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# Mechanical Analysis of an AM Fabricated Viscoelastic Shaft under Torsion by Rigid Disks

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***Abstract*—Mechanical analysis of an additive manufacturing (AM) fabricated viscoelastic shaft under torsion by rigid disks is under consideration. An approach of growing solids mechanics for the determination of the stress-strain state of a shaft is utilized. The shaft has the form of circular cylinder with two rigid disks attached to its end faces. The process of continuous surface growth of such a cylinder under the influence of twisting torques applied to the disks is studied. Dual series equations reflecting the mathematical content of the problem of different stages of the growing process are derived and investigated. The results of a numerical analysis and the singularities of the qualitative mechanical behaviour of the fundamental characteristics are discussed.**

***Index Terms*—additive manufacturing, shaft, shape, strength, torsion**

## I. Formulation the Torsion Problem

We will assume that a fairly long shaft (circular cylinder) of length $2l$ and radius $b_0$ (the ratio of $l$ to $b_0$ is fairly large) is fabricated from an ageing viscoelastic material at zero time. Both of the shaft endfaces are in perfect contact with circular disks with a flat bottom of radius $a < b_0$ . At a time $\tau_0$ a torques $M(t)$ starts to act upon the disks, rotating them through an angle $\gamma = 2\alpha(t)$ with respect to each other. The shaft side surface is stress-free.

At a time $\tau_1$ substance influx to the shaft side surface starts. The new incremental elements are not stressed and the time of their fabrication coincides with the time of initial body fabrication.

The law of shaft growth is given completely by the function $b(t)$ that characterizes the change in its radius with time. Naturally $b(\tau_1) = b_0$.

The growing ceases at a time $\tau_2$. At that time the shaft radius takes the value $b_1$ ($b(\tau_2) = b_1$), and its side surface is free of any action even at $t \geq \tau_2$. The contact growing problem is studied within the framework of a quasistatic approximation in the absence of volumetric forces (Fig. 1).

The shaft is considered to be relatively long during the growth process and after its cessation (the ratios $l/b(t)$ and $l/b_1$ are fairly large). Taking into account the symmetric property of the problem we consider only a half of the shaft with one end face clamped at a rigid base and the other end face coupled with the disk.

Let us use the general approach for the solution of growth problems developed and used in [1–8].

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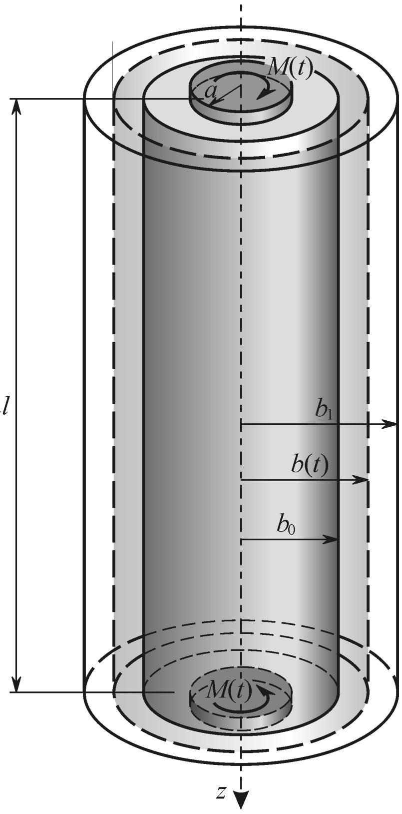


Fig. 1. AM fabricated shaft under torsion

Consider the fundamental relations of the problem in the time interval $t \in [\tau_0, \tau_1]$. We have for the initial viscoelastic ageing shaft

