Homogenization of an Optimal Control Problem in Fixed Domains

Jake Avila, Bituin Cabarrubias

***Abstract*—This work deals with the asymptotic behavior of an optimal control problem based on an elliptic boundary value problem with a linear Robin boundary condition posed in a fixed domain. We consider an L^2-cost functional and use the Periodic Unfolding Method to homogenize the problem. We also show that under this method, the energy corresponding to this cost functional converges. Moreover, we obtain estimates satisfied by the state and control variables.**

***Index Terms*—elliptic problem, homogenization, Robin condition, optimal control, unfolding method.**

I. Introduction

In this paper, we study the homogenization of an optimal control problem based on a linear elliptic boundary value problem with highly oscillating coefficients posed in a fixed domain. The boundary is assumed to be Lipschitz continuous with a prescribed linear Robin condition. An L^2-cost functional is considered in this work.

Optimal controls and homogenization both have a lot of applications in many different fields such as aerospace, process control, robotics, bioengineering, economics, finance, management science, filtration, geomechanics and soil mechanics, petroleum and construction engineering, geosciences, biomedicine and biophysics, among others. The goal of this paper is to yield a new model that can be used in describing some physical phenomenon or in producing a new material for advanced technologies. As far as the authors know, this study is the first in this area.

To study the asymptotic behavior of the problem, we use the Periodic Unfolding Method (PUM), a recently designed technique of homogenization that was first introduced in [6] (see [7] for the detailed proofs and to [14] for an elementary approach) for fixed domains. This technique was extended to perforated domains in [9] (see also [10] for the proofs and [11] for additional applications). One can also check [12] when one wants to consider more general situations and comprehensive presentation and [17] for time-dependent functions. This method was also further stretched to domains with small holes in [13] (see also [27]) and to domain with small holes for time-dependent functions in [5].

As for the homogenization of optimal control problems governed by a linear elliptic equation with linear Robin boundary condition on perforated domains via PUM by considering two types of cost functionals, the reader is referred to [4]. For the asymptotic behavior of the optimal controls based on the wave equation with homogeneous Neumann boundary condition in domains with oscillating boundary and via some compactness properties and evolution triples, see [15]. In [16], the asymptotic behavior of a quasilinear optimal control problem with thick multilevel junction was considered by passing to the limit in the adjoint problem and by using the $\Gamma-$convergence (see [2] and [3] for this technique). The reader is also referred to [18] for the homogenization of an optimal control problem posed in perforated and nonperforated domains via this convergence. As for the multi-scale convergence and $H-$convergence (see [1] for this method) one can check [19] and [20] and [22], respectively. For additional works and applications in this area, the reader is referred to [24], [25], [26], [28] and [30].

This paper is organized as follows: Section II is devoted to the short discussion on PUM for fixed domains, Section III presents the problem, the optimal control results and estimates while Section IV contains the convergence results.

Manuscript received April 13, 2017; revised April 21, 2017.

J. Avila is an undergraduate student of the Institute of Mathematics, University of the Philippines Diliman, 1101 Diliman, Quezon City, PHILIPPINES e-mail: jakeavila2015@gmail.com.

B. Cabarrubias is an assistant professor in the Institute of Mathematics, University of the Philippines Diliman, 1101 Diliman, Quezon City, PHILIPPINES e-mail: bituin@math.upd.edu.ph.

II. Periodic Unfolding Method for Fixed Domains

Let us recall here some notations, definition of the unfolding operator and some of its properties as given in [7].

Let Ω be an open set and $\mathbf{b} = (b_1, b_2, \ldots, b_N)$ be a basis of $\mathbb{R}^N$. Let Y be a reference cell or a set possessing the paving property with respect to $\mathbf{b}$.

