Minimizing Makespan in a Class of Two-Stage Chain Reentrant Hybrid Flow Shops

Lowell Lorenzo

Abstract—**A flow shop is said to be reentrant when a recirculation property occurs. One of the many types of reentrant flows is the two-stage chain reentrant flow shop which has the stage flow sequence, $\phi = (1, 2, 1)$. This stage flow sequence describes the stages on which each of the three operations of each job are performed. The third operation exhibits the reentrant flow characteristic. In the production system that we study, each stage consists of multiple identical machines in parallel and is referred to as a two-stage chain reentrant hybrid flow shop. The objective is to schedule *n* jobs so as to minimize the makespan. Since the problem is strongly NP-hard, we develop efficient heuristic solutions and derive error bounds.**

Index Terms— **scheduling, reentrant hybrid flow shop, heuristics, error bound**

I. Introduction

The growing complexity of manufacturing processes in some industries has led to the design of a process flow structure that is best suited for its production. By modifying the process flow structure of the traditional flow shop, production has been enabled to address these added manufacturing complexities. The reentrant flow shop is one such variant of the traditional flow shop and is the subject of this paper.

A recirculation property that occurs in the process flow in a flow shop is referred to as a reentrant flow. A reentrant flow shop is distinguished from a traditional flow shop by the requirement that a job may need to revisit or reenter a stage while it flows through the production system. Many instances of reentrant flows are encountered in various industries like the semiconductor industry. In the semiconductor industry, production can be categorized into two phases namely, the wafer fabrication phase and the assembly and testing phase. The following two examples illustrate a reentrant flow application in each phase of semiconductor production.

Photolithography is one of the most complex steps in the wafer fabrication process of semiconductor production. It is an optical process used to etch multiple layers of circuit patterns on the silicon wafer. Every layer is etched by visiting the photolithography stage several times. Between successive visits to this stage, wafers have to be processed on other stages as well. The photolithography stage therefore exhibits a reentrant flow characteristic.

In the assembly and testing phase of semiconductor production, some products have multiple electronic circuits (or dies) stacked or attached on top of each other. Every time a new die is attached, it revisits that stage before it proceeds to other stages for additional operations.

The outline of this paper is as follows. A formal definition of the problem and literature review is discussed in Section II. Several lower bounds are developed in Section III and then a heuristic solution and its error bound is derived in Section IV.

II. Problem Definition and Literature Review

Consider a simple flow shop with m stages. At every stage $i, i=1,\ldots,m$, there is a single machine M_i available to process an operation of a job. Let ϕ_k be the stage visited to perform the *k*th operation of a job where $\phi_k \in \{1,2,\ldots,m\}$. Then $\phi = (\phi_1, \phi_2, \ldots, \phi_m) = (1,2,\ldots,m)$ is the stage flow sequence for all jobs and consists of m elements or operations. In a simple flow shop, the number of operations a job undergoes is equal to the number of stages. In an m-stage chain reentrant flow shop, its stage flow sequence $\phi = (1,2,\ldots,m,1)$ has now *(m+1)* operations due to an occurrence of a single reentrant operation. The single reentrant characteristic occurs in the *(m+1)*th operation which is performed at stage 1 and is referred to as the finishing operation.

When there are m_i identical parallel machines available in stage *i*, the resulting system is referred to as a hybrid flow shop. Let this group of m_i machines in stage *i* be referred to as work center WC_i in stage *i*.

In the two-stage chain reentrant flow shop, each job $J_j, j = 1,\ldots,n$ has a stage flow sequence $\phi = (1,2,1)$. The processing time of the first operation of job J_j is a_j, its processing time in the second operation is b_j and the reentrant processing time for the finishing operation is c_j. Let the processing time vector for each job be (a_j, b_j, c_j) or simply referred to now as the processing times of J_j in the two-stage chain reentrant flow shop. Since each job is processed in every operation in the chain reentrant flow shop, then $A = (a_1,\ldots,a_n), B = (b_1,\ldots,b_n), C = (c_1,\ldots,c_n)$ are the vectors of processing times for each operation in ϕ respectively.

In the two-stage chain reentrant hybrid flow shop, there are two work centers WC_1 and WC_2 with m_1 and m_2 identical machines in parallel at stages 1 and 2 respectively. There are n jobs that have to be processed and the completion time of J_j occurs when the third or finishing operation at any of the m_1 machines in WC_1 is completed.

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Let CRF_{m_1,m_2} be a two-stage chain reentrant hybrid flow shop where our objective is to find a schedule that minimizes the maximum completion time. Using the three-tuple convention of defining scheduling problems proposed by Graham *et al*. (1979), minimizing makespan in CRF_{m_1,m_2} can be identified by $F(m_1, m_2|chain\ reentrant|C_{max})$ for which the optimal objective function value is $C^*_{CRF_{m_1,m_2}}$.

The CRF_{m_1,m_2} system is a general case of the two-stage chain reentrant flow shop studied by Wang *et al.* (1997). In their paper, they study the makespan minimization of $CRF_{1,1}$ and derive a Johnson based heuristic solution with complexity *O(nlogn)* and worst-case error bound of $3/2$ is derived. In Drobouchevitch and Strusevich (1999), another heuristic solution is presented for the same problem with complexity *O(nlogn)* and an improved worst-case error bound of $4/3$.

Lev and Adiri (1984) study the makespan minimization of an *m*-stage chain reentrant flow shop wherein the stage flow sequence vector is $\phi = (1,2, \dots, m-1, m, 1)$. They consider the special case when *m*=2 which is equivalent to $CRF_{1,1}$ and prove that it is NP-hard by using a reduction from *Partition*. However, the exact complexity of the problem is not yet completely determined because a pseudopolynomial algorithm has not yet been presented.

In Kubiak *et al.* (1996), another reentrant shop referred to as a reentrant job shop with a hub is studied wherein the stage flow sequence is $\phi = (1,2, \dots, m-1, m, 1)$ and the first stage is designated as the hub. Each job has *2m-1* operations and the objective function for this problem is to minimize the mean flow time. An optimal shortest-processing time (SPT) based schedule derived from a dynamic programming algorithm is presented when two key assumptions are made for the results to hold.

Lu and Kumar (1991) consider a problem encountered in scheduling wafer parts in a semiconductor facility with reentrant flows. They are interested in evaluating the performance and stability of various scheduling policies based on due date and buffer priorities. They have an intuitive insight that some of these policies may perform well based on the following performance measures namely: (1) mean delay or manufacturing cycle time and (2) variance of the delay. Simulations are used to confirm the intuitive insight of these policies based on these performance measures.