$$\frac{\partial \sigma_{r\varphi}}{\partial r} + \frac{\partial \sigma_{\varphi z}}{\partial z} + \frac{2\sigma_{r\varphi}}{r} = 0 \quad (\nabla \cdot \mathbf{T} = \mathbf{0}), \tag{1}$$
$$z = 0,\ 0 \leq r \leq a\colon u_\varphi = \alpha(t) r;$$
$$z = 0,\ a \leq r \leq b_0\colon \sigma_{\varphi z} = 0;$$
$$r = b_0,\ 0 \leq z \leq l\colon \sigma_{r\varphi} = 0;$$
$$z = l,\ 0 \leq r \leq b_0\colon u_\varphi = 0,$$
$$\varepsilon_{r\varphi} = \frac{1}{2}\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right),$$
$$\varepsilon_{\varphi z} = \frac{1}{2}\frac{\partial u_\varphi}{\partial z} \quad \left(\boldsymbol{\varepsilon} = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]\right),$$
$$\mathbf{T} = 2G(t)(\mathbf{I} + \mathbf{L}(\tau_0, t))\mathbf{E},$$
$$(\mathbf{I} - \mathbf{L}(\tau_0, t)) = (\mathbf{I} + \mathbf{N}(\tau_0, t))^{-1},$$
$$\mathbf{L}(\tau_0, t) f(t) = \int_{\tau_0}^{t} f(\tau) K_1(t, \tau)\, d\tau,$$
$$K_1(t, \tau) = G(\tau)\frac{\partial}{\partial \tau}\left[\frac{1}{G(\tau)} + \omega(t, \tau)\right],$$

where $\mathbf{T}$ and $\mathbf{E}$ are the stress and strain tensors with non-zero components $\sigma_{r\varphi}$, $\sigma_{\varphi z}$ and $\varepsilon_{r\varphi}$, $\varepsilon_{\varphi z}$ respectively, $\mathbf{u}$ is the displacement vector with a single non-zero component $u_\varphi$, $K_1(t,\tau)$, $\omega(t,\tau)$, and $G(t)$ is the creep kernel, the measure of creep, and the modulus of elastic deformation under pure shear.

We set

$$\mathbf{T}^\circ = (\mathbf{I} - \mathbf{L}(\tau_0, t))\mathbf{T}G^{-1} \tag{2}$$

and we act on the expression from (1) containing $\mathbf{T}$ and its components with the operator $(\mathbf{I} - \mathbf{L}(\tau_0, t))$. Then taking (2) into account, we obtain the following boundary-value problem

$$\begin{gathered}\frac{\partial \sigma^\circ_{r\varphi}}{\partial r} + \frac{\partial \sigma^\circ_{\varphi z}}{\partial z} + \frac{2\sigma^\circ_{r\varphi}}{r} = 0 \quad (\nabla \cdot \mathbf{T}^\circ = \mathbf{0}), \qquad (3)\\ z = 0,\ 0 \le r \le a\colon u_\varphi = \alpha(t)r;\\ z = 0,\ a \le r \le b_0\colon \sigma^\circ_{\varphi z} = 0;\\ r = b_0,\ 0 \le z \le l\colon \sigma^\circ_{r\varphi} = 0;\\ z = l,\ 0 \le r \le b_0\colon u_\varphi = 0,\\ \mathbf{E} = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T], \quad \mathbf{T}^\circ = 2\mathbf{E}.\end{gathered}$$

On the basis of (3) we determine that the displacement $u_\varphi$ satisfies the equation

$$\boldsymbol{\mathcal{V}} u_\varphi = \frac{\partial^2 u_\varphi}{\partial r^2} + \frac{\partial^2 u_\varphi}{\partial z^2} + \frac{1}{r}\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r^2} = 0. \tag{4}$$

Following [10] we take the solution of (4) in the form (e.g., see [4])

$$u_\varphi(r,z,t) = \frac{l d_0(t) r}{b_0}\left(1 - \frac{z}{l}\right) + \sum_{n=1}^{\infty} \frac{d_n(t)}{\delta_n} J_1(r\delta_n) \frac{\sinh[\delta_n(l-z)]}{\sinh(\delta_n l)}, \tag{5}$$

where $d_k(t)$ $(k = 0, \ldots, \infty)$ are unknown functions of time, $\delta_n$ $(n = 1, \ldots, \infty)$ are undetermined constants, and $J_\nu(x)$ is the Bessel function of order $\nu$.

