Estimation of the Vehicle Sideslip Angle by Means of the State Dependent Riccati Equation Technique

S. Strano, M. Terzo

Abstract— **The knowledge of the vehicle sideslip angle represents a fundamental condition for all the actual vehicle dynamics control systems. Since the measurement of the sideslip angle is expensive and unsuitable for common vehicles, its estimation is nowadays an important task. To this aim, several techniques have been adopted and in many cases their limits are emerged due to the nonlinear nature of the vehicle system.**

In order to overcome these limits, this paper focuses on an alternative nonlinear estimation method based on the State-Dependent-Riccati-Equation (SDRE). Simulations have been conducted and comparisons with the largely used Extended Kalman Filter (EKF) are illustrated.

Index Terms— **Vehicle dynamics, Sideslip angle, Model based estimation, State dependent Kalman filter, Extended Kalman filter.**

I. Introduction

The vehicle sideslip angle is an important physical variable strongly linked to the directional behaviour and stability of the vehicle. As a consequence, the knowledge of the sideslip angle is requested from the vehicle dynamics control systems that establish their intervention on the basis of a difference between a target and a current value [1]. The measurement of the vehicle sideslip angle can be obtained by means of devices that are very expensive and not functional for an easy installation on the car. The well-known industrial solutions typically rely on observers, which are based on heavily simplified dynamic vehicle models in combination with kinematic models. Methodologies for vehicle sideslip angle estimation can be found in literature. Many of them are generally based on the extended Kalman filter (EKF) algorithm [2]. In any case, the nonlinear nature of the vehicle system strongly limits the performance of linear and linearization based approaches [3] that inevitably give estimation errors that affect the performance of the vehicle dynamics controllers employing the estimated variables as feedback. In order to overcome this limit, this paper investigates on a nonlinear technique for the estimation of the vehicle sideslip angle. The approach is based on the State-Dependent-Riccati-Equation (SDRE) nonlinear filtering formulation. The SDRE techniques are recently emerging for optimal nonlinear control and filtering techniques. The SDRE filter (SDREF) originates from a suboptimal nonlinear regulator technique that uses parameterization to bring the nonlinear system into a linear-like structure with state-dependent coefficients (SDC).

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II. Vehicle model

The model adopted in a state observer has to be simple enough in order to limit the computational load but, at the same time, it has to take into account the not negligible dynamics that affect the real system and that are crucial for a good estimation. A single-track model has been assumed for the vehicle. It is characterized by two states referred to the in plane vehicle body motions (lateral and yaw motions) with the steering angle and longitudinal velocity representing the system parameters. Fig. 1 shows the vehicle model in the inertial reference frame Oxy and defines the body fixed reference frame Bxy. With reference to the same figure, v is the centre of mass absolute velocity referred to the earth-fixed axis system, and u (longitudinal velocity) and v (lateral velocity) are its components in the vehicle axis system; r is the yaw rate evaluated in the reference frame Oxy, Fy1 and Fy2 are the lateral interaction forces of the front and rear axle, respectively.

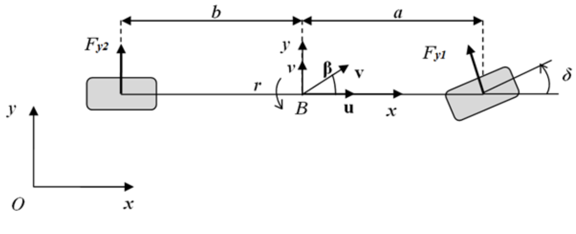

Fig. 1. Reference frames and vehicle model.

The distances from the front and rear axle to the centre of gravity are represented by a and b, respectively. The steering wheel angle of the front tyres is denoted by δ, while the rear tyres are assumed to be no-steering. The in-plane motion equations are the following:

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$$\begin{cases} m(\dot{v}+ur) = F_{y1}\cos\delta + F_{y2} \\ J_z\dot{r} = F_{y1}a\cos\delta - F_{y2}b \end{cases} \quad (1)$$

where m is the vehicle total mass, J_z is its moment of inertia respect to z axis. Taking into account that in the hypothesis of constant u

$$v = u\tan\beta \Rightarrow \dot{v} = u\dot{\beta}\left(1+\tan^2\beta\right)$$

the system (1) becomes:

$$\begin{cases} \dot{\beta} = \left[-r + \frac{1}{um}\left(F_{y1}\cos\delta + F_{y2}\right)\right]\frac{1}{1+\tan^2\beta} \\ \dot{r} = \frac{1}{J_z}\left(F_{y1}a\cos\delta - F_{y2}b\right) \end{cases} \quad (2)$$