For $z \in \mathbb{R}^N$, we denote by $[z]_Y = \sum_{j=1}^{N} k_j b_j$ the unique integer combination of periods with the property that $z - [z]_Y$ belongs to Y, and by $\{z\}_Y$ the difference

$$\{z\}_Y = z - [z]_Y.$$

This means,

$$z = \{z\}_Y + [z]_Y, \quad z \in \mathbb{R}^N.$$

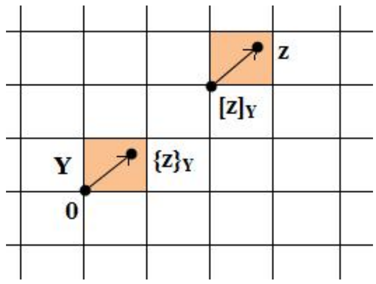

Fig. 1. $\{z\}_Y$ and $[z]_Y$.

Thus, we can write

$$x = \varepsilon \left(\left\{ \frac{x}{\varepsilon} \right\}_Y + \left[\frac{x}{\varepsilon} \right]_Y \right),$$

for all $x \in \mathbb{R}^N$ and for each $\varepsilon > 0$.

ISBN: 978-988-14047-4-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

We also use the notations

$$\begin{aligned}
\Xi_\varepsilon &= \left\{\xi \in \mathbb{Z}^N,\ \varepsilon(\xi + Y) \subset \Omega\right\},\\
\widehat{\Omega}_\varepsilon &= \text{interior}\left\{\bigcup_{\xi\in\mathbb{Z}^N} \varepsilon(\xi + \overline{Y})\right\},\\
\Lambda_\varepsilon &= \Omega \backslash \widehat{\Omega}_\varepsilon.
\end{aligned}$$

Here, one has $\widehat{\Omega}_\varepsilon$ the interior of the largest union of the cells $\varepsilon(\xi + \overline{Y})$ such that $\varepsilon(\xi + Y)$ is fully contained in Ω and Λ_ε represents the parts from the cells $\varepsilon(\xi + \overline{Y})$ that intersects the boundary $\partial\Omega$ (see the figure below).

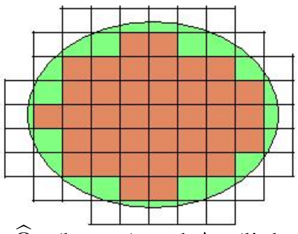

Fig. 2. $\widehat{\Omega}_\varepsilon$ (brown) and Λ_ε (light green)

We have the following definition of the unfolding operator.

Definition 1. Let φ be a *Lebesgue-measurable* function on Ω. Define the *unfolding operator* $\mathcal{T}_\varepsilon$ as

$$\mathcal{T}_\varepsilon(\varphi)(x, y) = \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right),$$

a.e. for $(x, y) \in \widehat{\Omega}_\varepsilon \times Y$ and

$$\mathcal{T}_\varepsilon(\varphi)(x, y) = 0,$$

a.e. for $(x, y) \in \Lambda_\varepsilon \times Y$.

Remark 2. From this definition, it follows that

$$\mathcal{T}_\varepsilon(v\, w)(x, y) = \mathcal{T}_\varepsilon(v)(x, y)\, \mathcal{T}_\varepsilon(w)(x, y),$$

for v and w Lebesgue-measurable functions.

Some properties of the unfolding operator needed in this work are given in the following theorem.

Theorem 3. *Let $p \geqslant 1$ and is finite, $\varphi \in L^1(\Omega)$, and $w \in L^p(\Omega)$. Then,*

1. *$\mathcal{T}_\varepsilon$ is linear and continuous from $L^p(\Omega)$ to $L^p(\Omega \times Y)$.*
2. $\displaystyle \frac{1}{|Y|}\int_{\Omega\times Y} \mathcal{T}_\varepsilon(\varphi)(x, y)\, dx\, dy = \int_\Omega \varphi(x)\, dx - \int_{\Lambda_\varepsilon} \varphi(x)\, dx = \int_{\widehat{\Omega}_\varepsilon} \varphi(x)\, dx.$
3. $\displaystyle \frac{1}{|Y|}\int_{\Omega\times Y} |\mathcal{T}_\varepsilon(\varphi)(x, y)|\, dx\, dy \leqslant \int_\Omega |\varphi(x)|\, dx.$
4. $\displaystyle \left|\int_\Omega \varphi(x)\, dx - \frac{1}{|Y|}\int_{\Omega\times Y} \mathcal{T}_\varepsilon(\varphi)(x, y)\, dx\, dy\right| \leqslant \int_{\Lambda_\varepsilon} |\varphi(x)|\, dx.$
5. $\|\mathcal{T}_\varepsilon(w)(x, y)\|_{L^p(\Omega\times Y)} = |Y|^{\frac{1}{p}} \|w\, 1_{\widehat{\Omega}_\varepsilon}\|_{L^p(\Omega)} \leqslant |Y|^{\frac{1}{p}} \|w\|_{L^p(\Omega)}.$

One has the following criterion called the unfolding criterion for integrals.