We note that expression (5) for the displacement $u_\varphi$ satisfies the boundary condition from (3) on the clamped endface of the shaft for $z = l$ and enables us to write the tensor components of the operator stresses $\mathbf{T}^\circ$ in the form (see (1) and (3))

$$\begin{aligned}\sigma^\circ_{\varphi z}(r,z,t) &= -\frac{d_0(t) r}{b_0} - \sum_{n=1}^{\infty} d_n(t) J_1(r\delta_n) \frac{\cosh[\delta_n(l-z)]}{\sinh(\delta_n l)},\\ \sigma^\circ_{r\varphi}(r,z,t) &= -\sum_{n=1}^{\infty} d_n(t) J_2(r\delta_n) \frac{\sinh[\delta_n(l-z)]}{\sinh(\delta_n l)}.\end{aligned} \tag{6}$$

Utilizing the boundary condition from (3) on the shaft side surface ($r = b_0$) and (6) , we find a set of constants $\delta_n$. Indeed, by equating the expression for $\sigma^\circ_{r\varphi}$ to zero for $r = b_0$ we obtain that $\delta_n = \lambda_n b_0^{-1}$, where $\lambda_n$ are roots of the equation $J_2(\lambda_n) = 0$.

Finally, satisfying the boundary conditions for $z = 0$, we will have the following dual series equations to seek the sequence of functions $d_k(t)$

$$\begin{gathered}\frac{l d_0(t) r}{b_0} + \sum_{n=1}^{\infty} \frac{b_0 d_n(t)}{\lambda_n} J_1\left(\frac{\lambda_n r}{b_0}\right) = \alpha(t) r\\ (0 \le r \le a),\\ \frac{d_0(t) r}{b_0} + \sum_{n=1}^{\infty} d_n(t) J_1\left(\frac{\lambda_n r}{b_0}\right) \coth\frac{\lambda_n l}{b_0} = 0\\ (a \le r \le b),\end{gathered} \tag{7}$$

Since $\lambda_n \ge \lambda_1 = 3.8317$ and $l b_0^{-1} = \varkappa_0 \gg 1$, then $\coth(b_0^{-1}\lambda_n l)$ can be set equal to one with a high degree of accuracy and (7) can be investigated in the form (see [10])

$$\begin{gathered}u_\varphi(r,0,t) = \varkappa_0 d_0(t) r + \sum_{n=1}^{\infty} \frac{b_0 d_n(t)}{\lambda_n} J_1\left(\frac{\lambda_n r}{b_0}\right) = \alpha(t) r\\ (0 \le r \le a),\\ \sigma^\circ_{\varphi z}(r,0,t) = \frac{d_0(t) r}{b_0} + \sum_{n=1}^{\infty} d_n(t) J_1\left(\frac{\lambda_n r}{b_0}\right) = 0\\ (a \le r \le b_0).\end{gathered} \tag{8}$$

The dual series Eqs. (8) describe the formulated contact problem in the interval $t \in [\tau_0, \tau_1]$, where the time itself occurs in it parametrically. We will now construct the solution of (8) below by first obtaining the resolving equations of the problem during continuous growth and after cessation of growth. We merely note that the true stresses can be restored from the formula

$$\mathbf{T}(r,z,t) = G(t)\left[\mathbf{T}^\circ(r,z,t) + \int_{\tau_0}^{t} \mathbf{T}^\circ(r,z,\tau) R_1(t,\tau)\, d\tau\right], \tag{9}$$

where $R_1(t,\tau)$ is the resolvent of the kernel $K_1(t,\tau)$.

