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Mathematical Model for the Study of the Equilibrium State of Renewable Systems Based on Functional Operators with Shift

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***Abstract*—In previous works we proposed a method for the study of systems with one renewable resource. The separation of the individual and the group parameters and the discretization of time led us to scalar linear functional equations with shift. Cyclic models, in which the initial state of the system coincides with the final state, were considered. In this work, we present cyclic models for systems with two renewable resources. In modelling, the interactions and the reciprocal influences between these two resources are taken into account. Analysis of the models is carried out in weighted Holder spaces. A method for the solution of the balance system of equations is proposed. The equilibrium state of the system is found.**

***Index Terms*—renewable resources, degenerate kernel, Holder space, invertibility, equilibrium state.**

I. Introduction

Systems whose state depends on time and whose resources are renewable form an important class of general systems. A great number of works has been dedicated to systems with renewable resources. The core of the mathematical apparatus used for the study of such systems consists of differential equations in which the sought for function is dependent on time.

Our approach presupposes discretization of the processes with respect to time.We move away from tracking the changes in the system continuously to tracking the changes at fixed time points. This discretization and the identification of the individual parameter and the group parameter lead us to functional equations with shift.

II. Cyclic model of a system with two renewable resources.

Let S be a system with two resources λ_1, λ_2 and let T be a time interval. The choice of T is related to periodic processes taking place in the system and to human interferences.

Let these resources λ_1, λ_2 have the same individual parameter but scales of measurement of values of the individual parameter may be different:

$$x_{min} = x_1 < x_2 < ... < x_{n_1} = x_{max},$$

$$y_{min} = y_1 < y_2 < ... < y_{n_2} = y_{max}.$$

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We introduce the group parameters by functions $v(x_i,t)$, $w(y_i,t)$ which express a quantitative estimate of the elements of resources λ_1, λ_2 with the individual parameter $x_i, i = 1,2,...,n_1$ and $y_i, i = 1,2....,n_2$ at the time t.

Let t_0 be the initial time and S the system under consideration.

As in our previous works [1], [2] on modelling the system, we will hold the following principles:

I. The description of changes that occur on the interval $(t_0, t_0 + T)$ will be substituted by the fixing of the final results at the moment $t_0 + T$;

II. The separation of parameters into individual parameters, group parameters and the study of dependence of group parameters from individual parameters.

The initial state of the system S at time t_0 is represented as density functions of a distribution of the group parameter by the individual parameter for each resource

$$v(x,t_0) = v(x), 0 < x < x_{max},$$

$$w(y,t_0) = w(y), 0 < y < y_{max}.$$

We will now analyze the system's evolution. In the course of time, the elements of the system can change their individual parameter - e.g. fish can change their weight and length.

Modifications in the distribution of the group parameters by the individual parameters is represented by a displacement. The state of the system S at the time $t = t_0 + T$ is:

$$v(x,t_0+T) = \frac{d}{dx}\alpha(x)\cdot v(\alpha(x)), \tag{1}$$

$$w(y,t_0+T) = \frac{d}{dy}\beta(y)\cdot w(\beta(y)). \tag{2}$$

In the article [1], the appearance of derivatives in (1), (2) was explained.

Over the period $j_0 = [t_0, t_0 + T]$, extractions might be taken from the system as a result of human economic activity; these are represented by summands $\rho(x)$, $\delta(y)$. If an artificial entrance of elements into the system has taken place, it shall be accounted for by adding terms $\zeta(x)$, $\xi(y)$.

We take natural mortality into account with the coefficients $c(x)$, $d(y)$.

The process of reproduction will be represented by

$$\sum_{i=1}^{n} P_i p_i(x),$$

where

$$P_1 = \int_{\nu_0}^{\nu_1} v(x)dx,\ P_2 = \int_{\nu_1}^{\nu_2} v(x)dx, ..., P_n = \int_{\nu_{n-1}}^{\nu_n} v(x)dx,$$