The estimation model of the vehicle dynamics includes the equations of motion (2); the tyre forces are considered as variables to be estimated and the system of equations (2) is augmented to include differential equations for each force. So, a random walk model [4] is considered to model each force to be determined:

$$\begin{bmatrix} \dot{f}_0 \\ \dot{f}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} f_0 \\ f_1 \end{bmatrix} + \mathbf{w}_f \quad (3)$$

where f_0 represents the force to be estimated, f_1 its first time derivative, and $\mathbf{w}_f$ is the random white noise.

The model adopted in the estimators is:

$$\begin{cases} \dot{\beta} = \left[-r + \frac{1}{um}\left(F_{y1}\cos\delta + F_{y2}\right)\right]\frac{1}{1+\tan^2\beta} \\ \dot{r} = \frac{1}{J_z}\left(F_{y1}a\cos\delta - F_{y2}b\right) \\ \dot{F}_{y1} = F_{y11} \\ \dot{F}_{y11} = 0 \\ \dot{F}_{y2} = F_{y21} \\ \dot{F}_{y21} = 0 \end{cases} \quad (4)$$

where F_{y11} and F_{y12} are the first time derivative of the lateral interaction forces of the front and rear axle, respectively. The state and the input vectors of equations (4) are defined respectively by $\mathbf{x} = \left[\beta, r, F_{y1}, F_{y11}, F_{y2}, F_{y21}\right]^T$ and $\mathbf{u} = [u,\delta]^T$.

The nonlinear equations of the system (4) have been adopted in order to derive the EKF and the SDREF starting from the measurements of signals like the lateral acceleration a_y and the yaw rate r (common on current commercial vehicles) with the objective of estimating the sideslip angle of the vehicle.

III. EXTENDED KALMAN FILTER

The linearization procedure is at the basis of the EKF approach, which is briefly recalled in the following, since it has been adopted for a comparative analysis. The system and the measurement equations can be generically represented by:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) + \boldsymbol{\psi}_k \quad (5)$$

$$\mathbf{z}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)) + \mathbf{g}_k \quad (6)$$

being $\mathbf{x}$ the state vector, $\mathbf{f}$ and $\mathbf{h}$ non-linear functions, $\mathbf{u}$ the input vector, $\boldsymbol{\psi}_k$ the process noise with covariance $\mathbf{Q}_k$, $\mathbf{z}$ the measurement vector, and $\mathbf{g}_k$ the Gaussian white measurement noise with covariance $\mathbf{R}_k$.

The estimator can be implemented in a discrete time form, integrating the system equation from time t_{k-1} to time t_k. The EKF methodology is conceptually based on two fundamental steps, namely estimates and updates steps.

Denoting the estimates as $(\hat{\bullet})$, the following initializing conditions are applied to the state estimates (7) and to the error covariance (8):

$$\hat{\mathbf{x}}_0^+ = E(\mathbf{x}_0) \quad (7)$$

$$P_0^+ = E\left[\left(\mathbf{x}_0 - \hat{\mathbf{x}}_0^+\right)\left(\mathbf{x}_0 - \hat{\mathbf{x}}_0^+\right)^T\right] \quad (8)$$

being E the expected value.
The state estimates and the estimation of the error covariance are given by (9) and (10) respectively:

$$\hat{\mathbf{x}}_k^- = \mathbf{f}_{k-1}(\hat{\mathbf{x}}^+{}_{k-1}, \mathbf{u}_{k-1}) \quad (9)$$

$$\mathbf{P}_k^- = \mathbf{A}_{k-1}\mathbf{P}_{k-1}^+\mathbf{A}_{k-1}^T + \mathbf{L}_{k-1}\mathbf{Q}_{k-1}\mathbf{L}_{k-1}^T \quad (10)$$

with

$$\mathbf{A}_{k-1} = \left.\frac{\partial \mathbf{f}_{k-1}}{\partial \hat{\mathbf{x}}}\right|_{\hat{\mathbf{x}}^+{}_{k-1}} \quad (11)$$

$$\mathbf{L}_{k-1} = \left.\frac{\partial \mathbf{f}_{k-1}}{\partial \boldsymbol{\psi}_k}\right|_{\hat{\mathbf{x}}^+{}_{k-1}} \quad (12)$$