Let $t \in [\tau_1, \tau_2]$. Then the boundary-value problem for a growing shaft being twisted by a disk has the form (see [5-8])

$$\begin{gathered}\frac{\partial s_{r\varphi}}{\partial r} + \frac{\partial s_{\varphi z}}{\partial z} + \frac{2 s_{r\varphi}}{r} = 0 \quad (\nabla \cdot \mathbf{S} = \mathbf{0}), \qquad (10)\\ z = 0,\ 0 \le r \le a\colon v_\varphi = \dot{\alpha}(t) r;\\ z = 0,\ a \le r \le b(t)\colon s_{\varphi z} = 0;\\ r = b(t),\ 0 \le z \le l\colon s_{r\varphi} = 0,\ t = \tau^*(r),\\ z = l,\ 0 \le r \le b(t)\colon v_\varphi = 0,\\ \mathbf{D} = \frac{1}{2}[\nabla \mathbf{v} + (\nabla \mathbf{v})^T],\\ \mathbf{S} = 2\mathbf{D}, \quad \frac{\partial \mathbf{T}^\circ}{\partial t} = \mathbf{S}.\end{gathered}$$

It is seen that the rate of displacement $v_\varphi$ satisfies the equation $\boldsymbol{\mathcal{V}} v_\varphi = 0$ (see (4)) while the expression for $v_\varphi$ and the rates of the operator stresses $s_{r\varphi}$ and $s_{\varphi z}$ can be written in the form

$$\begin{aligned}v_\varphi(r,z,t) &= \frac{l d_0^\circ(t) r}{b(t)}\left(1 - \frac{z}{l}\right) + \sum_{n=1}^{\infty} \frac{d_n^\circ(t)}{\eta_n(t)} J_1[r\eta_n(t)] \frac{\sinh[\eta_n(t)(l-z)]}{\sinh[\eta_n(t) l]},\\ s_{\varphi z}(r,z,t) &= -\frac{d_0^\circ(t) r}{b(t)} - \sum_{n=1}^{\infty} d_n^\circ J_1[r\eta_n(t)] \frac{\cosh[\eta_n(t)(l-z)]}{\sinh[\eta_n(t) l]}\\ s_{r\varphi}(r,z,t) &= -\sum_{n=1}^{\infty} d_n^\circ J_2[r\eta_n(t)] \frac{\sinh[\eta_n(t)(l-z)]}{\sinh[\eta_n(t) l]}.\end{aligned} \tag{11}$$


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Here $d_k^\circ(t)$ ($k=0,\ldots,\infty$) and $\eta_n(t)$ ($n=1,\ldots,\infty$) are sequences of functions to be determined.

By satisfying the boundary conditions from (13), taking into account that $lb^{-1}(t) \gg 1$ we arrive at dual series equations for finding $d_k^\circ(t)$

$$\begin{aligned} &v_\varphi(r,0,t) = \varkappa(t)d_0^\circ(t)r + \sum_{n=1}^\infty \frac{b(t)d_n^\circ(t)}{\lambda_n} J_1\left[\frac{\lambda_n r}{b(t)}\right] = \dot\alpha(t) r \\ &\quad (0 \le r \le a), \\ &s_{\varphi z}(r,0,t) = \frac{d_0^\circ(t) r}{b(t)} + \sum_{n=1}^\infty d_n^\circ J_1\left[\frac{r\lambda_n}{b(t)}\right] = 0 \\ &\quad (a \le r \le b(t)), \\ &\eta_n(t) = \frac{\lambda_n}{b(t)}, \quad \varkappa(t) = \frac{l}{b(t)}, \quad \tau_1 \le t \le \tau_2. \end{aligned} \tag{12}$$

If the $d_k^\circ(t)$ are found, meaning $\mathbf{S}$ and $\mathbf{v}$ also, the stress tensor $\mathbf{T}$ and the displacement vector $\mathbf{u}$ are established according to the formulas

$$\begin{aligned} \mathbf{T}(r,z,t) &= G(t)\left\{\frac{\mathbf{T}(r,z,\tau_0(r))}{G(\tau_0(r))}\left[1 + \int_{\tau_0(r)}^t R_1(t,\tau)\,d\tau\right]\right. \\ &\left. + \int_{\tau_0(r)}^t \left[\mathbf{S}(r,z,\tau) + \int_{\tau_0(r)}^\tau \mathbf{S}(r,z,\zeta)\,d\zeta\, R_1(t,\tau)\right] d\tau\right\}, \\ \mathbf{u}(r,z,t) &= \mathbf{u}(r,z,\tau_0(r)) + \int_{\tau_0(r)}^t \mathbf{v}(r,z,\tau)\,d\tau. \end{aligned} \tag{13}$$