Bispo and Tayur (2001) study a cyclic reentrant flow shop with *m* stages with a stage flow sequence $\phi = (1,2, \dots, m)$ which repeats *L* times. Therefore, there are *mL* operations performed on each job. In their paper, they look at an approach of combining various capacity allocation, inventory management and production control rules in managing this reentrant flow shop using a simulation based optimization approach. The performance of these rules are then evaluated based on the holding and backlog costs they generate.

Aldakhilallah and Ramesh (2001) develop two heuristics for a reentrant job shop wherein a single product is produced repetitively on a set of machines wherein cycle time and flow time are simultaneously minimized.

A specific application of minimizing makespan in a reentrant job shop for a truck manufacturing company is studied by Hwang and Sun (1997). The job shop has two machines with the first machine having the reentrant flow characteristic. A dynamic programming approach is used to solve the problem with the different possible work flows and precedence constraints encountered in the production system.

Hall *et al.* (2002) study a two-stage cyclic job shop with one machine at each stage. Each job has three operations and there are two types of jobs. Type 1 jobs have a stage flow sequence $\phi = (1,2,1)$ while Type 2 jobs have a stage flow sequence $\phi = (2,1,2)$. A pseudopolynomial time algorithm is developed for a special case of this cyclic job shop whose objective function is to minimize the cycle time.

Gupta and Tunc (1994) consider the two-stage hybrid flow shop with setup and removal times with a minimization makespan objective. Since setup and removal times are considered in this two-stage hybrid flow shop, a definition of makespan with respect to a defined reference point is clearly made. In this two-stage hybrid flow shop, they consider the case when $m_1 = 1$ and $m_2 \geq n$ for which an optimal polynomial algorithm is developed. When $m_2 < n$, the problem becomes NP-hard and several heuristics are developed with the conduct of corresponding computational experiments to verify the efficiency of the heuristics.

Lee and Vairaktarakis (1994) develop heuristic solutions and error bounds for a hybrid flow shop with *k*-stages with the objective of minimizing makespan. In the two-stage hybrid flow shop, the heuristic has a worst-case error bound of $2 - 1/\max\{m_1, m_2\}$. Guinet and Solomon (1996) also study this type of hybrid flow shop but with the objective of minimizing maximum tardiness or makespan when the due dates are equal to zero. They develop several heuristics and computational experiments are then conducted to evaluate their performance against lower bounds.

Koulamas and Kyparisis (2000) consider the two-stage and three-stage hybrid flow shops with the minimization makespan objective and propose linear time algorithm heuristics and derive their corresponding absolute worst-case error bounds.

In Hoogeveen *et al.* (1996), the complexity of the problems $F(1,2||C_{max})$ and $F(2,1||C_{max})$ are shown to be strongly NP-hard. Therefore, the $F(m_1, m_2|chain\ reentrant|C_{max})$ problem is strongly NP-hard as well. Due to its complexity, we consider an approach of reducing the problem to a simpler form. In Buten and Shen (1973), they reduce a two-stage hybrid flow shop into a two-stage flow shop which we refer to as an auxiliary flow shop (AF). In our analysis, we make use of an auxiliary flow shop as follows. An auxiliary two-stage flow shop can be constructed from any hybrid two-stage flow shop by the following procedure. Consider a two-stage hybrid flow shop FS_{m_1,m_2} with two work centers WC_1 and WC_2 with m_1 and m_2 machines in stages one and two respectively. The processing times for job J_j are (a_j, b_j) in stages one and two respectively. The equivalent auxiliary flow shop $AF_{1,1}$ can be constructed with processing times $\left(\frac{1}{m_1} a_j, \frac{1}{m_2} b_j\right)$ for stages one and two respectively.

For the two-stage chain reentrant hybrid flow shop CRF_{m_1,m_2}, we construct two auxiliary two-stage flow shops. These two auxiliary two-stage flow shops are $AF1_{1,1}$ and

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$AF2_{1,1}$ with their respective processing times $\left(\frac{1}{m_1}a_j, \frac{1}{m_2}b_j\right)$ and $\left(\frac{1}{m_2}b_j, \frac{1}{m_1}c_j\right)$ and their corresponding makespans $C_{AF1_{1,1}}$ and $C_{AF2_{1,1}}$. The AFs just introduced help in the development of lower bounds and this is the focus of the next section.

III. Lower Bounds for $C^*_{CRF_{m_1,m_2}}$

Lower bounds for $C^*_{CRF_{m_1,m_2}}$ can be developed from the constructed auxiliary two-stage flow shops described in the previous section. It is well known that minimizing the makespan of a two-stage flow shop using Johnson's Algorithm (JA) yields the optimal solution in *O(nlogn)* time. In brief, the JA algorithm schedules job J_i before job J_j if $\min(a_i, b_j) \le \min(a_j, b_i)$. By applying JA to the auxiliary flow shops $AF1_{1,1}$ and $AF2_{1,1}$, the following set of lower bounds using two of the three processing times is established.

Lemma 1. Let $LB_1 = max\left(C_{AF1_{1,1}}, C_{AF2_{1,1}}, \frac{1}{m_1}\sum_{j=1}^{n}(a_j + c_j)\right)$. Then $LB_1 \le C^*_{CRF_{m_1,m_2}}$.

Proof: Consider the two-stage hybrid flow shops FS_{m_1,m_2} with processing times (a_j, b_j) and FS_{m_2,m_1} with processing times (b_j, c_j). Let $C^*_{FS1_{m_1,m_2}}$ and $C^*_{FS2_{m_2,m_1}}$ be the optimal makespans for FS_{m_1,m_2} and FS_{m_2,m_1} respectively. Clearly, $C^*_{FS1_{m_1,m_2}} \le C^*_{CRF_{m_1,m_2}}$ and $C^*_{FS2_{m_2,m_1}} \le C^*_{CRF_{m_1,m_2}}$.