Theorem 4. *If the sequence $\{\varphi_\varepsilon\}$ in $L^1(\Omega)$ satisfies*

$$\int_{\Lambda_\varepsilon} |\varphi_\varepsilon(x)|\, dx \to 0,$$

then

$$\int_\Omega \varphi_\varepsilon(x)\, dx - \frac{1}{|Y|}\int_{\Omega\times Y} \mathcal{T}_\varepsilon(\varphi_\varepsilon)(x, y)\, dx\, dy \to 0.$$

Moreover, we write

$$\int_\Omega \varphi_\varepsilon(x)\, dx \overset{\mathcal{T}_\varepsilon}{\simeq} \frac{1}{|Y|}\int_{\Omega\times Y} \mathcal{T}_\varepsilon(\varphi_\varepsilon)(x, y)\, dx\, dy.$$

The following convergence properties of the unfolding operator are very essential in the homogenization of the problem.

Theorem 5. *Let $1 \leq p < \infty$.*

1. *For $w \in L^p(\Omega)$,*
$$\mathcal{T}_\varepsilon(w) \to w \text{ in } L^p(\Omega \times Y).$$
2. *Let $\{w_\varepsilon\}$ be a sequence in $L^p(\Omega)$ such that*
$$w_\varepsilon \to w \text{ in } L^p(\Omega).$$
Then,
$$\mathcal{T}_\varepsilon(w) \to w \text{ in } L^p(\Omega \times Y).$$

Corollary 6. *Let $1 < p < \infty$, and $\{w_\varepsilon\}$ be a sequence converging weakly in $W^{1,p}(\Omega)$ to w. Then*

$$\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup w \quad w\text{-}L^p(\Omega; W^{1,p}(Y)).$$

Theorem 7. *Let $p > 1$ and is finite. If $w_\varepsilon \rightharpoonup w$ weakly in $W^{1,p}(\Omega)$, then there exists $\widehat{w} \in L^p(\Omega; W^{1,p}_{per}(Y))$ with $\mathcal{M}_Y(\widehat{w}) = 0$ such that up to a subsequence,*

$$\mathcal{T}_\varepsilon(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \widehat{w} \ w\text{-}L^p(\Omega \times Y).$$

III. Statement of the Problem and Optimal Control Results

Let $M(\alpha, \beta, \Omega)$ denote the set of $N \times N$ matrix fields

$$A = (a_{ij})_{1\leq i,j\leq N} \in (L^\infty(\Omega))^{N\times N},$$

satisfying

$$(A(y)\lambda, \lambda) \geq \alpha|\lambda|^2 \quad \text{and} \quad |A(y)\lambda| \leq \beta|\lambda|,$$

for all $\lambda \in \mathbb{R}^N$ and a.e. in Ω with $0 < \alpha < \beta$ for real numbers α and β.

Consider the space $H^1(\Omega)$ equipped with the norm

$$\|v\|_{H^1(\Omega)} = \left(\|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}. \tag{1}$$

Let $\varepsilon > 0$. We want to study the asymptotic behaviour of the optimal control governed by the linear elliptic problem with linear Robin boundary condition given by,

$$\begin{cases} -\text{div}(A^\varepsilon \nabla u_\varepsilon) = f + \theta & \text{in } \Omega,\\ A^\varepsilon \nabla u_\varepsilon \cdot \vec{n} + h_\varepsilon u_\varepsilon = g & \text{on } \partial\Omega, \end{cases} \tag{2}$$

situated in the fixed domain Ω where $\vec{n}$ is the exterior unit normal vector to Ω. We suppose that the boundary $\partial\Omega$ is Lipschitz continuous and the data satisfy:

- $A1.$ f and θ are functions in $L^2(\Omega)$.
- $A2.$ g is a function in $L^2(\partial\Omega)$.
- $A3.$ h_ε is a nonnegative function such that $h_\varepsilon \in L^\infty(\partial\Omega)$.

