On Computing the Vandermonde Matrix Inverse

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***Abstract* — A simple and efficient approach for computing the inverse of Vandermonde matrix is presented in this paper, which is based on synthesis division of polynomials. This approach does not involve matrix multiplication, computation of determinant or solving a system of linear equations for determining the unknown elements of the inverse matrix. Some illustrative examples are provided.**

***Index Terms* — Vandermonde matrix, matrix inverse, synthetic division.**

I. Introduction

THE Vandermonde matrix (VDM) has important applications in various areas such as polynomial interpolation, signal processing, curve fitting, coding theory and control theory [2, 4, 6, 9], etc. Since the last decade, the study of more efficient approaches for computing the inverse of VDM or its generalized version has been an important research topic in many mathematics and sciences disciplines. In this paper, we present a novel simple and efficient approach for finding the inverse of VDM, based on synthesis division of polynomials and some previous works of the author and the others [5, 7, 8, 10].

The whole paper is organized like this. The basic mathematical background is described in section 2. Then, the new computational approach is discussed in section 2, followed by some numerical examples in section 3. Finally, some concluding remarks are provided in section 4.

II. Mathematical Background

Consider the polynomial:

$$f(x) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

where a_i, λ_i are known constants. We are interested to find the inverse of the following Vandermonde matrix:

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ {\lambda_1}^2 & {\lambda_2}^2 & \cdots & {\lambda_n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ {\lambda_1}^{n-1} & {\lambda_2}^{n-1} & \cdots & {\lambda_n}^{n-1} \end{pmatrix}$$

According to [10], the formula $V^{-1} = W \times A$ can be applied to compute the inverse of V, where the matrices W and A are defined by:

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$$W = \begin{pmatrix} \frac{{\lambda_1}^{n-1}}{\prod_{j\neq 1}(\lambda_1-\lambda_j)} & \frac{{\lambda_1}^{n-2}}{\prod_{j\neq 1}(\lambda_1-\lambda_j)} & \cdots & \frac{1}{\prod_{j\neq 1}(\lambda_1-\lambda_j)} \\ \frac{{\lambda_2}^{n-1}}{\prod_{j\neq 2}(\lambda_2-\lambda_j)} & \frac{{\lambda_2}^{n-2}}{\prod_{j\neq 2}(\lambda_2-\lambda_j)} & \cdots & \frac{1}{\prod_{j\neq 2}(\lambda_2-\lambda_j)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{{\lambda_n}^{n-1}}{\prod_{j\neq n}(\lambda_n-\lambda_j)} & \frac{{\lambda_n}^{n-2}}{\prod_{j\neq n}(\lambda_n-\lambda_j)} & \cdots & \frac{1}{\prod_{j\neq n}(\lambda_n-\lambda_j)} \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & 1 \end{pmatrix}.$$

with $a_1 = -\sum \lambda_j$, $a_2 = \sum_{j\neq m} \lambda_j\lambda_m$, $a_3 = -\sum_{j\neq m\neq s} \lambda_j\lambda_m\lambda_s$, ..., and $a_n = (-1)^n \prod \lambda_j$. However, the computational cost could be high if direct matrix multiplications are applied to compute V^{-1}. In the next section, we will introduce a new approach to compute V^{-1} by means of synthetic division of polynomials.

III. A New Approach

Let us see how the elements of V^{-1} look like, when the formula $V^{-1} = W \times A$ is applied to the cases $n = 2$ and $n = 3$.

(i) When n= 2, we have:

$$V = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix},\ A = \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix},\ W = \begin{pmatrix} \frac{\lambda_1}{\lambda_1-\lambda_2} & \frac{1}{\lambda_1-\lambda_2} \\ \frac{\lambda_2}{\lambda_2-\lambda_1} & \frac{1}{\lambda_2-\lambda_1} \end{pmatrix}$$

where $a_1 = -\lambda_1 - \lambda_2$.