Consider $AF1_{1,1}$ and $AF2_{1,1}$, with processing times $\left(\frac{1}{m_1}a_j, \frac{1}{m_2}b_j\right)$ and $\left(\frac{1}{m_2}b_j, \frac{1}{m_1}c_j\right)$ respectively and their corresponding JA schedules S_1 and S_2. Jobs r_k, k=1,2 are the critical jobs of the JA schedules S_k. The critical job in a two-stage flow shop is defined to be the last job wherein the stage 2 operation starts immediately after the stage 1 operation. Lee and Vairaktarakis (1994), show that, $C_{AF1_{1,1}} = \frac{1}{m_1}\sum_{j=1}^{r_1} a_j + \frac{1}{m_2}\sum_{j=r_1}^{n} b_j \le C^*_{FS1_{m_1,m_2}} \le C^*_{CRF_{m_1,m_2}}$, and similarly by symmetry, $C_{AF2_{1,1}} = \frac{1}{m_2}\sum_{j=1}^{r_2} b_j + \frac{1}{m_1}\sum_{j=r_2}^{n} c_j \le C^*_{FS2_{m_2,m_1}} \le C^*_{CRF_{m_1,m_2}}$. Because the makespan $C^*_{CRF_{m_1,m_2}}$ is always attained in WC_1, then $\frac{1}{m_1}\sum_{j=1}^{n} a_j + \frac{1}{m_1}\sum_{j=1}^{n} c_j \le C^*_{CRF_{m_1,m_2}}$. The last three inequalities yield the desired result.■

The following four lemmas are now used to develop other lower bounds to the problem by incorporating the processing times a_j, b_j and c_j simultaneously in the bound.

Lemma 2. There is a permutation 1,2,…,n associated with an arbitrary schedule S such that for every $1 \le i \le n$, there is a $1 \le k_i \le i$ such that $\frac{1}{2}\sum_{j=1}^{i}\left(\frac{a_j}{m_1} + \frac{b_j}{m_2}\right) \le \frac{1}{m_1}\sum_{j=1}^{k_i} a_j + \frac{1}{m_2}\sum_{j=k_i}^{i} b_j$.

Proof: Consider an auxiliary two-stage flow shop with i jobs and processing times $\left(\frac{1}{2m_1}a_j, \frac{1}{2m_2}b_j\right)$. Let J_{k_i} be the critical job for schedule S. Since $k_i \ge 1$, $\frac{1}{2m_1}a_1 + \sum_{j=1}^{i}\frac{1}{2m_2} b_j \le \frac{1}{2m_1}\sum_{j=1}^{k_i} a_j + \frac{1}{2m_2}\sum_{j=k_i}^{i} b_j$. Similarly, since $k_i \le i$, $\sum_{j=1}^{i}\frac{1}{2m_1} a_j + \frac{1}{2m_1}b_i \le \frac{1}{2m_1}\sum_{j=1}^{k_i} a_j + \frac{1}{2m_2}\sum_{j=k_i}^{i} b_j$. Adding these last two expressions and removing the terms $\frac{1}{2m_1}a_1$ and $\frac{1}{2m_1}b_i$ completes the proof.■

Lemma 3. There is a permutation 1,2,…,n associated with an arbitrary schedule S such that for every $1 \le i \le n$, there is a $1 \le k_i \le i$, such that $\frac{1}{m_1+m_2}\sum_{j=1}^{i}(a_j + b_j) \le \frac{1}{m_1}\sum_{j=1}^{k_i} a_j + \frac{1}{m_2}\sum_{j=k_i}^{i} b_j$.

Proof: Let $\frac{1}{m_1}\sum_{j=1}^{k_i} a_j + \frac{1}{m_2}\sum_{j=k_i}^{i} b_j$ be the completion time of job i in schedule S for the auxiliary two-stage flow shop with processing times $\left(\frac{1}{m_1}a_j, \frac{1}{m_2}b_j\right)$. Let J_{k_i} be the critical job for that schedule and let C_{i1} and C_{i2} be the completion times of the stage 1 and 2 operations of job i respectively. We will assume that an operation in stage 1 starts as early as possible so that there is no idle time between operations in stage 1. Thus,

$$C_{i1} = \frac{1}{m_1}\sum_{j=1}^{i} a_j,\ i = 1,2,\dots,n \qquad (1)$$

$$C_{i-1,2} = \frac{1}{m_1}\sum_{j=1}^{k_i} a_j + \frac{1}{m_2}\sum_{j=k_i}^{i} b_j,\ 1 \le k_i \le i - 1 \qquad (2)$$

Due to the flow shop constraints,

$$C_{i2} = max(C_{i1}, C_{i-1,2}) + \frac{1}{m_2}b_i. \qquad (3)$$

We will now show that for every i, there is a $1 \le k_i \le i$ such that, $D_{1i} = \frac{1}{m_1}\sum_{j=1}^{k_i} a_j + \frac{1}{m_2}\sum_{j=k_i}^{i} b_j - \frac{1}{m_1+m_2}\sum_{j=1}^{i}(a_j + b_j) \ge 0$.

We distinguish two cases.

Case 1 : $k_i = i$.

Applying the definition of D_{1i} when $k_i = i$ we have, $D_{1i} = \frac{1}{m_1}\sum_{j=1}^{i} a_j + \frac{1}{m_2}b_i - \frac{1}{m_1+m_2}\sum_{j=1}^{i} a_j - \frac{1}{m_1+m_2}\sum_{j=1}^{i} b_j = \frac{m_2}{m_1(m_1+m_2)}\sum_{j=1}^{i} a_j + \frac{m_1 b_i}{m_2(m_1+m_2)} - \frac{1}{m_1+m_2}\sum_{j=1}^{i-1} b_j$.

Since job J_i is the critical job, $max(C_{i1}, C_{i-1,2}) = C_{i1}$. Thus $C_{i1} \ge C_{i-1,2}$ or $\frac{1}{m_1}\sum_{j=1}^{i} a_j \ge \frac{1}{m_1}a_1 + \frac{1}{m_2}\sum_{j=1}^{i-1} b_j$ or $\frac{1}{m_1}\sum_{j=2}^{i} a_j \ge \frac{1}{m_2}\sum_{j=1}^{i-1} b_j$. Multiplying both sides by $\frac{m_2}{(m_1+m_2)}$ we get,

$$\frac{m_2}{m_1(m_1+m_2)}\sum_{j=2}^{i} a_j - \frac{1}{m_1+m_2}\sum_{j=1}^{i-1} b_j \ge 0. \qquad (4)$$

Adding $\frac{m_2}{m_1(m_1+m_2)}a_1 + \frac{m_1 b_i}{m_2(m_1+m_2)}$ on the left hand side of (4) we get,

$$\frac{m_2}{m_1(m_1+m_2)}\sum_{j=1}^{i} a_j + \frac{m_1 b_i}{m_2(m_1+m_2)} - \frac{1}{m_1+m_2}\sum_{j=1}^{i-1} b_j = D_{1i} \ge 0. \qquad (5)$$

Case 2 : $k_i = i_0, 1 \le i_0 \le i - 1$.

