Solving Linear Schrödinger Equation through Perturbation Iteration Transform Method

Grace O. Akinlabi, Sunday O. Edeki, *Member, IAENG*

Abstract— **This paper applies Perturbation Iteration Transform Method: a combined form of the Perturbation Iteration Algorithm and the Laplace Transform Method to linear Schrödinger equations for approximate-analytical solutions. The results converge rapidly to the exact solutions.**

Index Terms— **Schrödinger equations, perturbation iteration algorithm, Laplace transform, perturbation iteration transform method, linear PDEs**

I. Introduction

Schrödinger equation is a partial differential equation used in the description of the way the quantum state of a physical system changes with time [1]. This offers a way on how the calculation of the associated wave function is done. These equations have wide applications in physics, and other areas of applied sciences, which include but not limited to hydrodynamics, superconductivity, nonlinear optics, and plasma physics [2].

The Schrödinger equation is of the form:

$$\begin{cases} iu_t + \phi u_{xx} + h(x)u + \eta |u|^2 u = 0, \\ u(x,0) = p(x). \end{cases} \quad (1.1)$$

where $u = u(x,t)$ is a complex function, $h(x)$ a function of x, $i^2 = -1$, ϕ, η are constants.

In this paper, we shall consider a special case of the Schrödinger equation in (1.1) based on some initial conditions for the consideration of the time evolution of a free particle.

The Schrödinger equation has been studied by many researchers with various solution techniques. Bulut, et al in [3] applied the Sumudu transform method (STM) to the Schrödinger equation with variable coefficients. Adomian decomposition method (ADM) was applied to these equations in [4] while a coupling of the Homotopy Perturbation Method (HPM) with the ADM was adopted in [5] for the solution of both linear and nonlinear Schrödinger equations. Other methods include: Finite Difference Method (FDM) [6], Modified Variational Iteration Method (VIM) [7], and differential transformation method (DTM) [8]. For more work on the Schrödinger equation, see [9 – 11].

The Perturbation Iteration Transform Method (PITM) involves the fusion of the Perturbation Iteration Algorithm and the Laplace Transform. This idea was introduced in [14], where the method was used to solve linear and nonlinear Klein-Gordon equations. For other articles on the use of the PITM to solve linear and nonlinear PDEs, see [13-16].

The remaining part of the paper will be structured as follows: in section II and III, the overviews of the PIA and PITM are presented respectively. In section IV, we apply the PITM to solve some illustrative examples while in section V, we give a concluding remark.

Manuscript received March 24, 2017. Revised: March 31, 2017.

G. O. Akinlabi (e-mail: grace.akinlabi@covenantuniversity.edu.ng) is with the Department of Mathematics, Covenant University, Nigeria.

S. O. Edeki (e-mail: soedeki@yahoo.com) is with the Department of Mathematics, Covenant University, Nigeria.

II. Perturbation Iteration Algorithm [12], [14]

In this section, we illustrate how the Perturbation Iteration Algorithm works. Suppose a perturbation algorithm is been developed by taking the correction terms of the first derivatives in the Taylor series expansion and also one correction term in the perturbation expansion. This algorithm will be named: PIA(1,1).

We now consider a partial differential equation of the form:

$$F(\dot{u}, u'', u, \varepsilon) = 0, \quad (2.1)$$

where $u = u(x,t)$, $\dot{u} = \frac{\partial u}{\partial t}$, $u'' = \frac{\partial^2 u}{\partial x^2}$ and ε is the introduced perturbation parameter.

