_{2xz}.
\end{aligned} \quad (14)$$

We will define matrix B elements according to a condition:

$$\begin{aligned} &\bar{x}^T \left[B + C + 0.5(\hat{\delta}\dot{I}_2\hat{\delta}^{-1} - \dot{\hat{\delta}} I_2 \hat{\delta}^{-1} + \hat{\delta} I_2 \dot{\hat{\delta}}^{-1}) \right] \bar{x} \geq \\ &\geq b_0 \left\| \bar{x} \right\|^2, \quad b_0 = \text{const} > 0. \end{aligned} \quad (15)$$

Then the matrix $\left[B + C + 0.5(\hat{\delta}\dot{I}_2\hat{\delta}^{-1} - \dot{\hat{\delta}} I_2 \hat{\delta}^{-1} + \hat{\delta} I_2 \dot{\hat{\delta}}^{-1}) \right]$ will be positive definite, and the derivative (13) of Lyapunov function will be negative definite determined by speeds. The set on which the derivative is equal to zero, is a set $\{x = 0\}$. The system limit to system (8), (10) with (9) on a set $\{x = 0\}$ has no other decisions, except (11).

Therefore on the basis of the theorem from paper [10] we will receive that equilibrium position $\bar{x} = 0,\ \bar{s}_{0i} = \bar{r}_{0i}$ is asymptotically stable. Any other equilibrium positions are unstable. Thereby we have that the control torque $\bar{M}_C = \bar{M}_p + \bar{M}_s$ defined from formulas (6), (9), (14) by condition (15) solves the task of realization of program motion of the gyrostat with variable inertia moments.

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