ISBN: 978-988-14047-4-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$A4.$ $A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right)$ such that $A \in \mathcal{M}(\alpha, \beta, \Omega)$.

The variational formulation of (2) is given by

$$\begin{cases} \text{For every } \varepsilon > 0, \text{ find } u_\varepsilon \in H^1(\Omega) \text{ such that} \\ \displaystyle\int_\Omega A^\varepsilon \nabla u_\varepsilon \nabla v\, dx + \int_{\partial\Omega} h_\varepsilon u_\varepsilon v\, ds = \int_\Omega f v\, dx \\ \displaystyle + \int_\Omega \theta v\, dx + \int_{\partial\Omega} g v\, ds, \quad \forall v \in H^1(\Omega). \end{cases} \tag{3}$$

We have the following property due to Lax-Milgram Theorem.

THEOREM 8. *The variational problem (3) admits a unique solution.*

Let us now consider the L^2-cost functional given by

$$J_\varepsilon(u_\varepsilon, \theta) = \frac{1}{2}\int_\Omega (u_\varepsilon - u_d)^2\, dx + \frac{\nu}{2}\int_\Omega \theta^2\, dx,$$

where $u_d \in L^2(\Omega)$ is the desired state independent of ε, and $\nu > 0$ is the functional's regularization parameter. The optimal control problem for $J_\varepsilon(u_\varepsilon, \theta)$ is given by

$$F_\varepsilon : \inf\left\{J_\varepsilon(u_\varepsilon, \theta) | \theta \in L^2(\Omega),\, (u_\varepsilon, \theta) \text{ satisfies } (3)\right\}.$$

This problem is standard and we have:

PROPOSITION 9. *For each $\varepsilon > 0$, problem F_ε admits a unique solution.*

We also have the following property which characterizes the optimal control.

PROPOSITION 10. *Let $(\bar{u}_\varepsilon, \bar{\theta})$ be the optimal solution of F_ε and let the adjoint state $\bar{p}_\varepsilon$ satisfy the problem*

$$\begin{cases} -\text{div}({}^tA^\varepsilon \nabla \bar{p}_\varepsilon) = \bar{u}_\varepsilon - u_d & \text{in} \quad \Omega, \\ {}^tA^\varepsilon \nabla \bar{p}_\varepsilon \cdot \vec{n} + h_\varepsilon \bar{p}_\varepsilon. = 0 & \text{on } \partial\Omega. \end{cases}$$

Then the optimal control $\bar{\theta}_\varepsilon$ is given by $\bar{\theta}_\varepsilon = -\frac{1}{\nu}\,\bar{p}_\varepsilon$.

Conversely, if the pair $(\widehat{u}_\varepsilon, -\frac{1}{\nu}\,\widehat{p}_\varepsilon) \in H^1(\Omega) \times H^1(\Omega)$ solves the optimality system

$$\begin{cases} -\text{div}(A^\varepsilon \nabla \widehat{u}_\varepsilon) = f - \frac{1}{\nu}\,\widehat{p}_\varepsilon & \text{in} \quad \Omega, \\ A^\varepsilon \nabla \widehat{u}_\varepsilon \cdot \vec{n} + h_\varepsilon \widehat{u}_\varepsilon = g & \text{on } \partial\Omega, \\ -\text{div}({}^tA^\varepsilon \nabla \widehat{p}_\varepsilon) = \widehat{u}_\varepsilon - u_d & \text{in} \quad \Omega, \\ {}^tA^\varepsilon \nabla \widehat{p}_\varepsilon \cdot \vec{n} + h_\varepsilon \widehat{p}_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

then the pair $(\widehat{u}_\varepsilon, -\frac{1}{\nu}\,\widehat{p}_\varepsilon)$ is the optimal solution of F_ε.