The boundary-value problem for a growing shaft has the form (10) after the cessation of growth $t \ge \tau_2 = \tau^*(b_1)$, where only $b(t) = b_1$ and the usual boundary conditions $\tau_{r\varphi} = 0$ is specified on the shaft surface. Just as before, it can be reduced to a boundary-value problem in the rates of displacement and operator stresses with a solution in the form (14) under the condition $b(t) = b_1$. The resolving dual series equations retain the form (15), where $b(t) = b_1$, $\varkappa(t) = \varkappa_1 = lb_1^{-1}$, $\eta_n(t) = \eta_n = \lambda_n b_1^{-1}$, $t \ge \tau_2$. After their solution, the stresses $\sigma_{r\varphi}$, $\sigma_{\varphi z}$, and the displacement $u_\varphi$ are determined by using (16). It should be noted that the dependence of $\mathbf{S}$ and $\mathbf{v}$ on the time $t$ is parametric.

The condition of disk equilibrium that holds in the whole time interval must be added to the dual series equations obtained

$$M(t) = -2\pi\int_0^a \sigma(\rho,t)\rho^2\,d\rho, \quad \sigma(\rho,t) = \sigma_{\varphi z}(\rho,0,t). \tag{14}$$

On the basis of (17) the following conditions can also be obtained

$$\begin{aligned} &M^\circ(t) = (\mathbf{I} - \mathbf{L}(\tau_0,t))\frac{M(t)}{G(t)} = -2\pi\int_0^a \sigma^\circ(\rho,t)\rho^2\,d\rho \\ &\quad (\tau_0 \le t \le \tau_1), \end{aligned} \tag{15}$$

$$\begin{aligned} &\frac{\partial M^\circ(t)}{\partial t} = \frac{1}{G(t)}\frac{\partial M(t)}{\partial t} + \int_{\tau_0}^t \frac{\partial M(\tau)}{\partial \tau}\frac{\partial \omega(t,\tau)}{\partial t}d\tau \\ &\quad + M(\tau_0)\frac{\partial\omega(t,\tau_0)}{\partial t} = -2\pi\int_0^a s(\rho,t)\rho^2\,d\rho \\ &\quad (t \ge \tau_1), \end{aligned} \tag{16}$$

which are more convenient for constructing the solution of the contact problem in a number of cases.

## II. Solution of the Torsion Problem

The resolving dual equations of the problem can be represented in three fundamental time intervals by the single relationships

$$\begin{aligned} &\zeta\varphi_0 x + \sum_{n=1}^\infty \frac{\varphi_n}{\lambda_n} J_1(\lambda_n x) = \psi x \quad (0 \le x \le c), \\ &-p(x) = \varphi_0 x + \sum_{n=1}^\infty \varphi_n J_1(\lambda_n x) = 0 \quad (c \le x \le 1), \end{aligned} \tag{17}$$

where we set $\zeta = \varkappa_0$, $\varphi_k = d_k(t)$ ($k = 0,\ldots,\infty$), $p(x) = \sigma^\circ(xb_0,t)$, $\psi = \alpha(t)$, $c = ab_0^{-1}$, $x = rb_0^{-1}$, for $t \in [\tau_0,\tau_1]$, we have $\zeta = \varkappa(t)$, $\varphi_k = d_k^\circ(t)$, $\psi = \dot\alpha(t)$, $p(x) = s(xb(t),t)$, $c = ab^{-1}(t)$, $x = rb^{-1}(t)$, for $t \in [\tau_1,\tau_2]$ and unlike the preceding $\zeta = \varkappa_1$, $b(t) = b_1$ for $t \ge \tau_2$.