Applying the definition of D_{1i} when $k_i = i_0$ we have, $D_{1i} = \frac{1}{m_1}\sum_{j=1}^{i_0} a_j + \frac{1}{m_2}\sum_{j=i_0}^{i} b_j - \frac{1}{m_1+m_2}\sum_{j=1}^{i} a_j - \frac{1}{m_1+m_2}\sum_{j=1}^{i} b_j = \frac{m_2}{m_1(m_1+m_2)}\sum_{j=1}^{i_0} a_j - \frac{1}{m_1+m_2}\sum_{j=i_0+1}^{i} a_j + \frac{m_1}{m_2(m_1+m_2)}\sum_{j=i_0}^{i} b_j - \frac{1}{m_1+m_2}\sum_{j=1}^{i_0-1} b_j$.

Since job J_{i_0} is the critical job, $max(C_{i1}, C_{i-1,2}) = C_{i-1,2}$. Thus $C_{i1} \le C_{i-1,2}$ or $\frac{1}{m_1}\sum_{j=1}^{i_0} a_j + \frac{1}{m_2}\sum_{j=i_0}^{i-1} b_j \ge \frac{1}{m_1}\sum_{j=1}^{i} a_j$ or $\frac{1}{m_2}\sum_{j=i_0}^{i-1} b_j - \frac{1}{m_1}\sum_{j=i_0+1}^{i} a_j \ge 0$. Multiplying all terms by $\frac{m_1}{(m_1+m_2)}$ we get,

$$\frac{m_1}{m_2(m_1+m_2)}\sum_{j=i_0}^{i-1} b_j - \frac{1}{m_1+m_2}\sum_{j=i_0+1}^{i} a_j \ge 0 \quad (6)$$

Because J_{i_0} is the critical job, it also implies that $\frac{1}{m_1}\sum_{j=1}^{i_0} a_j + \frac{1}{m_2}\sum_{j=i_0}^{i-1} b_j \ge \frac{1}{m_1}a_1 + \frac{1}{m_2}\sum_{j=1}^{i-1} b_j$ or $\frac{1}{m_1}\sum_{j=2}^{i_0} a_j - \frac{1}{m_2}\sum_{j=1}^{i_0-1} b_j \ge 0$. Multiplying both sides by $\frac{m_2}{(m_1+m_2)}$ we get,

$$\frac{m_2}{m_1(m_1+m_2)}\sum_{j=2}^{i_0} a_j - \frac{1}{m_1+m_2}\sum_{j=1}^{i_0-1} b_j \ge 0 \quad (7)$$

Adding $\frac{m_2}{m_1(m_1+m_2)}a_1$on the left hand side of (7) and $\frac{m_1}{m_2(m_1+m_2)}b_i$on the lefthand side of (6) and then adding these two inequalities together yields,

$$\frac{m_2}{m_1(m_1+m_2)}\sum_{j=1}^{i_0} a_j - \frac{1}{m_1+m_2}\sum_{j=i_0+1}^{i} a_j + \frac{m_1}{m_2(m_1+m_2)}\sum_{j=i_0}^{i} b_j - \frac{1}{m_1+m_2}\sum_{j=1}^{i_0-1} b_j = D_{1i} \ge 0. \quad (8)$$

This completes the proof.■

Lemma 4. There is a permutation 1,2,…,n associated with an arbitrary schedule S such that for every $1 \le i \le n$, there is a $1 \le k_i \le i$ such that $\frac{1}{m_1+1}\sum_{j=1}^{i}\left(a_j + \frac{b_j}{m_2}\right) \le \frac{1}{m_1}\sum_{j=1}^{k_i} a_j + \frac{1}{m_2}\sum_{j=k_i}^{i} b_j$.

Proof: We will now show that for every i, there is a $1 \le k_i \le i$ such that, $D_{2i} = \frac{1}{m_1}\sum_{j=1}^{k_i} a_j + \frac{1}{m_2}\sum_{j=k_i}^{i} b_j - \frac{1}{m_1+1}\sum_{j=1}^{i}\left(a_j + \frac{b_j}{m_2}\right) \ge 0$.

We distinguish two cases.

Case 1 : $k_i = i$.

Applying the definition of D_{2i} when $k_i = i$ we have, $D_{2i} = \frac{1}{m_1}\sum_{j=1}^{k_i} a_j + \frac{1}{m_2}\sum_{j=k_i}^{i} b_j - \frac{1}{m_1+1}\sum_{j=1}^{i}\left(a_j + \frac{b_j}{m_2}\right) = \frac{1}{m_1(m_1+1)}\sum_{j=1}^{i} a_j + \frac{m_1 b_i}{m_2(m_1+1)} - \frac{1}{m_2(m_1+1)}\sum_{j=1}^{i-1} b_j$.

Since job J_i is the critical job, $max(C_{i1}, C_{i-1,2}) = C_{i1}$ or $C_{i1} \ge C_{i-1,2}$. Thus, $\frac{1}{m_1}\sum_{j=1}^{i} a_j \ge \frac{1}{m_1}a_1 + \frac{1}{m_2}\sum_{j=1}^{i-1} b_j$ or $\frac{1}{m_1}\sum_{j=2}^{i} a_j \ge \frac{1}{m_2}\sum_{j=1}^{i-1} b_j$. Multiplying both sides by $\frac{1}{(m_1+1)}$ we get,

$$\frac{1}{m_1(m_1+1)}\sum_{j=2}^{i} a_j - \frac{1}{m_2(m_1+1)}\sum_{j=1}^{i-1} b_j \ge 0. \quad (9)$$

Adding $\frac{1}{m_1(m_1+1)}a_1 + \frac{m_1 b_i}{m_2(m_1+1)} \ge 0$ on the left hand side of (9) we get,

$$\frac{1}{m_1(m_1+1)}\sum_{j=1}^{i} a_j + \frac{m_1 b_i}{m_2(m_1+1)} - \frac{1}{m_2(m_1+1)}\sum_{j=1}^{i-1} b_j = D_{2i} \ge 0. \quad (10)$$

Case 2 : $k_i = i_0, 1 \le i_0 \le i-1$.