With the computation of the filter gain (13) and evaluating the measurement residual, the updates of the state estimates (14) and of the estimation of the error covariance (15) can be determined:

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$$\mathbf{K}_k = \mathbf{P}_k^- \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T + \mathbf{M}_k \mathbf{R}_k \mathbf{M}_k^T)^{-1} \quad (13)$$

$$\hat{\mathbf{x}}^+{}_k = \hat{\mathbf{x}}^-{}_k + \mathbf{K}_k \left[\mathbf{z}_k - \mathbf{h}_k(\hat{\mathbf{x}}^-{}_k, \mathbf{u}_k)\right] \quad (14)$$

$$\mathbf{P}_k^+ = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)\mathbf{P}_k^- \quad (15)$$

where:

$$\mathbf{H}_k = \left.\frac{\partial \mathbf{h}_k}{\partial \hat{\mathbf{x}}}\right|_{\hat{\mathbf{x}}^-{}_k} \quad (16)$$

$$\mathbf{M}_k = \left.\frac{\partial \mathbf{h}_k}{\partial \mathbf{g}_k}\right|_{\hat{\mathbf{x}}^-{}_k} \quad (17)$$

IV. STATE DEPENDENT RICCATI EQUATION FILTER

The SDRE techniques are used as control and filtering design methods and are based on state dependent coefficient (SDC) factorization [5]. Infinite-horizon nonlinear regulator problem is a generalization of time invariant infinite horizon linear quadratic regulator problem where all system coefficient matrices are state-dependent. When the coefficient matrices are constant, the SDRE control method changes into the steady-state linear regulator. Filtering counterpart of the SDRE control algorithm is obtained by taking the dual system of the steady-state linear regulator and then allowing coefficient matrices of the dual system to be state-dependent. Starting from the nonlinear system (4), there are infinite solutions to transform this nonlinear system into an SDC form as:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \mathbf{u}(t))\mathbf{x} + \boldsymbol{\psi} \quad (18)$$

$$\mathbf{z}(t) = \mathbf{H}(\mathbf{x}(t), \mathbf{u}(t))\mathbf{x} + \mathbf{g} \quad (19)$$

where

$\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) = \mathbf{F}(\mathbf{x}(t), \mathbf{u}(t))\mathbf{x}$ and

$$\mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)) = \mathbf{H}(\mathbf{x}(t), \mathbf{u}(t))\mathbf{x} \quad (20)$$

$\boldsymbol{\psi}$ is the process noise with covariance $\mathbf{Q}$, and $\mathbf{g}$ is the Gaussian white measurement noise with covariance $\mathbf{R}$.

Starting from the SDC form, the derivative of the state estimate is given by:

$$\dot{\hat{\mathbf{x}}} = \mathbf{F}(\hat{\mathbf{x}}, \mathbf{u})\hat{\mathbf{x}} + \mathbf{K_f}(\hat{\mathbf{x}})[\mathbf{z}(\mathbf{x}) - \mathbf{H}(\hat{\mathbf{x}}, \mathbf{u})\hat{\mathbf{x}}] \quad (21)$$

where

$$\mathbf{K_f}(\hat{\mathbf{x}}) = \mathbf{P}(\mathbf{x})\mathbf{H^T}(\hat{\mathbf{x}}, \mathbf{u})\mathbf{R^{-1}} \quad (22)$$

and $\mathbf{P}$ is the positive definite solution of the algebraic Riccati equation (23).

$$\begin{aligned}&\mathbf{F}(\hat{\mathbf{x}}, \mathbf{u})\mathbf{P}(\hat{\mathbf{x}}) + \mathbf{P}(\hat{\mathbf{x}})\mathbf{F^T}(\hat{\mathbf{x}}, \mathbf{u}) + \\ &- \mathbf{P}(\hat{\mathbf{x}})\mathbf{H^T}(\hat{\mathbf{x}}, \mathbf{u})\mathbf{R^{-1}}\mathbf{H}(\hat{\mathbf{x}}, \mathbf{u})\mathbf{P}(\hat{\mathbf{x}}) + \mathbf{Q} = \mathbf{0}\end{aligned} \quad (23)$$