For the proofs of Proposition 9 and Proposition 10, one can follow the arguments for e.g., in [21] and [29].

PROPOSITION 11. *Let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ be the optimal solution of F_ε, and $\bar{p}_\varepsilon$ be the adjoint state. Then we have the estimates,*

$$\|\bar{p}_\varepsilon\|_{H^1(\Omega)} \leqslant C,$$

$$\|\bar{\theta}_\varepsilon\|_{L^2(\Omega)} \leqslant C,$$

and

$$\|\bar{u}_\varepsilon\|_{H^1(\Omega)} \leqslant C,$$

where C is a generic positive constant independent of ε.

IV. HOMOGENIZATION RESULTS

In this section, we present the homogenization results we obtained via periodic unfolding method. We start with the limit problem corresponding to problem (2).

Consider the space $H_0^1(\Omega)$ together with the norm defined in (1) and the problem

$$\begin{cases} -\text{div}(A^0 \nabla u) = f + \theta & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

with the following assumptions:

$H1.$ f and θ are functions in $L^2(\Omega)$.

$H2.$ $A^0 \in \mathcal{M}(\alpha_0, \beta_0, \Omega)$, where $\alpha_0, \beta_0 \in \mathbb{R}$ with $0 < \alpha_0 < \beta_0$.

An immediate consequence of the Lax-Milgram Theorem is the following proposition.

PROPOSITION 12. *Problem (4) admits a unique solution u in $H_0^1(\Omega)$ that satisfies the a priori estimate*

$$\|u\|_{H_0^1(\Omega)} \leqslant C,$$

for some positive constant C.

Now, let us have the L^2-cost functional given by

$$J(u, \theta) = \frac{1}{2}\int_\Omega (u - u_d)^2\, dx + \frac{\nu}{2}\int_\Omega \theta^2\, dx,$$

where $u_d \in L^2(\Omega)$ is the desired state, and $\nu > 0$ is the functional's regularization parameter. The optimal control problem for $J(u, \theta)$ is

$$F : \inf\left\{J(u, \theta) | \theta \in L^2(\Omega),\, (u, \theta) \text{ satisfies } (4)\right\}.$$

PROPOSITION 13. *Problem F admits a unique optimal solution $(\bar{u}, \bar{\theta})$, where $\bar{u}$ is the optimal state and $\bar{\theta}$ is the optimal control.*

To characterize the optimal control, we have:

PROPOSITION 14. *Let $(\bar{u}, \bar{\theta})$ be the optimal solution of F and let the adjoint state $\bar{p}$ satisfy the problem*

$$\begin{cases} -\text{div}({}^tA^0 \nabla \bar{p}) = \bar{u} - u_d & \text{in } \Omega, \\ \bar{p} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

then the optimal control is given by $\bar{\theta} = -\frac{1}{\nu}\,\bar{p}$.

Conversely, if the pair $(\widehat{u}, -\frac{1}{\nu}\,\widehat{p}) \in H_0^1(\Omega) \times H_0^1(\Omega)$ solves the optimality system

$$\begin{cases} -\text{div}(A^0 \nabla \widehat{u}) = f - \dfrac{1}{\nu}\widehat{p} & \text{in } \Omega, \\ \widehat{u} = 0 & \text{on } \partial\Omega, \\ -\text{div}({}^tA^0 \nabla \widehat{p}) = \widehat{u} - u_d & \text{in } \Omega, \\ \widehat{p} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

then the pair $(\widehat{u}, -\frac{1}{\nu}\,\widehat{p})$ is the optimal solution of F.

One can prove Proposition 13 and Proposition 14, again, by following the steps done in in [21] and [29].

We now have the following convergence results obtained via PUM.