And if we use just one correction term in the perturbation expansion, we have:

$$u_{n+1} = u_n + \varepsilon (u_c)_n. \quad (2.2)$$

Putting (2.2) into (2.1) and expanding in a Taylor series with first derivatives will yield.

$$\begin{cases} 0 = F(\dot{u}, u'', u, 0) + F_{\dot{u}}(\dot{u}, u'', u, 0)\varepsilon(\dot{u}_c)_n \\ \quad + F_{u''}(\dot{u}, u'', u, 0)\varepsilon(u''_c)_n + F_u(\dot{u}, u'', u, 0)\varepsilon(u_c)_n \\ \quad + F_\varepsilon(\dot{u}, u'', u, 0)\varepsilon \quad , \end{cases} \quad (2.3)$$

where $u = u(x,t)$, $F_{\dot{u}} = \frac{\partial F}{\partial \dot{u}}$, $F_{u''} = \frac{\partial F}{\partial u''}$, $F_u = \frac{\partial F}{\partial u}$, $F_\varepsilon = \frac{\partial F}{\partial \varepsilon}$ and ε is the perturbation parameter to be evaluated at zero.

Reorganizing (2.3), we have

$$(\dot{u}_c)_n + \frac{F_{u''}}{F_{\dot{u}}}(u''_c)_n = -\frac{F_\varepsilon + \frac{F}{\varepsilon}}{F_{\dot{u}}} - \frac{F_u}{F_{\dot{u}}}(u_c)_n. \quad (2.4)$$

Starting with an initial guess, u_0, evaluate the term, $(u_c)_0$ from (2.4) and then substitute the result into (2.2) for u_1. We continue this iteration procedure by using Equations (2.4) and (2.2) until a satisfactory result is obtained.

III. Analysis of The PITM [14,15]

In this section, we demonstrate the basic idea of the PITM. Consider the general nonlinear PDE of the form:

$$Lu(x,t)+Mu(x,t)+Nu(x,t)=g(x,t) \tag{3.1}$$

with the associated initial condition:

$$u(x,0)=f(x) \tag{3.2}$$

where $L=\frac{\partial}{\partial t}$ is the first order linear differential operator, $M=\frac{\partial^2}{\partial x^2}$ is the second order linear differential operator, $Nu(x,t)$ represents both the linear and the nonlinear terms, and $g(x,t)$ is the source term.

We take the Laplace transform of both sides of (3.1) to have

$$L\left[Lu(x,t)\right]+L\left[Mu(x,t)\right]+L\left[Nu(x,t)\right]=L\left[g(x,t)\right] \tag{3.3}$$

On using the differential property of Laplace transform in (3.3), we get

$$L\left[Lu(x,t)\right]=\frac{f(x)}{s}+\frac{1}{s}L\left[h(x,t)\right]-\frac{1}{s}L\left[Mu(x,t)\right]-\frac{1}{s}L\left[Nu(x,t)\right]. \tag{3.4}$$

Applying the Inverse Laplace Transform to both sides of (3.4) gives

$$u(x,t)=E(x,t)-L^{-1}\left[\frac{1}{s}L\left[Mu(x,t)+Nu(x,t)\right]\right], \tag{3.5}$$

where $E(x,t)$ is the term obtained from the source term and the associated initial condition.

Now, by using the PITM, (3.5) becomes:

$$u(x,t)-E(x,t)+u_c(x,t)\varepsilon-L^{-1}\left[\frac{1}{s}L\left[Mu(x,t)+Nu(x,t)\right]\right]\varepsilon=0. \tag{3.6}$$

Hence,

$$u_c(x,t)=\frac{E(x,t)-u(x,t)}{\varepsilon}-L^{-1}\left[\frac{1}{s}L\left[Mu(x,t)+Nu(x,t)\right]\right]. \tag{3.7}$$

Equation (3.7) is the combined form of the Laplace transform method and the perturbation iteration method. From (3.7), the term, $(u_c)_0$ is then calculated and substituted into (2.2) to obtain u_1. This iteration procedure is repeated for u_2, u_3 and so on. The approximate solution is thus obtained by the formula:

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t) \tag{3.8}$$

That is,

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots \tag{3.9}$$

IV. Numerical Illustrative Examples

In this section, we apply the PITM to the Linear Schrödinger Equations.