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$$0=\nu_0<\nu_1<...<\nu_n=x_{max},$$

and

$$\sum_{i=1}^{m} Q_i q_i(y),$$

where

$$Q_1=\int_{\mu_0}^{\mu_1} w(y)dy,\ Q_2=\int_{\mu_1}^{\mu_2} w(y)dy,...,Q_m=\int_{\mu_{m-1}}^{\mu_m} w(y)dy,$$

$$0=\mu_0<\mu_1<...<\mu_m=y_{max}.$$

We obtain

$$v(x,t_0+T)=$$

$$c(x)\frac{d}{dx}\alpha(x)v(\alpha(x))+\rho(x)+\zeta(x)+\sum_{i=1}^{n} P_i p_i(x)$$

and

$$w(y,t_0+T)=$$

$$w(y,t_0+T)=d(y)\frac{d}{dy}\beta(y)w(\beta(y))+\delta(y)+\xi(y)+\sum_{i=1}^{m} Q_i q_i(y).$$

Resources λ_1 and λ_2 are not independent. We will account for reciprocal influence by terms

$$\sum_{i=1}^{k} R_i r_i(x),$$

where

$$R_1=\int_{\gamma_0}^{\gamma_1} w(y)dy, R_2=\int_{\gamma_1}^{\gamma_2} w(y)dy,...,R_k=\int_{\gamma_{k-1}}^{\gamma_k} w(y)dy,$$

$$0=\gamma_0<\gamma_1<,...,<\gamma_k=y_{max},$$

and

$$\sum_{i=1}^{l} F_i f_i(y),$$

where

$$F_1=\int_{\epsilon_0}^{\epsilon_1} v(x)dx, F_2=\int_{\epsilon_1}^{\epsilon_2} v(x)dx,...,F_l=\int_{\epsilon_{l-1}}^{\epsilon_l} v(x)dx,$$

$$0=\epsilon_0<\epsilon_1<,...,<\epsilon_l=x_{max}.$$

Thereby, the final state of the system at the moment $[t_0+T]$ is described as follows:

$$v(x,t_0+T)=$$

$$c(x)\frac{d}{dx}\alpha(x)v(\alpha(x))+\rho(x)+\zeta(x)+\sum_{i=1}^{n} P_i p_i(x)+\sum_{i=1}^{k} R_i r_i(x), \quad (3)$$

$$w(y,t_0+T)=$$

$$d(y)\frac{d}{dy}\beta(y)w(\beta(y))+\delta(y)+\xi(y)+\sum_{i=1}^{m} Q_i q_i(y)+\sum_{i=1}^{l} F_i f_i(y). \quad (4)$$

Let our goal be to find the equilibrium state of system S, that is, to find such an initial distribution of group parameters by the individual parameter $v(x,t_0)$, $w(x,t_0)$, that after all transformations during the time interval (t_0, t_0+T)), it would coincide with the final distribution:

$$v(x)=v(x,t_0+T), \quad (5)$$

$$w(y)=w(y,t_0+T). \quad (6)$$

From here, substituting relations (3) and (4) into (5), (6), it follows that

$$v(x)=$$

$$c(x)\frac{d}{dx}\alpha(x)v(\alpha(x))+\rho(x)+\zeta(x)+\sum_{i=1}^{n} P_i p_i(x)+\sum_{i=1}^{k} R_i r_i(x), \quad (7)$$

$$w(y)=$$

$$d(y)\frac{d}{dy}\beta(y)w(\beta(y))+\delta(y)+\xi(y)+\sum_{i=1}^{m} Q_i q_i(y)+\sum_{i=1}^{l} F_i f_i(y). \quad (8)$$

Equations (7), (8) are called equilibrium proportions or balance equations. A model is called cyclic if the state of system S at the initial time t_0 coincides with the state of system S at the final time t_0+T.

The application of principles I and II leads us to functional operators with shift.

We recall the definition of spaces of Holder functions with weight and the conditions of invertibility for scalar linear functional operators with shift.

III. Conditions of invertibility in the space of Holder functions with weight

A function $\varphi(x)$ that satisfies the condition on $J=[0,x_{max}]$:

$$|\varphi(x_1)-\varphi(x_2)|\le C\,|x_1-x_2|^{\varsigma},\ x_1\in J, x_2\in J,\ \varsigma\in(0,\ 1),$$

is called a Holder function with exponent ς and constant C on J.