Hence,

$$V^{-1} = W \times A = \begin{pmatrix} \frac{\lambda_1+a_1}{\lambda_1-\lambda_2} & \frac{1}{\lambda_1-\lambda_2} \\ \frac{\lambda_2+a_1}{\lambda_2-\lambda_1} & \frac{1}{\lambda_2-\lambda_1} \end{pmatrix}$$

If we apply synthetic division to the $[f(x) - a_2] \div (x - \lambda_1)$, where $f(x) = (x-\lambda_1)(x-\lambda_2) = x^2 + a_1x + a_2$, we have

$$\begin{array}{c|cc} \lambda_1 & 1 & a_1 \\ & & \lambda_1 \\ \hline & 1 & \lambda_1 + a_1 \end{array}$$

We can see that the elements obtained by synthetic division is equal to the numerators of the elements in the first row of V^{-1},

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except in the reverse order. By applying a further synthetic division to the answer obtained, we get

λ_1	1	a_1
		λ_1
	1	$\lambda_1 + a_1$
		λ_1
	1	$\lambda_1 - \lambda_2$

since $2\lambda_1 + a_1 = \lambda_1 - \lambda_2$. Notice that the element in bracket, namely $\lambda_1 - \lambda_2$, is equal to the denominators of the elements in the first row of V^{-1}. Similarly, if we apply synthetic divisions to $[f(x) - a_2] \div (x - \lambda_2)$, we have

λ_2	1	a_1
		λ_2
	1	$\lambda_2 + a_1$
		λ_2
	1	$\lambda_2 - \lambda_1$

since $2\lambda_2 + a_1 = \lambda_2 - \lambda_1$. In other words, all the elements of V^{-1} can be determined completely by using synthetic division of polynomials only, without having to use matrix multiplications or computation of determinant.

(i) When n= 3, we have:

$$V = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix}, A = \begin{pmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_2 & a_1 & 1 \end{pmatrix}$$

$$W = \begin{pmatrix} \frac{\lambda_1^2}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} & \frac{\lambda_1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} & \frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} \\ \frac{\lambda_2^2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} & \frac{\lambda_2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} & \frac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} \\ \frac{\lambda_3^2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} & \frac{\lambda_3}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} & \frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} \end{pmatrix}$$

where $a_1 = -\lambda_1 - \lambda_2 - \lambda_3$ and $a_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$.

Now,

$$V^{-1} = \begin{pmatrix} \frac{\lambda_1^2 + a_1\lambda_1 + a_2}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} & \frac{\lambda_1 + a_1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} & \frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} \\ \frac{\lambda_2^2 + a_1\lambda_2 + a_2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} & \frac{\lambda_2 + a_1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} & \frac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} \\ \frac{\lambda_3^2 + a_1\lambda_3 + a_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} & \frac{\lambda_3 + a_1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} & \frac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} \end{pmatrix}$$

Applying synthetic divisions to the $[f(x) - a_3] \div (x - \lambda_1)$, where $f(x) = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3) = x^3 + a_1x^2 + a_2x + a_3$, we have

λ_1	1	a_1	a_2
		λ_1	$\lambda_1^2 + a_1\lambda_1$
	1	$\lambda_1 + a_1$	$\lambda_1^2 + a_1\lambda_1 + a_2$
		λ_1	$\lambda_1^2 - \lambda_1\lambda_2 - \lambda_1\lambda_3$
	1	$\lambda_1 - \lambda_2 - \lambda_3$	$(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)$

since $2\lambda_1^2 + a_1\lambda_1 + a_2 - \lambda_1\lambda_2 - \lambda_1\lambda_3 = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)$.

Next,

λ_2	1	a_1	a_2
		λ_2	$\lambda_2^2 + a_1\lambda_2$
	1	$\lambda_2 + a_1$	$\lambda_2^2 + a_1\lambda_2 + a_2$
		λ_2	$\lambda_2^2 - \lambda_1\lambda_2 - \lambda_2\lambda_3$
	1	$\lambda_2 - \lambda_1 - \lambda_3$	$(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)$

since $2\lambda_2^2 + a_1\lambda_2 + a_2 - \lambda_1\lambda_2 - \lambda_1\lambda_3 = (\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)$.