The nonlinear equations (10) have been parameterized in SDC form with the following choice [6, 7]:

$$\mathbf{F(x,u)} = \begin{bmatrix} 0 & -\frac{1}{1+\tan^2\beta} & \frac{\cos\delta}{um(1+\tan^2\beta)} & 0 & \frac{1}{um(1+\tan^2\beta)} & 0 \\ 0 & 0 & \frac{a\cos\delta}{J_z} & 0 & -\frac{b}{J_z} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (24)$$

Moreover, with reference to the measurement equation, taking into account that

$$a_y = \dot{v} + ur = u\dot{\beta}(1 + \tan^2\beta) + ur = \frac{F_{y1}\cos\delta + F_{y2}}{m} \quad (25)$$

it results:

$$\mathbf{H(x,u)} = \mathbf{H(u)} = \begin{bmatrix} 0 & 0 & \frac{\cos\delta}{m} & 0 & \frac{1}{m} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (26)$$

V. SIMULATION TEST

Simulation tests have been carried out in CarSim®, that constitutes a reference in vehicle simulation environment. The first simulated manoeuvre consists of an open loop steering pad, functional to evaluate the steady state circular driving behaviour. It is based on a constant longitudinal speed of 120 km/h, a steering angle smoothly and linearly increasing with the time and high adhesion conditions. The characteristics of the simulated vehicle are listed in Table 1. The performance of the SDREF has been evaluated by means of comparison with the EKF estimator and the simulated state given by CarSim. The observers have been designed taking into account as input the longitudinal speed (u), the steering angle (δ), the measurements given by the lateral acceleration a_y and the yaw rate r. Moreover, the two estimators have been identically parameterized and a sampling time of 10 ms has been selected for both.

Table 1. Vehicle parameters for simulation test.

Distance of the centre of gravity from the front axle (m)	1.15
Distance of the centre of gravity from the rear axle (m)	1.15
Height of the centre of gravity (m)	0.35
Front and rear track width (m)	1.6
Vehicle mass (kg)	1600
Yaw moment of inertia (kg m^2)	1800
Wheel radius (m)	0.3
Wheel moment of inertia (kg m^2)	1

Numerical simulations have performed in Matlab/Simulink environment adopting an integration algorithm with a fixed step size. The first important result is showed in Fig. 2, where the capability of the SDREF is easily visible: indeed, the comparison with the EKF technique highlights substantial differences due to the linearization process. The nonlinearities of the system are fully considered in the SDREF and, consequently, the estimation gives a value that is superimposed to the simulated one. This result finds an important application in all the vehicle dynamics control systems based on the sideslip angle adopted as feedback. It has to be highlighted the better performance of the SDREF respect to the EKF, as it can be observed in Fig. 3 that shows an estimation error for the EKF greater of one order of magnitude. By comparing the estimation error with the simulated value of β (Figs. 2, 3) it results that the EKF estimation causes an error corresponding to about 12% of the simulated value while the SDREF gives an error lower than 1%. This important result demonstrates the effectiveness of the proposed technique applied to the estimation of the vehicle sideslip angle.

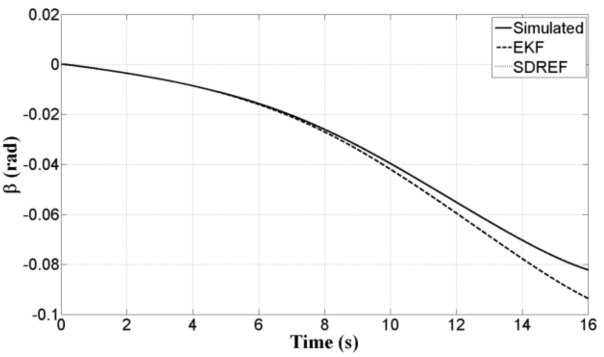

Fig. 2. Sideslip angle – steering pad manoeuvre.