THEOREM 15. *Let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ be the optimal solution of F_ε with the adjoint state $\bar{p}_\varepsilon$. If there exists a matrix field A in $\mathcal{M}(\alpha, \beta, \Omega \times Y)$ such that*

$$\mathcal{T}_\varepsilon(A^\varepsilon) \to A(x, y) \qquad \text{in } \Omega \times Y, \tag{7}$$

ISBN: 978-988-14047-4-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

then there exist $u, p \in H_0^1(\Omega)$ and $\theta \in L^2(\Omega)$ such that up to subsequences,

$$\begin{cases} \widetilde{u_\varepsilon} \rightharpoonup u & w\text{-}H^1(\Omega), \\ \widetilde{p_\varepsilon} \rightharpoonup p & w\text{-}H^1(\Omega), \\ \widetilde{\theta_\varepsilon} \rightharpoonup \theta & w\text{-}L^2(\Omega), \end{cases} \tag{8}$$

and $(u, p, \theta) \in H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$ is the unique solution to the optimality system

$$\begin{cases} -\mathrm{div}(A^0 \nabla u) = f + \theta & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \\ -\mathrm{div}({}^tA^0 \nabla p) = u - u_d & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

The homogenized matrix fields A^0 and ${}^tA^0$ are constant and elliptic, and are given by

$$\begin{cases} A^0 = \mathcal{M}_Y(a_{ij}) - \mathcal{M}_Y\left(\displaystyle\sum_{k=1}^n a_{ik} \frac{\partial \widehat{\chi_j}(y)}{\partial y_k}\right) \\ {}^tA^0 = \mathcal{M}_Y(a_{ij}) - \mathcal{M}_Y\left(\displaystyle\sum_{k=1}^n a_{kj} \frac{\partial \chi_i(y)}{\partial y_k}\right), \end{cases} \tag{10}$$

for almost every $y \in Y$, and for all $\lambda \in \mathbb{R}^N$, $\widehat{\chi}_j$ is the solution to

$$\begin{cases} -\mathrm{div}(A \nabla \widehat{\chi}_\lambda) = -\mathrm{div}(A\lambda) & \text{in } Y \\ A\nabla(\widehat{\chi}_\lambda - \lambda \cdot y) \cdot n = 0 & \text{on } \partial\Omega \\ \widehat{\chi}_\lambda \text{ be } Y\text{-periodic} \\ \mathcal{M}_Y(\widehat{\chi}_\lambda) = 0, \end{cases} \tag{11}$$

and χ_i is the solution to

$$\begin{cases} -\mathrm{div}({}^tA \nabla \chi_\lambda) = -\mathrm{div}({}^tA\lambda) & \text{in } Y \\ {}^tA\nabla(\chi_\lambda - \lambda \cdot y) \cdot n = 0 & \text{on } \partial\Omega \\ \chi_\lambda \text{ be } Y\text{-periodic} \\ \mathcal{M}_Y(\chi_\lambda) = 0. \end{cases} \tag{12}$$

Moreover, there exist $\widehat{u}, \widehat{p} \in L^2(\Omega; H^1_{per}(Y))$ with $\mathcal{M}_Y(\widehat{u}) = 0$ and $\mathcal{M}_Y(\widehat{p}) = 0$, such that up to subsequences,

$$\begin{cases} \mathcal{T}_\varepsilon(\bar{u}_\varepsilon) \rightharpoonup u & w\text{-}L^2(\Omega; H^1(Y)) \\ \mathcal{T}_\varepsilon(\nabla \bar{u}_\varepsilon) \rightharpoonup \nabla_x u + \nabla_y \widehat{u} & w\text{-}L^2(\Omega \times Y) \\ \mathcal{T}_\varepsilon(\bar{p}_\varepsilon) \rightharpoonup p & w\text{-}L^2(\Omega; H^1(Y)) \\ \mathcal{T}_\varepsilon(\nabla \bar{p}_\varepsilon) \rightharpoonup \nabla_x p + \nabla_y \widehat{p} & w\text{-}L^2(\Omega \times Y) \\ \mathcal{T}_\varepsilon(\bar{\theta}_\varepsilon) \rightharpoonup \theta & w\text{-}L^2(\Omega; H^1(Y)), \end{cases} \tag{13}$$

where $(u, p, \widehat{u}, \widehat{p}, \theta)$ is the unique solution to the limit equation