Applying the definition of D_{2i} when $k_i = i_0$ we have, $D_{2i} = \frac{1}{m_1}\sum_{j=1}^{i_0} a_j + \frac{1}{m_2}\sum_{j=i_0}^{i} b_j - \frac{1}{m_1+1}\sum_{j=1}^{i}\left(a_j + \frac{b_j}{m_2}\right) = \frac{1}{m_1(m_1+1)}\sum_{j=1}^{i_0} a_j + \frac{1}{(m_1+1)}\sum_{j=i_0+1}^{i} a_j + \frac{m_1}{m_2(m_1+1)}\sum_{j=i_0}^{i} b_j - \frac{1}{m_2(m_1+1)}\sum_{j=1}^{i_0-1} b_j$.

Since job J_{i_0} is the critical job, $max(C_{i1}, C_{i-1,2}) = C_{i-1,2}$. Thus $C_{i1} \le C_{i-1,2}$ or $\frac{1}{m_1}\sum_{j=1}^{i_0} a_j + \frac{1}{m_2}\sum_{j=i_0}^{i-1} b_j \ge \frac{1}{m_1}\sum_{j=1}^{i} a_j$ or $\frac{1}{m_2}\sum_{j=i_0}^{i-1} b_j - \frac{1}{m_1}\sum_{j=i_0+1}^{i} a_j \ge 0$. Multiplying both sides by $\frac{m_1}{(m_1+1)}$ we get,

$$\frac{m_1}{m_2(m_1+1)}\sum_{j=i_0}^{i-1} b_j - \frac{1}{m_1+1}\sum_{j=i_0+1}^{i} a_j \ge 0 \quad (11)$$

Because J_{i_0} is the critical job, it also implies that $\frac{1}{m_1}\sum_{j=1}^{i_0} a_j + \frac{1}{m_2}\sum_{j=i_0}^{i-1} b_j \ge \frac{1}{m_1}a_1 + \frac{1}{m_2}\sum_{j=1}^{i-1} b_j$ or $\frac{1}{m_1}\sum_{j=2}^{i_0} a_j - \frac{1}{m_2}\sum_{j=1}^{i_0-1} b_j \ge 0$. Multiplying both sides by $\frac{1}{(m_1+1)}$ we get,

$$\frac{1}{m_1(m_1+1)}\sum_{j=2}^{i_0} a_j - \frac{1}{m_2(m_1+1)}\sum_{j=1}^{i_0-1} b_j \ge 0 \quad (12)$$

Adding $\frac{1}{m_1(m_1+1)}a_1$ on the left hand side of (12) and $\frac{m_1}{m_2(m_1+1)}b_i$ on the lefthand side of (11) and then adding these two inequalities together yields,

$$\frac{1}{m_1(m_1+1)}\sum_{j=1}^{i_0} a_j - \frac{1}{(m_1+1)}\sum_{j=i_0+1}^{i} a_j + \frac{m_1}{m_2(m_1+1)}\sum_{j=i_0}^{i} b_j - \frac{1}{m_2(m_1+1)}\sum_{j=1}^{i_0-1} b_j = D_{2i} \ge 0. \quad (13)$$

This completes the proof.■

Lemma 5. There is a permutation 1,2,…,n associated with an arbitrary schedule S such that for every $1 \le i \le n$, there is a $1 \le k_i \le i$ such that $\frac{1}{m_2+1}\sum_{j=1}^{i}\left(\frac{a_j}{m_1} + b_j\right) \le \frac{1}{m_1}\sum_{j=1}^{k_i} a_j + \frac{1}{m_2}\sum_{j=k_i}^{i} b_j$.

Proof: We will now show that for every i, there is a $1 \le k_i \le i$ such that, $D_{3i} = \frac{1}{m_1}\sum_{j=1}^{k_i} a_j + \frac{1}{m_2}\sum_{j=k_i}^{i} b_j - \frac{1}{m_2+1}\sum_{j=1}^{i}\left(\frac{a_j}{m_1} + b_j\right) \ge 0$.

We distinguish two cases.

Case 1 : $k_i = i$.

Applying the definition of D_{3i} when $k_i = i$ we have, $D_{3i} = \frac{1}{m_1}\sum_{j=1}^{k_i} a_j + \frac{1}{m_2}b_i - \frac{1}{m_2+1}\sum_{j=1}^{i}\left(\frac{a_j}{m_1} + b_j\right) = \frac{m_2}{m_1(m_2+1)}\sum_{j=1}^{i} a_j + \frac{b_i}{(m_2+1)} - \frac{1}{(m_2+1)}\sum_{j=1}^{i-1} b_j$.

Since job J_i is the critical job, $max(C_{i1}, C_{i-1,2}) = C_{i1}$. Thus, $C_{i1} \ge C_{i-1,2}$ or $\frac{1}{m_1}\sum_{j=1}^{i} a_j \ge \frac{1}{m_1}a_1 + \frac{1}{m_2}\sum_{j=1}^{i-1} b_j$ or $\frac{1}{m_1}\sum_{j=2}^{i} a_j \ge \frac{1}{m_2}\sum_{j=1}^{i-1} b_j$. Multiplying both sides by $\frac{m_2}{(m_2+1)}$ we get,

$$\frac{m_2}{m_1(m_2+1)}\sum_{j=2}^{i} a_j - \frac{1}{(m_2+1)}\sum_{j=1}^{i-1} b_j \ge 0. \quad (14)$$

Adding $\frac{m_2}{m_1(m_2+1)}a_1 + \frac{b_i}{(m_2+1)} \ge 0$ on the left hand side of (14) we get,

$$\frac{m_2}{m_1(m_2+1)}\sum_{j=1}^{i} a_j + \frac{b_i}{(m_2+1)} - \frac{1}{(m_2+1)}\sum_{j=1}^{i-1} b_j = D_{3i} \ge 0. \quad (15)$$

Case 2 : $k_i = i_0, 1 \le i_0 \le i-1$.