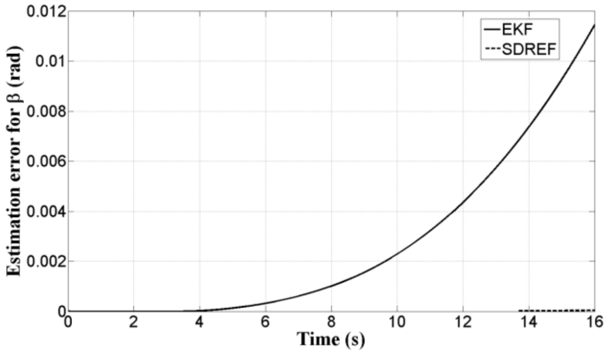

Fig. 3. Estimation error for sideslip angle – steering pad manoeuvre.

With the particular reference to the vehicle dynamics, the nonlinearities involve the sideslip angle equation of (4). As a consequence, the differences between the EKF and the SDREF are evident with the particular reference to the estimate of β. In any case, with the aim of completeness of the discussion, the comparisons with the estimated measurements are given because they are fundamental to evaluate the coherence of the estimator with the system involved in the observation procedure. The results show the better performance of the SDREF in terms of estimation of the lateral acceleration measurement (Figs. 4 and 5) and substantially comparable results in terms of estimation of the yaw rate measurement (Figs 6, 7). This is due to the presence of nonlinearities in the first measurement equation (25).

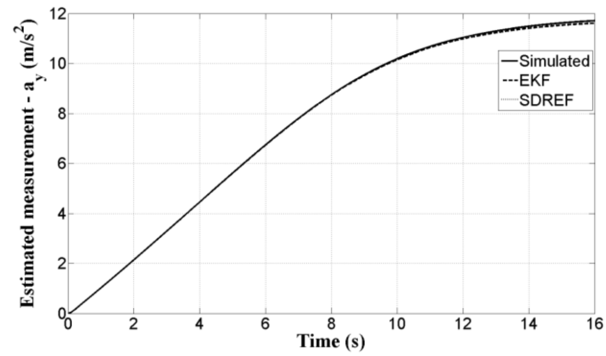

Fig. 4. Lateral acceleration – steering pad manoeuvre.

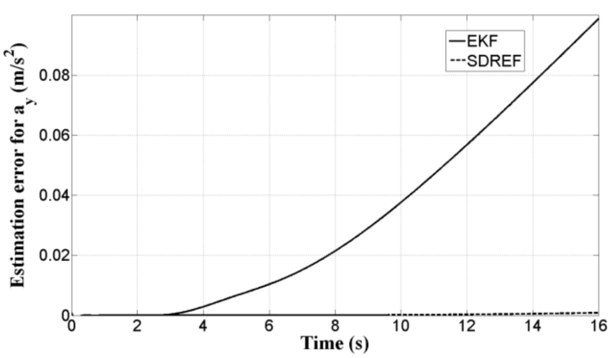

Fig. 5. Estimation error for lateral acceleration – steering pad manoeuvre.

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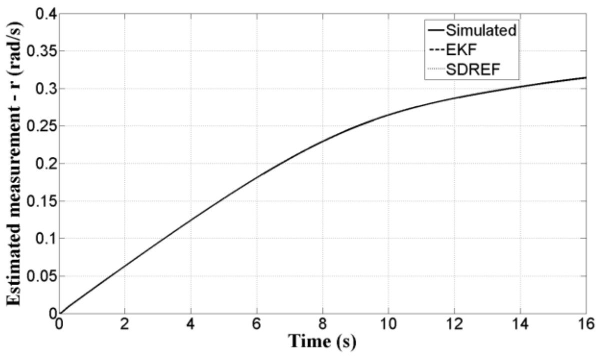

Fig. 6. Yaw rate – steering pad manoeuvre.

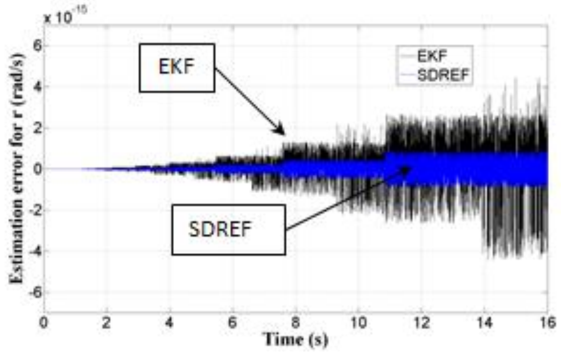

Fig. 7. Estimation error for yaw rate – steering pad manoeuvre.