$$\begin{cases} \forall \varphi \in H^1(\Omega), \psi \in L^2(\Omega; H^1_{per}(Y)), \\ \displaystyle\frac{1}{|Y|}\int_{\Omega \times Y} A(\nabla_x u + \nabla_y \widehat{u})(\nabla_x \varphi + \nabla_y \psi(x,y))\, dx\, dy \\ \qquad = \displaystyle\int_\Omega (f + \theta)\varphi\, dx \\ \displaystyle\frac{1}{|Y|}\int_{\Omega \times Y} {}^tA(\nabla_x p + \nabla_y \widehat{p})(\nabla_x \varphi + \nabla_y \psi(x,y))\, dx\, dy \\ \qquad = \displaystyle\int_\Omega (u - u_d)\varphi\, dx. \end{cases} \tag{14}$$

Outline of the Proof: The convergences in (8) follow from the estimates given in Proposition 10. Those in (13) are due to Proposition 11, Corollary 6 and Theorem 7. The limit equation in (14) can be derived by considering two types of test functions in the variational problem and applying the unfolding operator together with its properties given in Section II. □

Finally, we have the following convergence of the energy corresponding to the L^2-cost functional $J_\varepsilon(u_\varepsilon, \theta)$.

THEOREM 16. *Let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ and $\bar{p}_\varepsilon$, and $(\bar{u}, \bar{\theta})$ and $\bar{p}$ be the optimal solutions and adjoint states of F_ε and F, respectively. Then,*

$$\begin{cases} (i) \ \widetilde{u_\varepsilon} \rightharpoonup \bar{u} & w\text{-}H^1(\Omega), \\ (ii) \ \widetilde{p_\varepsilon} \rightharpoonup \bar{p} & w\text{-}H^1(\Omega), \\ (iii) \ \widetilde{\theta_\varepsilon} \rightharpoonup \bar{\theta} & w\text{-}L^2(\Omega). \end{cases} \tag{15}$$

Moreover,

$$\lim_{\varepsilon \to 0} J_\varepsilon(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) = J(\bar{u}, \bar{\theta}). \tag{16}$$

REFERENCES

[1] M. Briane, A. Damlamian and P. Donato, "H-convergence in perforated domains" in Non-linear partial Differential Equations and Their Applications, Collége de France seminar vol. XIII, D. Cioranescu and J. L. Lions eds., Pitman Research Notes in Mathematics Series, Longman, New York, (391) (1998), 62-100.

[2] G. Buttazzo and G. Dal Maso, "Γ-convergence and optimal control problems", Journal of Optimization Theory and Applications, Vol 38, No. 3, (1982), 385-407.

[3] G. Buttazzo, "Γ-convergence and its applications to some problem in the calculus of variations", in School on Homogenization, ICTP, Trieste, 1993 (1994) 3861.

[4] B. Cabarrubias, "Homogenization of optimal control problems in perforated domains via periodic unfolding method", Applicable Analysis, 95:11, (2016) 2517-2534, DOI:10.1080/00036811.2015.1094799.

[5] B. Cabarrubias and P. Donato, "Homogenization of some evolution problems in domains with small holes", Electronic Journal of Differential Equations, Vol. 2016, No. 169 (2016), 1-26.

[6] D. Cioranescu, A. Damlamian and G. Griso, "Periodic unfolding and homogenization", *C.R. Acad. Sci. Paris*, Ser. I 335, pp. 99-104, 2002.

[7] D. Cioranescu, A. Damlamian and G. Griso, "The periodic unfolding and homogenization", *SIAM J. Math. Anal.*, Vol. 40, No. 4, pp. 1585-1620, 2008.

[8] D. Cioranescu and P. Donato, "Homogénésation du problème de Neumann non homogène dans des ouverts perforés", *Asymptotic Anal.* Vol. 1, pp. 115-138, 1988.

[9] D. Cioranescu, P. Donato and R. Zaki, "The Periodic Unfolding and Robin problems in perforated domains", *C.R.A.S. Paris*, Série 1, 342, pp. 467-474, 2006.

[10] D. Cioranescu, P. Donato and R. Zaki, "The Periodic Unfolding Method in Perforated Domains",*Portugal Math* 63(4), pp. 476-496, 2006.