Problem I:
Consider the linear Schrödinger equation [11]:

$$u_t(x,t)+iu_{xx}(x,t)=0, \tag{4.1}$$

subject to:

$$u(x,0)=1+\cosh 2x. \tag{4.2}$$

Solution to Problem I:
We take the Laplace Transform of equation (4.1) with the initial condition (4.2) to get:

$$L\left[u(x,t)\right]=\frac{1+\cosh 2x}{s}-\frac{1}{s}L\left[iu_{xx}(x,t)\right] \tag{4.3}$$

We take the Inverse Laplace Transform of both sides of (4.3) to have:

$$u(x,t)=1+\cosh 2x-L^{-1}\left[\frac{1}{s}L\left[iu_{xx}(x,t)\right]\right]. \tag{4.4}$$

Applying the PITM to (4.4), we get:

$$u(x,t)-(1+\cosh 2x)+u_c(x,t)\varepsilon+L^{-1}\left[\frac{1}{s}L\left[iu_{xx}(x,t)\right]\right]\varepsilon=0. \tag{4.5}$$

Thus, we have

$$u_c(x,t)=\frac{-u(x,t)+1+\cosh 2x}{\varepsilon}-L^{-1}\left[\frac{1}{s}L\left[iu_{xx}(x,t)\right]\right] \tag{4.6}$$

Hence,

$$u_0(x,t)=1+\cosh 2x,$$
$$u_1(x,t)=-4it\cosh 2x,$$
$$u_2(x,t)=-8t^2\cosh 2x,$$
$$u_3(x,t)=\frac{32}{3}it^3\cosh 2x,$$
$$u_4(x,t)=\frac{32}{3}t^4\cosh 2x,$$
$$u_5(x,t)=-\frac{128}{15}it^5\cosh 2x,$$
$$u_6(x,t)=-\frac{256}{45}t^6\cosh 2x,$$
$$u_7(x,t)=\frac{1024}{315}it^7\cosh 2x,$$
$$\vdots.$$

Therefore, the solution $u(x,t)$ is given by:

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots$$
$$=1+\cosh 2x-4it\cosh 2x-8t^2\cosh 2x$$

ISBN: 978-988-14047-4-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$+\frac{32}{3}it^3\cosh 2x+\frac{32}{3}t^4\cosh 2x-\frac{128}{15}it^5\cosh 2x$$
$$-\frac{256}{45}t^6\cosh 2x+\frac{1024}{315}it^7\cosh 2x+\cdots\ .$$
$$=1+\left\{1-8t^2+\frac{32}{3}t^4-\frac{256}{45}t^6+\cdots\right\}\cosh 2x$$
$$+\left\{-4t+\frac{32}{3}t^3-\frac{128}{15}t^5+\frac{1024}{315}t^7-\cdots\right\}i\cosh 2x$$
$$=1+\left\{\frac{(4t)^0}{0!}-\frac{(4t)^2}{2!}+\frac{(4t)^4}{4!}-\frac{(4t)^6}{6!}+\cdots\right\}\cosh 2x$$
$$+\left\{-\frac{(4t)}{1!}+\frac{(4t)^3}{3!}-\frac{(4t)^5}{5!}+\frac{(4t)^7}{7!}-\cdots\right\}i\cosh 2x$$
$$=1+\left[\left\{\frac{(-1)^n(4t)^{2n}}{(2n)!}\right\}-\left\{\frac{(-1)^n(4t)^{2n+1}i}{(2n+1)!}\right\}\right]\cosh 2x,\ n\geq 0$$
$$=1+\left[\cos 4t-i\sin 4t\right]\cosh 2x$$
$$=1+e^{-4it}\cosh 2x \qquad (4.7)$$

Equation (4.7) is the exact solution of Problem 1.

Equation (4.8) is the approximate solution of Case I.