Applying the definition of D_{3i} when $k_i = i_0$ we have, $D_{3i} = \frac{1}{m_1}\sum_{j=1}^{i_0} a_j + \frac{1}{m_2}\sum_{j=i_0}^{i} b_j - \frac{1}{m_2+1}\sum_{j=1}^{i}\left(\frac{a_j}{m_1} + b_j\right) = \frac{m_2}{m_1(m_2+1)}\sum_{j=1}^{i_0} a_j + \frac{1}{m_1(m_2+1)}\sum_{j=i_0+1}^{i} a_j + \frac{1}{m_2(m_2+1)}\sum_{j=i_0}^{i} b_j - \frac{1}{(m_2+1)}\sum_{j=1}^{i_0-1} b_j$.

Since job J_{i_0} is the critical job, $max(C_{i1}, C_{i-1,2}) = C_{i-1,2}$. Thus $C_{i1} \le C_{i-1,2}$ or $\frac{1}{m_1}\sum_{j=1}^{i_0} a_j + \frac{1}{m_2}\sum_{j=i_0}^{i-1} b_j \ge \frac{1}{m_1}\sum_{j=1}^{i} a_j$ or $\frac{1}{m_2}\sum_{j=i_0}^{i-1} b_j - \frac{1}{m_1}\sum_{j=i_0+1}^{i} a_j \ge 0$. Multiplying both sides by $\frac{1}{(m_2+1)}$ we get,

$$\frac{1}{m_2(m_2+1)}\sum_{j=i_0}^{i-1} b_j - \frac{1}{m_1(m_2+1)}\sum_{j=i_0+1}^{i} a_j \ge 0 \quad (16)$$

Because J_{i_0} is the critical job, it also implies that $\frac{1}{m_1}\sum_{j=1}^{i_0} a_j + \frac{1}{m_2}\sum_{j=i_0}^{i-1} b_j \ge \frac{1}{m_1}a_1 + \frac{1}{m_2}\sum_{j=1}^{i-1} b_j$ or $\frac{1}{m_1}\sum_{j=2}^{i_0} a_j - \frac{1}{m_2}\sum_{j=1}^{i_0-1} b_j \ge 0$. Multiplying both sides by

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$\frac{m_2}{(m_2+1)}$ we get,

$$\frac{m_2}{m_1(m_2+1)}\sum_{j=2}^{i_0} a_j - \frac{1}{(m_2+1)}\sum_{j=1}^{i_0-1} b_j \geq 0 \quad (17)$$

Adding $\frac{m_2}{m_1(m_2+1)}a_1$ on the left hand side of (17) and $\frac{1}{m_2(m_2+1)}b_i$ on the lefthand side of (16) and then adding these two inequalities together yields,

$$\frac{m_2}{m_1(m_2+1)}\sum_{j=1}^{i_0} a_j + \frac{1}{m_1(m_2+1)}\sum_{j=i_0+1}^{i} a_j + \frac{1}{m_2(m_2+1)}\sum_{j=i_0}^{i} b_j - \frac{1}{(m_2+1)}\sum_{j=1}^{i_0-1} b_j = D_{3i} \geq 0. \quad (18)$$

This completes the proof.■

Lemma 6. There is an optimal schedule S^* for CRF_{m_1,m_2} such that, for every $1 \leq k_1 \leq k_2 \leq n$, $\frac{1}{m_1}\sum_{j=1}^{k_1} a_j + \frac{1}{m_2}\sum_{j=k_1}^{k_2} b_j + \frac{1}{m_1}\sum_{j=k_2}^{n} c_j \leq C^*_{CRF_{m_1,m_2}}$.

Proof: Let $S_a = \{J_1, J_2, \ldots, J_n\}$ be the set of jobs ordered in nondecreasing order of completion times in the first operation in S^*. Let $C_{i,k}$, $1 \leq i \leq n$; k=1,2,3, be the completion time of job i in operation k. Consider the first k_1 jobs being processed in the first operation, then for i=k_1,

$$\frac{1}{m_1}\sum_{j=1}^{k_1} a_j \leq C_{k_1,1}. \quad (19)$$

Because of the flow shop constraints in S^*, the next $k_2 - k_1 + 1$ jobs in the second operation start no earlier than the completion time of the first k_1 jobs in the first operation. Hence,

$$\frac{1}{m_2}\sum_{j=k_1}^{k_2} b_j \leq max_{k_1 \leq r \leq k_2}\{C_{r,2}\} - C_{k_1,1}. \quad (20)$$

For the remaining $n - k_2 + 1$ jobs in the third operation, they can start no earlier than its completion time in the second operation. Because of this,

$$\frac{1}{m_1}\sum_{j=k_2}^{n} c_j \leq C^*_{CRF_{m_1,m_2}} - min_{k_1 \leq r \leq k_2}\{C_{r,2}\}. \quad (21)$$

Adding (19), (20) and (21) by parts yields the desired result.■

Lemma 7. Let C_{JA_1} be the makespan derived by JA for the auxiliary two-stage flow shop problem with processing times $(a'_j + b'_j, c'_j)$. For any of the following set of values of a'_j, b'_j and c'_j,

$$a'_j = \frac{1}{2m_1}a_j,\ b'_j = \frac{1}{2m_2}b_j,\ c'_j = \frac{1}{m_1}c_j \text{ or} \quad (22)$$

$$a'_j = \frac{1}{m_1+m_2}a_j, b'_j = \frac{1}{m_1+m_2}b_j,\ c'_j = \frac{1}{m_1}c_j \text{ or} \quad (23)$$

$$a'_j = \frac{1}{m_1+1}a_j, b'_j = \frac{1}{m_2(m_1+1)}b_j,\ c'_j = \frac{1}{m_1}c_j \text{ or} \quad (24)$$

$$a'_j = \frac{1}{m_1(m_2+1)}a_j, b'_j = \frac{1}{m_2+1}b_j,\ c'_j = \frac{1}{m_1}c_j, \quad (25)$$

$C_{JA_1} \leq C^*_{CRF_{m_1,m_2}}$.

Let C_{JA_2} be the makespan derived by JA for the auxiliary two-stage flow shop problem with processing times $(b''_j + c''_j, a''_j)$, For any of the following set of values of a''_j, b''_j, and c''_j,

$$a''_j = \frac{1}{m_1}a_j,\ b''_j = \frac{1}{2m_2}b_j,\ c''_j = \frac{1}{2m_1}c_j \text{ or} \quad (26)$$

$$a''_j = \frac{1}{m_1}a_j,\ b''_j = \frac{1}{m_1+m_2}b_j,\ c''_j = \frac{1}{m_1+m_2}c_j \text{ or} \quad (27)$$

$$a''_j = \frac{1}{m_1}a_j, b''_j = \frac{1}{m_2(m_1+1)}b_j,\ c''_j = \frac{1}{m_1+1}c_j \text{ or} \quad (28)$$

$$a''_j = \frac{1}{m_1}a_j, b''_j = \frac{1}{m_2+1}b_j,\ c''_j = \frac{1}{m_1(m_2+1)}c_j, \quad (29)$$

$C_{JA_2} \leq C^*_{CRF_{m_1,m_2}}$.