A second simulated result concerns a transient manoeuvre given by a double lane change at a constant longitudinal speed of 120 km/h, steering amplitude of 50° in high adhesion conditions. As in the previous test, the comparisons concerning the estimates of the vehicle sideslip angle are plotted. Also in this case, the goodness of the SDREF can be easily highlighted (Fig. 8) since the estimate is practically superimposed to the simulated value.

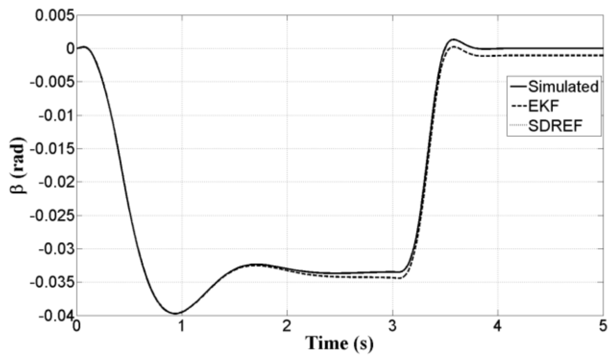

Fig. 8. Sideslip angle – double lane change manoeuvre.

The SDREF gives an estimation error (Fig. 9) reduced of an order of magnitude if compared to the EKF. This result strongly validates the nonlinear estimation technique also for this transient manoeuvre.

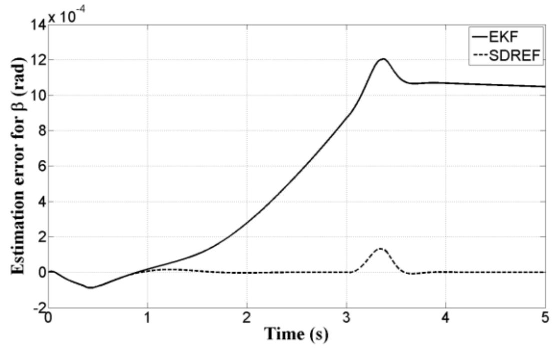

Fig. 9. Estimation error for sideslip angle – double lane change manoeuvre.

Fig. 10 represents the measurement a_y with its estimates, while Fig. 11 shows the estimation error.

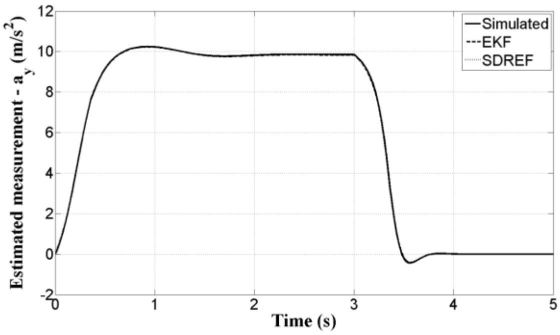

Fig. 10. Lateral acceleration – double lane change manoeuvre.

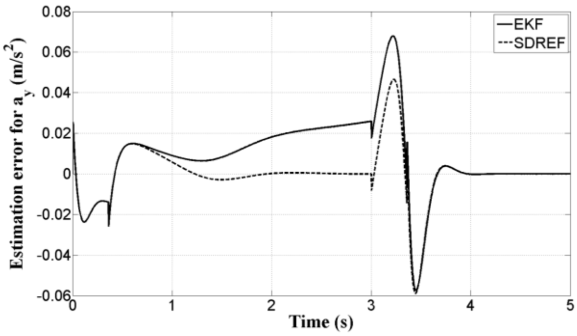

Fig. 11. Estimation error for lateral acceleration – double lane change manoeuvre.

The SDREF shows a better performance, while comparable results can be seen for the yaw rate measurement, highlighting a functional parameterisation of the two filters (Figs. 12, 13).