Problem II:

Consider the linear Schrödinger equation [11]:

$$u_t(x,t)+iu_{xx}(x,t)=0\,, \qquad (4.8)$$

subject to:

$$u(x,0)=e^{3ix}\,. \qquad (4.9)$$

Solution to Problem II:

We take the Laplace Transform of equation (4.8) with the initial condition (4.9) to get:

$$L\left[u(x,t)\right]=\frac{e^{3ix}}{s}-\frac{1}{s}L\left[iu_{xx}(x,t)\right]. \qquad (4.10)$$

We take the Inverse Laplace Transform of both sides of (4.10) to have:

$$u(x,t)=e^{3ix}-L^{-1}\left[\frac{1}{s}L\left[iu_{xx}(x,t)\right]\right]. \qquad (4.11)$$

Applying the PITM to (4.11), we get:

$$u(x,t)-e^{3ix}+u_c(x,t)\varepsilon+L^{-1}\left[\frac{1}{s}L\left[iu_{xx}(x,t)\right]\right]\varepsilon=0\,. \qquad (4.12)$$

Thus, we have

$$u_c(x,t)=\frac{-u(x,t)+e^{3ix}}{\varepsilon}-L^{-1}\left[\frac{1}{s}L\left[iu_{xx}(x,t)\right]\right] \qquad (4.13)$$

Hence,

$$u_0(x,t)=e^{3ix}\,,$$
$$u_1(x,t)=9ite^{3ix}\,,$$
$$u_2(x,t)=-\frac{81}{2}t^2e^{3ix}\,,$$
$$u_3(x,t)=-\frac{243}{2}it^3e^{3ix}\,,$$
$$u_4(x,t)=\frac{2187}{8}t^4e^{3ix}\,,$$
$$u_5(x,t)=\frac{19683}{40}it^5e^{3ix}\,,$$
$$u_6(x,t)=-\frac{59049}{80}t^6e^{3ix}\,,$$
$$u_7(x,t)=-\frac{531441}{560}it^7e^{3ix}\,,$$
$$\vdots\,.$$

Therefore, the solution $u(x,t)$ is given by:

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots$$
$$=e^{3ix}+9ite^{3ix}-\frac{81}{2}t^2e^{3ix}-\frac{243}{2}it^3e^{3ix}+\frac{2187}{8}t^4e^{3ix}$$
$$+\frac{19683}{40}it^5e^{3ix}-\frac{59049}{80}t^6e^{3ix}-\frac{531441}{560}it^7e^{3ix}+\cdots$$
$$=\left\{1-\frac{81}{2}t^2+\frac{2187}{8}t^4-\frac{59049}{80}t^6+\cdots\right\}e^{3ix}$$
$$+\left\{9t-\frac{243}{2}t^3+\frac{19683}{40}t^5-\frac{531441}{560}t^7+\cdots\right\}ie^{3ix}$$
$$=\left\{\frac{(9t)^0}{0!}-\frac{(9t)^2}{2!}+\frac{(9t)^4}{4!}-\frac{(9t)^6}{6!}+\cdots\right\}e^{3ix}$$
$$+\left\{\frac{(9t)}{1!}+\frac{(9t)^3}{3!}-\frac{(9t)^5}{5!}+\frac{(9t)^7}{7!}-\cdots\right\}ie^{3ix}$$
$$=\left[\left\{\frac{(-1)^n(9t)^{2n}}{(2n)!}\right\}+\left\{\frac{(-1)^n(9t)^{2n+1}i}{(2n+1)!}\right\},\ \ n\geq 0\right]e^{3ix}$$
$$=\left(\cos 9t+i\sin 9t\right)e^{3ix}$$
$$=\left(e^{9it}\right)e^{3ix}$$
$$=e^{3(x+3t)i} \qquad (4.14)$$

Equation (4.14) is the exact solution of Problem 2.

V. Conclusion

In this paper, we applied the Perturbation Iteration Transform Method to the linear Schrödinger equations. This method resulted from the combined form of the PIA and the LTM. The solutions are in series forms, and converged rapidly to the exact forms. The method is therefore, proven to be very efficient and reliable.