Also,

λ_3	1	a_1	a_2
		λ_3	$\lambda_3^2 + a_1\lambda_3$
	1	$\lambda_3 + a_1$	$\lambda_3^2 + a_1\lambda_3 + a_2$
		λ_3	$\lambda_3^2 - \lambda_1\lambda_3 - \lambda_2\lambda_3$
	1	$\lambda_3 - \lambda_1 - \lambda_2$	$(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)$

since $2\lambda_3^2 + a_1\lambda_3 + a_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3 = (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)$. Thus, the synthetic division approach works equally well for computing V^{-1} for n = 3.

IV. NUMERICAL EXAMPLES

Example1. Find the inverse of the following Vandermonde matrix.

$$V = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}.$$

Solution. Let us define the following polynomial:

$$f(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6.$$

Applying synthetic divisions, we have:

1	1	-6	11
		1	-5
	1	-5	6
		1	-4
	1	-4	2

So, the first row of V^{-1} is (3 -5/2 1/2).

Next,

2	1	-6	11
		2	-8
	1	-4	3
		2	-4
	1	-2	-1

So, the second row of V^{-1} is (-3 4 -1).

Also,

3	1	-6	11
		3	-9
	1	-3	2
		3	0
	1	0	2

So, the third row of V^{-1} is (1 -3/2 1/2).

Hence,

$$V^{-1} = \begin{pmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{pmatrix}.$$

Example2. Find a quadratic interpolation polynomial which passes through the points (-1, 10), (1, 0) and (2, 4).

Solution. Let the interpolation polynomial be:

$$s(x) = a_1 + a_2x + a_3x^2.$$

Since $s(-1)$=10, $s(1)$=0, $s(2)$=4, so we have:

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$$\begin{pmatrix}10\\0\\4\end{pmatrix}=\begin{pmatrix}1 & -1 & 1\\1 & 0 & 0\\1 & 2 & 4\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}.$$

Consider the following Vandermonde matrix:

$$V=\begin{pmatrix}1 & 1 & 1\\-1 & 0 & 2\\1 & 0 & 4\end{pmatrix}.$$

Let us define a polynomial as follows:

$$f(x)=(x+1)(x-1)(x-2)=x^3-2x^2-x-2.$$

Applying synthetic divisions, we have:

-1	1	-2	-1
		-1	3
	1	-3	2
		-1	4
	1	-4	[6]

So, the first row of V^{-1} is (1/3 -1/2 1/6).

Next,

1	1	-2	-1
		1	-1
	1	-1	-2
		1	0
	1	0	[-2]

So, the second row of V^{-1} is (1 1/2 -1/2).

Also,

2	1	-2	-1
		2	0
	1	0	-1
		2	4
	1	2	[3]

So, the third row of V^{-1} is (-1/3 0 1/3).

Hence,

$$V^{-1}=\begin{pmatrix}1/3 & -1/2 & 1/6\\1 & 1/2 & -1/2\\-1/3 & 0 & 1/3\end{pmatrix}$$

Therefore,

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=\begin{pmatrix}1/3 & 1 & -1/3\\-1/2 & 1/2 & 0\\1/6 & -1/2 & 1/3\end{pmatrix}\begin{pmatrix}10\\0\\4\end{pmatrix}=\begin{pmatrix}2\\-5\\3\end{pmatrix}.$$

The required interpolation polynomial is $s(x)=2-5x+3x^2$.

V. CONCLUDING REMARKS

In this paper, we have introduced a simple and efficient method for computing the inverse of VDM via synthetic division of polynomials, without having to involve matrix multiplication, computation of determinant or solving a system of linear equations for determining the elements of the inverse of VDM. Although we have discussed how to apply this new approach to the cases for n = 2 or 3 only, this approach can be easily adapted to deal with the inverse of VDM for n > 3 without much modifications. Also, it is not hard to see that the total number of arithmetic operations, namely multiplications and additions, involved in using synthetic divisions to compute the elements of the inverse of VDM is comparatively less than that by applying direct matrix multiplication to W×A, it means this new approach has less computational cost. More detail analysis of the complexity of the synthetic division approach and its applications to compute the inverse of the general VDM with size $n \times n$ will be our continued research topic and the findings will be presented or published elsewhere.

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