Proof: Consider the first set of values when $a'_j = \frac{1}{2m_1}a_j$, $b'_j = \frac{1}{2m_2}b_j$, $c'_j = \frac{1}{m_1}c_j$. In Lemma 2, we have shown that there is a schedule S such that, for every $1 \leq i \leq k_2$ there is a $1 \leq k_1 \leq i$, such that $\frac{1}{2}\sum_{j=1}^{k_2}\left(\frac{a_j}{m_1} + \frac{b_j}{m_2}\right) \leq \frac{1}{m_1}\sum_{j=1}^{k_1} a_j + \frac{1}{m_2}\sum_{j=k_1}^{k_2} b_j$. Adding $\frac{1}{m_1}\sum_{j=k_2}^{n} c_j$ to both sides, we get, $\frac{1}{2}\sum_{j=1}^{k_2}\left(\frac{a_j}{m_1} + \frac{b_j}{m_2}\right) + \frac{1}{m_1}\sum_{j=k_2}^{n} c_j \leq \frac{1}{m_1}\sum_{j=1}^{k_1} a_j + \frac{1}{m_2}\sum_{j=k_1}^{k_2} b_j + \frac{1}{m_1}\sum_{j=k_2}^{n} c_j$.

In Lemma 6, we have shown that for an optimal solution, S^* for CRF_{m_1,m_2}, for every $1 \leq k_1 \leq k_2 \leq n$, $\frac{1}{m_1}\sum_{j=1}^{k_1} a_j + \frac{1}{m_2}\sum_{j=k_1}^{k_2} b_j + \frac{1}{m_1}\sum_{j=k_2}^{n} c_j \leq C^*_{CRF_{m_1,m_2}}$. Therefore, $\frac{1}{2}\sum_{j=1}^{k_2}\left(\frac{a_j}{m_1} + \frac{b_j}{m_2}\right) + \frac{1}{m_1}\sum_{j=k_2}^{n} c_j \leq \frac{1}{m_1}\sum_{j=1}^{k_1} a_j + \frac{1}{m_2}\sum_{j=k_1}^{k_2} b_j + \frac{1}{m_1}\sum_{j=k_2}^{n} c_j \leq C^*_{CRF_{m_1,m_2}}$.

Let $\sigma = \{J_{\sigma(1)}, J_{\sigma(2)}, \ldots, J_{\sigma(n)}\}$ be a permutation ordered in nondecreasing order of completion times of the first operations in S^*. This sequence σ is not necessarily optimal for the auxiliary two-stage flow shop problem with processing times $(a'_j + b'_j, c'_j)$. Hence, by applying the JA for this auxiliary problem, $C_{JA_1} \leq C^*_{CRF_{m_1,m_2}}$.

Similarly, $C_{JA_1} \leq C^*_{CRF_{m_1,m_2}}$ can also be obtained for the following set of values of a'_j, b'_j and c'_j by letting: $a'_j = \frac{1}{m_1+m_2}a_j, b'_j = \frac{1}{m_1+m_2}b_j$, $c'_j = \frac{1}{m_1}c_j$, by using Lemma 3, $a'_j = \frac{1}{m_1+1}a_j, b'_j = \frac{1}{m_2(m_1+1)}b_j$, $c'_j = \frac{1}{m_1}c_j$, by using Lemma 4, and $a'_j = \frac{1}{m_1(m_2+1)}a_j, b'_j = \frac{1}{m_2+1}b_j$, $c'_j = \frac{1}{m_1}c_j$, by using Lemma 5.

Likewise, $C_{JA_2} \leq C^*_{CRF_{m_1,m_2}}$ will also be obtained for the symmetric parameter values of a''_j, b''_j, and c''_j namely: $a''_j = \frac{1}{m_1}a_j$, $b''_j = \frac{1}{2m_2}b_j$, $c''_j = \frac{1}{2m_1}c_j$; $a''_j = \frac{1}{m_1}a_j$, $b''_j = \frac{1}{m_1+m_2}b_j$, $c''_j = \frac{1}{m_1+m_2}c_j$; $a''_j = \frac{1}{m_1}a_j, b''_j = \frac{1}{m_2(m_1+1)}b_j$, $c''_j = \frac{1}{m_1+1}c_j$; and $a''_j = \frac{1}{m_1}a_j, b''_j = \frac{1}{m_2+1}b_j$, $c''_j = \frac{1}{m_1(m_2+1)}c_j$.■

IV. Heuristic Algorithms for $F(m_1, m_2|chain\ reentrant|C_{max})$

Since makespan minimization in CRF_{m_1,m_2} is a general case of makespan minimization in $CRF_{1,1}$, then it is NP-hard as well. This motivates the development of heuristics that will enable in the formulation of a solution to the problem. Towards this end, we also make the following observations. The makespan of CRF_{m_1,m_2} is attained in WC_1, where the first and third operations are processed. The following lemma establishes an optimal property for the scheduling of these operations in WC_1.

Lemma 8. To minimize the makespan of CRF_{m_1,m_2}, it is sufficient to consider a schedule wherein all the first operations of all jobs always precede the third operations of all jobs in any of the m_1 machines in WC_1.

Proof: Consider any of the m_1 machines in WC_1 wherein a third operation of an arbitrary job immediately precedes a task of a first operation of another arbitrary job. Since none of the m_1 machines wait for a first operation, an interchange between the two operations will not violate the flow shop constraint for the third operation. The interchange will also not worsen the makespan.■

In the development of a chain reentrant flow shop heuristic, you first need a sequence to specify the schedule of the jobs. Aside from the sequence you also need to assign each operation on the available machines in the corresponding stage. To do these assignment of operations to machines, we will utilize the first available machine (FAM) and last busy machine (LBM) rules. As the name implies, the FAM rule assigns a job from a sequence based on the first machine that becomes available. The LBM rule on the other hand is a mirror image of the FAM rule. Specifically, the assignment of jobs to machines for example in the second operation using the LBM rule is as follows.