Acknowledgment

The authors are grateful to Covenant University for the provision of a good working environment, and financial support. They also wish to thank the anonymous reviewers for helpful and valuable remarks.

REFERENCES

[1] E. Schrödinger, "An Undulatory Theory of the Mechanics of Atoms and Molecules", *Physical Review* **28** (6): (1926), 1049–1070.

[2] M. M. Mousa, S. F. Ragab, and Z. Nturforsch, “Application of the homotopy perturbation method to linear and nonlinear Schr¨odinger equations,” *Zeitschrift Fur Naturforschung A*, **63**, no. 3-4, (2008), 140–144.

[3] H. Bulut, H.M. Baskonus, and S. Tuluce, “The solutions of partial differential equations with variable coefficient by Sumudu transform method”, *AIP Conf.Proc.* **1493,** 91 (2012); doi: 10.1063/1.4765475.

[4] S. A. Khuri, “A new approach to the cubic Schrödinger equation: an application of the decomposition technique”, *Appl Math Comput.*, **97** (1988) 251-4.

[5] A. Sadighi and D. D. Ganji, “Analytic treatment of linear and nonlinear Schrödinger equations: A study with homotopy perturbation and Adomian’s decomposition method”, *Phys. Lett. A*, **372,** (2008), 465-469.

[6] H. Wang, “Numerical studies on the split-step finite difference method for nonlinear Schrödinger equations”, Appl Math Comput., **170** (2005) 17-35.

[7] S. T. Mohyud-Din, M. A. Noor and K. I. Noor, “Modified Variational Iteration Method For Schrödinger Equations”, *Mathematical and Computational Applications*, **15**, (3), (2010), 309-317.

[8] A. S. V. Ravi Kanth, K. Aruna, “Two-dimensional differential transform method for solving linear and non-linear Schrödinger equation”, *Chaos, Solution and fractals*, **41** (2009), 2277-2281.

[9] S.O. Edeki, G.O. Akinlabi and S.A. Adeosun, “Analytic and Numerical Solutions of Time-Fractional Linear Schrödinger Equation”, *Comm Math Appl*, **7**(1), (2016): 1–10.

[10] G.O. Akinlabi and S.O. Edeki “On Approximate and Closed-form Solution Method for Initial-value Wave-like Models”, *International Journal of Pure and Applied Mathematics,* **107**(2), (2016): 449–456.

[11] S.O. Edeki, G.O. Akinlabi, S.A. Adeosun, “On a modified transformation method for exact and approximate solutions of linear Schrödinger equations”, AIP Conference proceedings 1705, 020048 (2016); doi: 10. 1063/1.4940296.

[12] Y. Aksoy and M. Pakdemirli, “New Perturbation-Iteration Solutions for Bratu-type Equations, *Computers and Mathematics with Applications*, **59**(8), (2010): 2802-2808.

[13] G.O. Akinlabi and S.O. Edeki “Perturbation Iteration Transform Method for the Solution of Newell-Whitehead-Segel Model Equations”, *Journal of Mathematics and Statistics,* (In press).

[14] M. Khalid, M. Sultana, F. Zaidi and A. Uroosa “Solving Linear and Nonlinear Klein-Gordon Equations by New Perturbation Iteration Transform Method”, *TWMS J. App. Eng. Math,* **6**(1), (2016): 115–125.

[15] G.O. Akinlabi and S. O. Edeki, “On the Solution of the Cahn-Hilliard Equation via the Perturbation Iteration Transform Method”*, World Congress on Engineering 2017*, WCE2017, London, U.K. (*Accepted*).

[16] G.O. Akinlabi and S. O. Edeki, “The solution of initial-value wave-like models via Perturbation Iteration Transform Method,” Lecture Notes in Engineering and Computer Science: *Proceedings of the International MultiConference of Engineers and Computer Scientists 2017,* Vol II, IMECS 2017, March 15-17, 2017, Hong Kong, pp 1015-1018.