Let us construct the solution of (20) by following [11]. Let

$$p(x) = \left[\frac{\partial}{\partial x}\int_x^c \frac{g(\xi)}{\sqrt{\xi^2 - x^2}}\,d\xi\right] h(c-x). \tag{18}$$

The series in the second equation of (17) is a Dini expansion [12] of the function $-p(x)$, whose coefficients $\varphi_k$ ($k = 0,\ldots,\infty$) are given by the formulas

$$\begin{aligned} \varphi_0 &= -4\int_0^1 x^2 p(x)\,dx = 8\int_0^c \xi g(\xi)\,d\xi, \\ \varphi_n &= -\frac{2}{J_1^2(\lambda_n)}\int_0^1 xp(x)J_1(\lambda_n x)\,dx \\ &= \frac{2}{J_1^2(\lambda_n)}\int_0^c g(\xi)\sin(\lambda_n\xi)\,d\xi \quad (n = 1,\ldots,\infty), \end{aligned} \tag{19}$$

when (18) is taken into account.

Substituting (19) into the first equation of (17) and using the technique from [11, 13-15], we obtain a Fredholm integral equation of the second kind to determine the function $g(x)$

$$\begin{aligned} &g(x) + \int_0^c g(\xi)k(x,\xi)\,d\xi = \frac{3\psi x}{\pi} \quad (1 \le x \le c), \\ &k(x,\xi) = \frac{16}{\pi}(1-2\zeta)x\xi \\ &\quad + \frac{4}{\pi^2}\int_0^\infty \frac{K_2(y)}{I_2(y)}[8x\xi I_2(y) - \sinh(xy)\sinh(\xi y)]\,dy, \end{aligned} \tag{20}$$

where $K_\nu(y)$, $I_\nu(y)$ are Bessel functions of imaginary argument of order $\nu$.

The solution of (20) obviously yields the complete solution of the contact problem in question also. It can be found by using methods of [16, 17]. We consider here one method, proposed in [10], for constructing the approximate solution of (23). We note that for $\zeta \ge 10$ the deviation of the approximate from the numerical solution does not exceed 8.5% for $c = 0.7, 7\%$ for $c = 0.6$, and 1% for $c \le 0.5$.

We will use the fact that the quantity $\zeta$ is fairly large and we will limit ourselves to the first term in the expression for the kernel $k(x,\xi)$ (see (23))

$$g(x) + \frac{16}{\pi}(1-2\zeta)x\int_0^c g(\xi)\xi\,d\xi = \frac{4\psi x}{\pi} \quad (1 \le x \le c). \tag{21}$$

Then substituting $g(x) = Ax$ into (24) and determining $A$, we will have by virtue of (21)

$$p(x) = -\frac{4\psi}{\pi + 16(2\zeta-1)c^3/3}\frac{x}{\sqrt{c^2 - x^2}} \quad (1 \le x \le c). \tag{22}$$

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The dependences of the operator contact stresses and their rates on the angle of disk

$$\sigma^\circ(r,t) = \alpha(t)W(r,b_0) \quad (\tau_0 \le t \le \tau_1), \tag{23}$$
$$s^\circ(r,t) = \dot\alpha(t)W(r,b(t)) \quad (\tau - 1 \le t \le \tau_2), \tag{24}$$
$$s^\circ(r,t) = \dot\alpha(t)W(r,b_1) \quad (t \ge \tau_2), \tag{25}$$
$$W(r,\xi) = -\frac{4}{\pi + 16(2l/\xi - 1)a^3/(3\xi^3)}, \quad \frac{r}{\sqrt{d^2 - r^2}}.$$

For a given angle of disk rotation $\sigma^\circ(r,t)$, $s^\circ(r,t)$ are found at once from (23)–(25) and by using the relationships described earlier the contact stresses $\sigma(r,t)$ are restored. The moment acting on the disk is calculated from (14). We note that for $\alpha(t) = \mathrm{const}$ the mutual influence of the initial shaft and its newly forming unstressed part does not appear. On the basis of (9), (15) and (23) we will have for a given torque $M(t)$

$$\sigma(r,t) = \frac{3M(t)}{4\pi a^3}\frac{r}{\sqrt{a^2 - r^2}},$$
$$\alpha(t) = B(b_0)(\mathbf{I} - \mathbf{L}(\tau_0,t))\frac{M(t)}{G(t)} \quad (\tau_0 \le t \le \tau_1), \tag{26}$$
$$B(\xi) = \frac{3}{16a^3} + \frac{2l - \xi}{\pi\xi^4}.$$