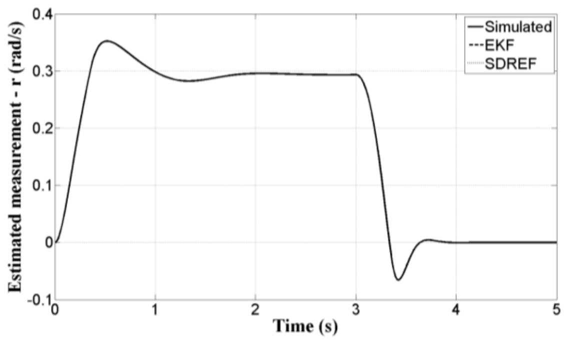

Fig. 12. Yaw rate – double lane change manoeuvre.

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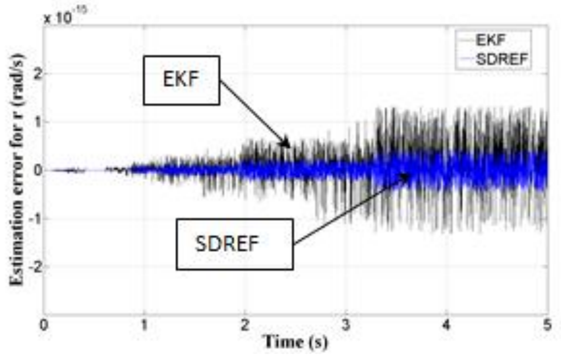

Fig. 13. Estimation error for yaw rate – double lane change manoeuvre.

The illustrated results allow to appreciate the goodness of the SDREF for the estimation of the vehicle sideslip angle.

VI. CONCLUSIONS

A state-dependent-Riccati-equation based Kalman filter has been proposed for the estimate of the vehicle sideslip angle. The estimator has been designed taking into account a single track vehicle model together with a random walk model for the lateral interaction forces. Consequently, the proposed approach is not based on specific interaction models that require the knowledge of parameters. The filter can be easily adapted to different vehicles, tyres and boundary conditions without a detailed knowledge of their characteristics and, differently from common approaches, no tuning procedure is required. The results show the advantages of the SDREF, able to fully capture all nonlinearities.

Nomenclature

Vehicle model

u Vehicle longitudinal velocity
v Vehicle lateral velocity
r Vehicle yaw rate
β Vehicle sideslip angle
a_y Vehicle lateral acceleration
Fy1 Lateral interaction force of the front axle
Fy2 Lateral interaction force of the rear axle
F_{y11} First time derivative of the lateral interaction force of the front axle
F_{y12} First time derivative of the lateral interaction force of the rear axle
a Distance from the front axle to the centre of gravity
b Distance from the rear axle to the centre of gravity
δ Steering wheel angle of the front tyres
m Vehicle total mass
J_z Vehicle moment of inertia respect to z axis
f_0 Force to be estimated
f_1 First time derivative of the force to be estimated
$\mathbf{w}_f$ Random white noise

EKF and SDREF

$\mathbf{u}$ Input vector
$\mathbf{f}$, $\mathbf{h}$ Nonlinear functions
$\mathbf{Q}_k$ Covariance of the process noise for the EKF
$\mathbf{z}$ Measurement vector
$\mathbf{g}_k$ Gaussian white measurement noise for the EKF
$\mathbf{R}_k$ Covariance of the measurement noise for the EKF
$\mathbf{K}_k$ Filter gain of the EKF
t Time
$(\hat{\bullet})$ Estimate
E Expected value
$\mathbf{x}_0$ Initial condition on the state vector
P_0 Initial condition on the error covariance
P_k Error covariance for the EKF
A_{k-1}, L_{k-1}, H_k, M_k Partial derivative matrices for the EKF
$\mathbf{F}(\mathbf{x}(t),\mathbf{u}(t))$, $\mathbf{H}(\mathbf{x}(t),\mathbf{u}(t))$ Input and state dependent matrices of the SDC form
$\mathbf{Q}$ Covariance of the process noise for the SDREF
$\mathbf{g}$ Gaussian white measurement noise for the SDREF
$\mathbf{R}$ Covariance of the measurement noise for the SDREF
$\mathbf{K_f}$ Filter gain of the SDREF
$\mathbf{P}$ Solution of the algebraic Riccati equation
$\boldsymbol{\psi}_k$ Process noise for the EKF
$\boldsymbol{\psi}$ Process noise for the SDREF

Superscripts

- *a priori* estimate
+ *a posteriori* estimate

Subscripts

k-1 related to time t_{k-1}
k related to time t_k

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