[11] D. Cioranescu, P. Donato and R. Zaki, "Asymptotic behavior of elliptic problems in perforated domains with nonlinear boundary conditions", *Asymptotic Analysis* 53, pp. 209-235, 2007.

[12] D. Cioranescu, A. Damlamian, P. Donato, G. Griso and R. Zaki, "The Periodic Unfolding Method in Domains With Holes", *SIAM J. Math. Anal.*, Vol. 44, No. 2, pp. 718-760, 2012.

[13] D. Cioranescu, A. Damlamian, G. Griso and D. Onofrei, The periodic unfolding method for perforated domains and Neumann sieve models, J. Math. Pures Appl. 89 (2008), 248-277.

[14] A. Damalamian, "An elementary introduction to periodic unfolding", *Gakuto International Series, Math.Sci. Appl.* Vol. 24, pp. 119-136, 2005.

[15] T. Durante, L. Faella and C. Perugia, "Homogenization and behavior of optimal controls for the wave equation in domains with oscillating boundary", Nonlinear Differ. Equ. Appl. 14 (2007), 455489.

[16] T. Durante and T.A. Melnyk, "Homogenization of quasilinear optimal control problem involving a thick multilevel junction of type 3:2:1", ESAIM Control Optim. Calc. Var. 18(2) (2012), 583610.

[17] P. Donato and Z. Yang, " The Periodic unfolding method for the wave equation in domains with holes", *Advances in Mathematical Sciences and Applications*, (22)2, pp. 521-551, 2012.

ISBN: 978-988-14047-4-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

[18] S. Kesavan and T. Muthukumar, "Homogenization of an optimal control problem with state constraints", Differ. Equ. Dyn. Syst. 19(4)(2011), 361-374.

[19] S. Kesavan and M. Rajesh, "Homogenization of periodic optimal control problems via multi-scale convergence", Proc. Indian Acad. Sci. (Math. Sci.), VOl. 108, No.2, (1998) 189-207.

[20]] S. Kesavan and J. Saint Jean Paulin, "Optimal control on perforated domains", J. Math. Anal. Appl. 229, (1999), 563-586.

[21] J.-L.Lions, *Optimal control of systems governed by partial differential equations*, Springer-Verlag, Berlin 1971.

[22] T. Muthukumar and A.K. Nandakumaran, "Homogenization of low-cost control problems on perforated domains", J. Math. Anal. Appl. 351 (2009), 29-42.

[23] A.K. Nandakumaran, R. Prakash and B.C. Sardar, "Homogenization of an optimal control problem in a domain with highly oscillating boundary using periodic unfolding method", MESA, Vol.4, No.3, (2013) 281-303.

[24] A.K. Nandakumaran and R. Prakash, "Homogenization of Boundary Optimal Control Problems with Oscillating Boundary", Nonlinear Studies-The International Journal, Vol. 20, No. 3, pp. 1-25, 2013.

[25] A.K. Nandakumaran, R. Prakash and J.P. Raymond, "Asymptotic analysis and error estimates for an optimal control problem with oscillating boundaries", Ann. Univ. Ferrara, 58 (2012), 143-166.

[26] A.K. Nandakumaran, R. Prakash and J-P Raymond, "Stokes System in a Domain with Oscillating Boundary: Homogenization and Error Analysis of an Interior Optimal Control Problem", Numerical Functional Analysis and Optimization, 20l3.

[27] D. Onofrei, "The unfolding operator near a hyperplane and its application to the Neumann Sieve model", AMSA, 16 (2006), 239-258.

[28] R. Prakash, "Optimal Control Problem for the Time-Dependent Kirchhoff-Love Plate in a Domain with Rough Boundary", Asymptotic Analysis 81 (2013) 337-355.

[29] J.P. Raymond, *Optimal control of partial differential equations*, Universitè Paul Sabatier, www.math.univ-toulouse.fr.

[30] H. Zoubairi, "Optimal control and homogenization in a mixture of fluids separated by a rapidly oscillating interface", Electr. J. Diff. Equ. No. 2, (2002), 1-32.