Using (16), (19), (24), and (25), we finally obtain the relationship (26) for the contact stresses for $t \ge \tau_1$, and the following expressions for the angle of rotation

$$\dot\alpha(t) = \frac{\partial M^\circ(t)}{\partial t}B(b(t)), \quad \alpha(t) = \alpha(\tau_1) + \int_{\tau_1}^{t}\dot\alpha(\tau)\,d\tau$$
$$(\tau_1 \le t \le \tau_2),$$
$$\dot\alpha(t) = \frac{\partial M^\circ(t)}{\partial t}B(b_1), \quad \alpha(t) = \alpha(\tau_2) + \int_{\tau_1}^{t}\dot\alpha(\tau)\,d\tau$$
$$(t \ge \tau_2).$$

It turns out that the growth of a shaft during torsion of a disk by a moment of forces has a slight influence on the contact stress distribution if the disk and shaft radii are not very close (specific ratios are given above). However, a substantial dependence of the angle of disk rotation on the time from when the shaft starts to grow and on the growth rate appears in this same case.

To obtain the stress-strain state of a shaft at some distance from its edge faces (Saint-Venant's principle) one can use the results of [18].

## III. Numerical Example

We examine the contact problem in question by considering the shaft to be fabricated from concrete with a modulus of elastically instantaneous shear strain $G(t) = G = \mathrm{const}$ and a measure of creep under shear in the form [19]

$$\omega(t,\tau) = (D_0 + Fe^{-\beta\tau})(1 - e^{-\gamma(t-\tau)}).$$

We will make a change of variables according to the formulas

$$r^* = \frac{r}{a}, \quad \rho^* = \frac{\rho}{a}, \quad t^* = \frac{t}{\tau_0}, \quad \sigma^*(r^*,t^*) = \frac{\sigma(r,t)}{G},$$
$$\tau_1^* = \frac{\tau_1}{\tau_0}, \quad \tau_2^* = \frac{\tau_2}{\tau_0}, \quad M^*(t^*) = \frac{M(t)}{Ga^3}, \quad \alpha^*(t^*) = \alpha(t),$$
$$\beta^* = \beta\tau_0, \quad \gamma^* = \gamma\tau_0, \quad b_0^* = \frac{b_0}{a}, \quad b_1^* = \frac{b_1}{a},$$
$$b^*(t^*) = \frac{b(t)}{a}, \quad l^* = \frac{l}{a}, \quad D_0^* = D_0G, \quad F^* = FG,$$

and omitting the asterisk in the notation, we give the following values of the functions and parameters:

$$b_0 = \frac{1}{0.7}, \quad l = \frac{290}{0.7}, \quad b(t) = \frac{b_0(t + \tau_2 - 2\tau_1)}{\tau_2 - \tau_1},$$
$$b_1 = 2b_0, \quad M(t) = 1, \quad D_0 = 0.251, \quad F = 1.818,$$
$$\beta = 0.31, \quad \gamma = 0.6, \quad \tau_0 = 10\,\text{days}.$$

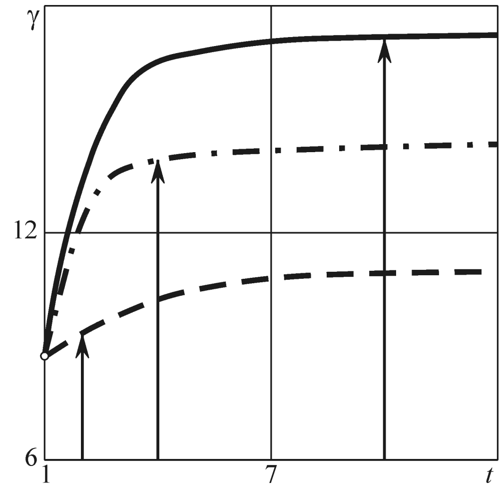

Fig. 2. Angle of the disk rotation with respect to various growth rates in the case of the first AM process

It is seen that during the time of growth the shaft radius doubles. The growth rate is constant and is determined only by the times of the beginning and cessation of growth. The torque acting on the disk does not change with time. Moreover, the ratio of the shaft length to its radius $\ge 20$ during the extent of the whole process, while the ratio of the disk and shaft radii $\le 0.7$, i.e., formulas of the approximate solution can be utilized.