Let ϱ be a function which has zeros at the endpoints $x=0$, $x=x_{max}$:

$$\varrho(x)=x^{\varsigma_0}(x_{max}-x)^{\varsigma_1},\quad \varsigma<\varsigma_0<1+\varsigma,\ \varsigma<\varsigma_1<1+\varsigma.$$

The functions that become Holder functions and turn into zero at the points $x=0$, $x=x_{max}$, after being multiplied by $\varrho(x)$, form a Banach space. Functions of this space $H^0_\varsigma(J,\varrho)$, are called Holder functions with weight ϱ.

The norm in the space $H^0_\varsigma(J,\varrho)$ is defined by

$$\|f(x)\|_{H^0_\varsigma(J,\varrho)}=\|\varrho(x)f(x)\|_{H_\varsigma(J)},$$

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where

$$\parallel \varrho(x)f(x) \parallel_{H_\varsigma(J)} = \|\rho(x)f(x)\|_C + \|\rho(x)f(x)\|_\varsigma,$$

$$\|\varrho(x)f(x)\|_C = \max_{x\in J}|\varrho(x)f(x)|,$$

$$\|\varrho(x)f(x)\|_\varsigma = \sup_{x_1,x_2\in J, x_1\neq x_2} |\varrho(x)f(x)|_\varsigma,$$

$$|\varrho(x)f(x)|_\varsigma = \frac{|\varrho(x_1)f(x_1) - \varrho(x_2)f(x_2)|}{|x_1 - x_2|^\varsigma}.$$

Let $\beta(x)$ be a bijective orientation-preserving displacement on J:
if $x_1 < x_2$ then $\beta((x_1) < \beta(x_2)$ for any $x_1 \in J,\ x_2 \in J$;
and let $\beta(x)$ have only two fixed points: $\beta(0) = 0$,
$\beta(x_{max}) = x_{max},\ \beta(x) \neq x,\ when\ \ x \neq 0,\ x \neq x_{max}$.

In addition, let $\beta(x)$ be a differentiable function and $\frac{d}{dx}\beta(x) \neq 0,\ x \in J$.

We consider the equation

$$(A\nu)(x) = f(x),$$

$$(A\nu)(x) \equiv a(x)(I\nu)(x) - b(x)\left(\Gamma_\beta\nu\right)(x), \quad x \in [0, x_{max}] \tag{9}$$

where I is the identity operator and Γ_β is the shift operator:

$$\left(I\nu\right)(x) = \nu(x).$$

$$\left(\Gamma_\beta\nu\right)(x) = \nu[\beta(x)].$$

Let functions $a(x),\ b(x)$ from the operator A belong to $H_\varsigma(J)$.

We will now formulate conditions of ivertibility for the operator A from (9) in the space of Holder class functions with weight [1].

Theorem

Operator A, acting in the Banach space $H^0_\varsigma(J,\varrho)$, is invertible if the following condition is fulfilled:

$$\theta_\beta[a(x), b(x), H^0_\varsigma(J,\varrho)] \neq 0, \quad x \in J,$$

where the function σ_β is defined by:

$$\theta_\beta[a(x), b(x), H^0_\varsigma(J,\varrho)] =$$

$$\begin{cases} a(x),\ when\ |a(0)| > [\beta'(0)]^{-\varsigma_0+\varsigma}\ |b(0)|; \\ \qquad and,\ |a(x_{max})| > [\beta'(x_{max})]^{-\varsigma_1+\varsigma}\ |b(x_{max})|; \\ b(x),\ when\ |a(0)| < [\beta'(0)]^{-\varsigma_0+\varsigma}\ |b(0)|; \\ \qquad and,\ |a(x_{max})| < [\beta'(x_{max})]^{-\varsigma_1+\varsigma}\ |b(x_{max})|; \\ 0 \qquad in\ other\ cases. \end{cases}$$