Given a constant $T > 0$ and a sequence S',

Step 1. Set $t_m = T$ for $m = 1, \dots, m_2$.

Step 2. Let J_j be the last unscheduled job in S' and $m = max_{1 \le m \le m_2}\{t_m\}$. Schedule J_j on machine m such that it finishes at time t_m.

Step 3. Set $t_m = t_m - b_j$. $S' = S' - \{j\}$. If $S' = \{\emptyset\}$, stop else go to step 2.

Three heuristics will now be presented for CRF_{m_1,m_2}.

Heuristic H1

Let S_1 be the JA schedule for the AF with processing times $\left(\frac{1}{m_1}a_j, \frac{1}{m_2}b_j\right)$ and S_2 be the JA schedule for the AF with processing times $\left(\frac{1}{m_2}b_j, \frac{1}{m_1}c_j\right)$.

Step 1. Using the sequence S_1,

a. Apply FAM on *A* on the stage 1 machines.

b. Apply LBM on *B* on the stage 2 machines and schedule these tasks as early as possible. Let T' be the largest completion time until the second operation.

c. Apply FAM on *C* on the stage 1 machines from T' and schedule these tasks as early as possible.

d. Calculate the makespan, C_{S_1}.

Step 2. Emulate Step 1 by using the sequence S_2 instead of S_1. Replace *A* with *C* in step 1a, replace *C* with *A* in step 1c and calculate the makespan, C_{S_2}.

Step 3. The makespan of the heuristic is $C_{H1} = min(C_{S_1}, C_{S_2})$.

Theorem 1. Let $C^* = C^*_{CRF_{m_1,m_2}}$, then $\frac{C_{H1}}{C^*} \le \frac{3}{2}\left(2 - \frac{1}{m}\right)$, where $m = max(m_1, m_2)$.

Proof: Let $C_{H1}(S_1)$ be the makespan of S_1 when heuristic H1 is applied. The following is an upper bound of $C_{H1}(S_1)$ because FAM is used for the third operations.

$$C_{H1}(S_1) \le C_{FS1_{m_1,m_2}} + \frac{1}{m_1}\sum_{j=1}^{n} c_j + \frac{m_1 - 1}{m_1} c_{n(S_1)}. \quad (30)$$

$C_{FS1_{m_1,m_2}}$ is the makespan obtained from steps 1a to 1b and $c_{n(S_1)}$ is the processing time of the of the *n*th job at the third operation in the schedule S_1.

Similarly, the following is an upper bound of the makespan, $C_{H1}(S_2)$ because FAM is used for the first operations.

$$C_{H1}(S_2) \le C_{FS2_{m_2,m_1}} + \frac{1}{m_1}\sum_{j=1}^{n} a_j + \frac{m_1 - 1}{m_1} a_{n(S_2)}. \quad (31)$$

$C_{FS2_{m_2,m_1}}$ is the makespan obtained in step 2 and $a_{n(S_2)}$ is the processing time of the *n*th job in the first operation in the schedule S_2.

Adding (30) and (31) yields $2C_{H1} \le C_{FS1_{m_1,m_2}} + C_{FS2_{m_2,m_1}} + \frac{1}{m_1}\sum_{j=1}^{n} a_j + \frac{1}{m_1}\sum_{j=1}^{n} c_j + \frac{m_1-1}{m_1} a_{n(S_2)} + \frac{m_1-1}{m_1} c_{n(S_1)}$. In addition, $\frac{m_1-1}{m_1} a_{n(S_2)} + \frac{m_1-1}{m_1} c_{n(S_1)} \le \frac{m-1}{m} a_{n(S_2)} + \frac{m-1}{m} c_{n(S_1)} \le \frac{m-1}{m} C^*$, where $m = max(m_1, m_2)$. From Lee and Vairaktarakis (1994), $C_{FSk_{m_1,m_2}} \le \left(2 - \frac{1}{m}\right) C^*$, k=1,2. Therefore, $2C_{H1} \le \left(2 - \frac{1}{m}\right) C^* + \left(2 - \frac{1}{m}\right) C^* + C^* + \left(1 - \frac{1}{m}\right) C^*$ or $\frac{C_{H1}}{C^*} \le \frac{3}{2}\left(2 - \frac{1}{m}\right)$.■

Heuristic H2

In Heuristic H2, Step 1c of Heuristic H1 is replaced by an LBM procedure namely:

Step 1c: Apply LBM on *C* on the stage 1 machines from $T' = \sum_{j=1}^{n}(a_j + b_j + c_j)$ and schedule these tasks as early as possible.

Theorem 2. Let $C^* = C^*_{CRF_{m_1,m_2}}$, then $\frac{C_{H2}}{C^*} \le \frac{3}{2}\left(2 - \frac{1}{m}\right)$, where $m = max(m_1, m_2)$.

Proof: The proof is similar to Theorem 1.■

Heuristic H3

Heuristics H1 and H2 use two symmetric JA sequences derived from two of the three processing times of the problem. With the lower bounds that have been derived in Lemma 7 based on the three processing times of the problem, we can modify the heuristic's input to now use two symmetric JA sequences based on all three processing times. In Lemma 7, the set of values (22), (23), (24) and (25) are symmetric to (26), (27), (28) and (29) respectively.

Consider the two AF problems with processing times $\left(a'_j + b'_j, c'_j\right)$ and its symmetric pair $\left(b''_j + c''_j, a''_j\right)$. Apply JA to these AF problems to obtain their corresponding schedule σ_k k=1,2. Replace S_1 and S_2 with σ_1 and σ_2 in steps 1 and 2 respectively in H1.

We can use any of the following four pairs of three processing time JA schedules in H3. We distinguish them by the following:

1. H3.1 when the pair of JA schedules is based on (22) and (26).
2. H3.2 when the pair of JA schedules is based on (23) and (27).
3. H3.3 when the pair of JA schedules is based on (24) and (28).
4. H3.4 when the pair of JA schedules is based on (25) and (29).

V. CONCLUSION

In this paper, we studied the problem of minimizing makespan for a class of two-stage chain reentrant hybrid flow shops. We discuss the CRF_{m_1,m_2} and develop Johnson-based heuristic solutions and lower bounds. One of our heuristic solutions has a worst-case error bound of $\frac{3}{2}\left(2 - \frac{1}{m}\right)$.

The next phase that will be explored will be the evaluation of these heuristics against the best derived lower bounds via computational experiments.

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