As regards the contact stress distribution, it is sufficient to refer to (26) to note that it (the distribution) is practically independent of the properties of the material and growth process in the case under consideration.

The behaviour of the angle of disks rotation with respect to each other as a function of the fundamental characteristics of the process of piecewise-continuous shaft growth requires a more detailed analysis.

The curves in Fig. 2 show the change in the angle of rotation $\gamma = 2\alpha$ in a time $t$ for the first case of AM process. In this case a shaft growth starts simultaneously with the application of the torque ($\tau_1 = 1$) for different growth rates $\dot b(t)$: $\dot b(t) = b_0/9$ ($\tau_2 = 10$) is the solid line $\dot b(t) = b_0/3$ ($\tau_2 = 4$) is the dash-dot line, and $\dot b(t) = b_0$ ($\tau_2 = 2$) is the dashed line. The times of the growth stops are marked by the vertical solid lines.

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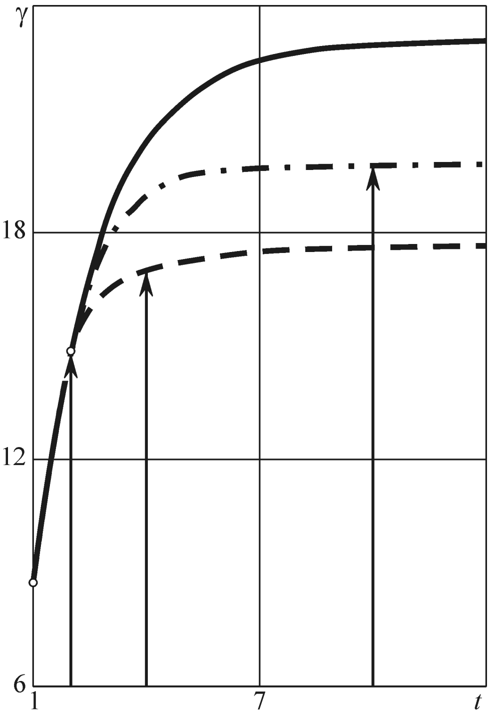


Fig. 3. Angle of the disk rotation with respect to various growth rates in the case of the second AM process

The curves in Fig. 3 correspond to dependencies of the angle of disk rotation on the time for the second case of AM process. In this case for a shaft loaded at the time 1 and starting to grow at the time $\tau_1 = 2$ we choose different growth rates $\dot{b}(t)$: $\dot{b}(t) = b_0/8$ ($\tau_2 = 10$) the dash-dot line, and $\dot{b}(t) = b_0/2$ ($\tau_2 = 4$) the dashed line. For comparison, the change in the angle of disk rotation that twists a shaft of fixed radius $b_0$ is shown by the solid line. The sections of the curves located between the vertical solid lines characterize the behavior of the angle of disk rotation in intervals of continuous shaft growth.

## IV. Conclusions

- The contact stresses which act on the shaft due to the interaction with rigid disks is practically independent of the material properties and growth process if the the shaft is comparatively long (the ratio of the shaft length to its radius is greater than or equal to 20), while the disks is not too large (the ratio of the disk and shaft radii does not exceed $0.7$) during the whole AM process.
- The essential dependence of the angle of rotation $\gamma(t)$ on the growth rate. Thus the limit value of the increment in the angle of disk rotation $\Delta(\infty)$ ($\Delta(t) = \gamma(t) - \gamma(\tau_0)$) during slow shaft growth can exceed the same value for rapid growth by a factor of $5$ and more.
- The limit value of the angle of disk rotation is significantly depends on the time interval between times of the beginning of loading and the beginning of growth.
- For constant torque the characteristic time exists, starting with which the influence of the AM process on the stress-strain state of a viscoelastic shaft can be neglected.

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