Corollary

If the following condition is fulfilled:

$$\theta_\beta[a(x), b(x), H^0_\varsigma(J,\varrho)] \neq 0, \quad x \in J,$$

then the operator

$$U = I - u\Gamma_\beta$$

is invertible in the space $H^o_\varsigma(J,\varrho)$ and its inverse operator is

$$U^{-1} = \left(I + u\Gamma_\beta + ... + \left(\prod_{j=0}^{n-2} u[\beta_j(x)]\right)\Gamma_\beta^{n-1}\right)\cdot$$

$$\left(I - \left(\prod_{j=0}^{n-1} u[\beta_j(x)]\right)\Gamma_\beta^n\right)^{-1}.$$

where

$$\beta_j(x) = (\Gamma_\beta^j x)(x)$$

and the number n is selected so that

$$\left\|\left(\prod_{j=0}^{n-1} u[\beta_j(x)]\right)\Gamma_\beta^n\right\|_{H^o_\varsigma(J,\varrho)} < 1.$$

IV. Analysis of solvability of the balance equations and finding of the equilibrium state of S

Let S be a system with two resources, considered in Section 2. We find the equilibrium state of the system in which the initial distribution of the group parameters by the individual parameters $v(x), w(y),\ x \in (0, x_{max})$ coincide with the final distribution, after all transformations during the time interval T.

Rewrite the balance equations of the cyclic model (7), (8) for system S

$$(Vv)(x) = \sum_{i=1}^{n} P_i p_i(x) + \sum_{i=1}^{k} R_i r_i(x) + g(x), \tag{10}$$

$$(Ww)(y) = \sum_{i=1}^{m} Q_i q_i(y) + \sum_{i=1}^{l} F_i f_i(y) + h(y), \tag{11}$$

where

$$(Vv)(x) = v(x) - c_\alpha(x)v(\alpha(x)),$$

$$g(x) = \rho(x) + \zeta(x), \quad x \in (0, x_{max}),$$

$$(Ww)(y) = w(x) - d_\beta(y)w(\beta(y)),$$

$$h(y) = \delta(y) + \xi(y), \quad y \in (0, y_{max})$$

and

$$c_\alpha(x) = c(x)\frac{d}{dx}\alpha(x),$$

$$d_\beta(y) = d(y)\frac{d}{dy}\beta(y).$$

Let us study the model in the space of Holder class functions with weight:

$$H^o_\varsigma(J,\varrho),\ J = 0, x_{max}],\ \varrho(x) = x^{\varsigma_0}(x_{max} - x)^{\varsigma_1},$$

$$\varsigma < \varsigma_0 < 1 + \varsigma,\ \varsigma < \varsigma_1 < 1 + \varsigma.$$

$$H^0_\vartheta(L,\sigma),\ L = [0, y_{max}],\ \sigma(y) = y^{\vartheta_0}(y_{max} - y)^{\vartheta_1},$$

$$\vartheta < \vartheta_0 < 1 + \vartheta,\ \vartheta < \vartheta_1 < 1 + \vartheta,$$

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considering conditions of invertibility of operators V and W fulfilled

$$\theta_\alpha[1, c_\alpha(x), H^0_\varsigma(J,\varrho)] \neq 0, \quad x \in J,$$

$$\theta_\beta[1, d_\beta(y), H^0_\vartheta(L,\sigma)] \neq 0, \quad y \in L.$$

Additionally, let us consider as known the integer positive constants N, M, for which the following inequalities are fulfilled:

$$\left\| \left(\prod_{j=0}^{N-1} c_\alpha(x)[\alpha_j(x)] \right) \Gamma_\alpha^N \right\|_{H^o_\varsigma(J,\varrho)} < 1,$$

$$\left\| \left(\prod_{j=0}^{M-1} d_\beta[\beta_j(y)] \right) \Gamma_\beta^M \right\|_{H^0_\vartheta(L,\sigma)} < 1,$$

where

$$(\Gamma_\alpha\varphi)(x) = \varphi[\alpha(x)], \quad \alpha_j(x) = (\Gamma_\alpha^j x)(x),$$
$$(\Gamma_\beta\varphi)(y) = \varphi[\beta(y)], \quad \beta_j(y) = (\Gamma_\beta^j y)(y)$$

From Theorem and Corollary from Section 3, operators inverse to operators V and W are:

$$V^{-1} = \left(I + c_\alpha\Gamma_\alpha + ... + \left(\prod_{j=0}^{N-2} c_\alpha[\alpha_j(x)] \right) \Gamma_\alpha^{N-1} \right) \cdot \left(I - \left(\prod_{j=0}^{N-1} c_\alpha[\alpha_j(x)] \right) \Gamma_\alpha^N \right)^{-1},$$

$$W^{-1} = \left(I + d_\beta\Gamma_\beta + ... + \left(\prod_{j=0}^{M-2} d_\beta[\beta_j(y)] \right) \Gamma_\beta^{M-1} \right) \cdot \left(I - \left(\prod_{j=0}^{M-1} d_\beta[\beta_j(y)] \right) \Gamma_\beta^M \right)^{-1}.$$

First, let us apply on the left side operators V^{-1}, W^{-1} to equations (10), (11); we have obtained a system of linear equations:

$$v(x) = \sum_{i=1}^{n} P_i (V^{-1}p_i)(x) + \sum_{i=1}^{k} R_i (V^{-1}r_i)(x) + (V^{-1}g)(x),$$

$$w(y) = \sum_{i=1}^{m} Q_i (W^{-1}q_i)(y) + \sum_{i=1}^{l} F_i (W^{-1}f_i)(y) + (W^{-1}h)(y).$$

For solving the system of equations, we use the idea for solution of integral equations of Fredholm of the second type with degenerate kernel [3], [4].

Having integrated the first equation of system over intervals $[\nu_{j-1}, \nu_j]$, $j = 1, 2, ..., n$ corresponding to constants

$$P_j = \int_{\nu_{j-1}}^{\nu_j} v(x)dx,$$

and over intervals $[\epsilon_{j-1}, \epsilon_j], j = 1, 2, ..., l$ corresponding to constants

$$F_j = \int_{\epsilon_{j-1}}^{\epsilon_j} v(x)dx,$$

and having subsequently integrated the second equation of system over intervals $[\mu_{j-1}, \mu_j]$, $j = 1, 2, ..., m$ corresponding to constants

$$Q_j = \int_{\mu_{j-1}}^{\mu_j} w(y)dy,$$

and over intervals $[\gamma_{j-1}, \gamma_j], j = 1, 2, ..., k$ corresponding to constants

$$R_j = \int_{\gamma_{j-1}}^{\gamma_j} w(y)dy$$

we have

$$P_j = \sum_{i=1}^{n} P_i \int_{\nu_{j-1}}^{\nu_j} (V^{-1}p_i)(x)dx + \sum_{i=1}^{k} R_i \int_{\nu_{j-1}}^{\nu_j} (V^{-1}r_i)(x)dx + \int_{\nu_{j-1}}^{\nu_j} (V^{-1}g)(x)dx, \; j=1,2,...,n,$$

$$F_j = \sum_{i=1}^{n} P_i \int_{\epsilon_{j-1}}^{\epsilon_j} (V^{-1}p_i)(x)dx + \sum_{i=1}^{k} R_i \int_{\epsilon_{j-1}}^{\epsilon_j} (V^{-1}r_i)(x)dx + \int_{\epsilon_{j-1}}^{\epsilon_j} (V^{-1}g)(x)dx, \; j=1,2,...,l,$$

$$Q_j = \sum_{i=1}^{m} Q_i \int_{\mu_{j-1}}^{\mu_j} (W^{-1}q_i)(y)dy + \sum_{i=1}^{l} F_i \int_{\mu_{j-1}}^{\mu_j} (W^{-1}f_i)(y)dy + \int_{\mu_{j-1}}^{\mu_j} (W^{-1}h)(y)dy, \; j=1,2,...,m,$$

$$R_j = \sum_{i=1}^{m} Q_i \int_{\gamma_{j-1}}^{\gamma_j} (W^{-1}q_i)(y)dy + \sum_{i=1}^{l} F_i \int_{\gamma_{j-1}}^{\gamma_j} (W^{-1}f_i)(y)dy + \int_{\gamma_{j-1}}^{\gamma_j} (W^{-1}h)(y)dy, \; j=1,2,...,k.$$

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V. Conclusions

On modelling systems with renewable resources, equations with shift appear [1], [2]. The theory of linear functional operators with shift is the adequate mathematical instrument for the investigation of such systems. In this work, we study systems with two renewable resources and our approach is based on functional operators with shift. We constructed the inverse operators with shift acting in the weighted Holder spaces and used these operators to find the equilibrium state of the considered systems.

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