Trigonometrically Fitted Semi-Implicit Fourth Order Hybrid Method for Solving Oscillatory Delay Differential Equations

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***Abstract*—A semi-implicit hybrid method of three-stage and fourth order which is suitable for solving special second order ordinary differential equations is constructed. The method is then trigonometrically fitted so that it is suitable for solving problems which are oscillatory in nature. The methods are then used for solving oscillatory delay differential equations. Numerical results clearly show the efficiency of the new method when compared to the existing explicit and implicit methods in the scientific literature.**

***Index Terms*— Delay differential equations, Oscillatory problems, Semi-Implicit Hybrid Method, Trigonometrically-fitted.**

I. Introduction

THERE has been a great interest in the research of new methods which can efficiently solve special second order ordinary differential equation (ODE) which has oscillatory solution. The special second order ODE in which the first derivative does not appear explicit can be written in the following form

$$y" = f(t,y),\ y(t_0) = t_0,\ y'(t_0) = y_0{}' \qquad (1)$$

This type of problems often arise in many fields of applied sciences such as mechanics, astrophysics, satellite tracking, quantum chemistry, molecular dynamic and electronics. Since we are also going to solve oscillatory delay differential equations (DDEs) using the method which will be derived, so here, we give a brief introduction to the special second order DDE. It can be written in the form of

$$y"(t) = f\big(t, y(t), y(t-\tau)\big),\ a \le t \le b,\ y(t_0) = y_0,$$
$$y'(t_0) = y_0',\ t \in [-\tau, a] \qquad (2)$$

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where τ is the delay term. There are many applications related to DDEs such as in population dynamics, epidemiology and reforestation. This kind of equation depends on the solution at prior times and best known as model that incorporating past history. It is a more realistic model which includes some of the past history of the system to determine the future behavior.

The most common methods that used to solve both (1) and (2) are usually Runge-Kutta (RK) method, Runge-Kutta Nyström (RKN) method, multistep method and hybrid method. Researchers have developed and modified the previously mention methods by focusing their research on developing methods with reduced dispersion (phase-lag) and dissipation (amplification) errors to improve the efficiency of the methods. In their work based on one-step method, Bursa and Nigro[1] introduced the analysis of dispersion error. D'Ambrosioet al.[2] used the exponentially fitting technique to construct Runge-Kutta (RK) methods which are suitable for oscillatory ODEs.While Senu et al.[3] derived an explicit RK method with phase-lag of order infinity based on the method by Dormand [4].

Solving (1) using RK methods means the equation need to be converted first into a system of first order ODEs, while Runge-Kutta Nystrom (RKN) method can directly solve the equation. Van de Vyver [5] in his paper proposed a symplectic RKN method with minimal phase-lag. Many authors incorporate the phase-lag of higher order into the construction of diagonally implicit RKN and diagonally implicit RKN methods, see: [6]-[9]. By modifying some of the coefficients of the existing RKN methods; authors such as Papadopoulos et al.[10] introduced a phase-fitted method, Kosti et al.[11] developed optimized method and Moo et al. [12] also developed phase-fitted and amplification-fitted methods. These authors show that, methods with higher order of dispersion and dissipation give a more accurate numerical results when used to solve oscillatory problems.

Franco [13] has proposed that (1) can be solved using a particular explicit hybrid algorithms or special multistep methods for solving second-order ODEs. He then continued this work (see: [14]) by developing explicit two-step hybrid methods of order four up to six for solving second-order IVPs based on the order condition developed by Coleman [15]. Work on developing and improving hybrid method using dispersion and dissipation properties for solving second order ODEs can also be seen in [16-21]. All the work mentioned above are focused on solving oscillatory ordinary differential equations.

In this paper, we are going to develop a new three-stage fourth-order SIHM then trigonometrically fitting the method so that it is suitable for solving oscillatory problems and applied it for solving oscillatory delay differential equations.

II. Derivation of the New Semi-Implicit Hybrid Methods

Semi-implicit hybrid method for the numerical integration of the IVPs in (1) is given as

$$Y_i = (1+c_i)y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{s} a_{ij} f\left(x_n + c_j h, Y_j\right), \quad (3)$$
$$y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i). \quad (4)$$

where $i = 1, \dots, s,$ and $i \geq j$. The equations of the form (3) and (4) are defined as

$$Y_1 = y_{n-1}, Y_2 = y_n, \quad (5)$$
$$Y_i = (1+c_i)y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{i} a_{ij} f\left(x_n + c_j h, Y_j\right), \quad (6)$$
$$y_{n+1} = 2y_n - y_{n-1} + h^2 (b_1 f_{n-1} + b_2 f_n + \sum_{i=3}^{s} b_i f(x_n + c_i h, Y_i)). \quad (7)$$

where the first two nodes are $c_1 = -1$, $c_2 = 0$ and $i = 3, \dots, s$, while functions $f_{n-1} = f(x_{n-1}, y_{n-1})$ and $f_n = f(x_n, y_n)$. The coefficients b_i, c_i , and a_{ij} can be represented in Butcher tableau as below:

Table 1: The s-stage semi-implicit hybrid methods

-1	0				
0	0	0			
c_3	$a_{3,1}$	$a_{3,2}$	γ		
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	
c_s	$a_{s,1}$	$a_{s,2}$	$\dots$	$a_{s,s-1}$	γ
	b_1	b_2	$\dots$	b_{s-1}	b_s

Note that the coefficients of the diagonal element (γ) are always equal for this method. In this section, we derive the three-stage fourth-order SIHMs using order conditions, simplifying conditions and minimization of the error constant C_{p+1} of the method. The error constant is defined

by $$C_{p+1} = \left\| \left(e_{p+1}(t_1)\right), \dots, e_{p+1}(t_k) \right\|_2 \quad (8)$$

Where k is the number of order $p+2(p(t_i) = p+2)$, for the pth – order method and $e_{p+1}(t_i)$ is the local truncation error defined in Coleman [15]. According to Coleman [15], the conditions up to order five are listed as follows:

(i) Order 2 : $\sum_{i=1}^{s} b_i = 1$. (9)

(ii) Order 3 : $\sum_{i=1}^{s} b_i c_i = 0$. (10)

(iii) Order 4 : $\sum_{i=1}^{s} b_i c_i^2 = \frac{1}{6}$, $\sum_{i=1}^{s} b_i a_{ij} = \frac{1}{12}$ (11)

(iv) Order 5: $\sum_{i=1}^{s} b_i c_i^3 = 0$, $\sum_{i=1}^{s} b_i c_i a_{ij} = \frac{1}{12}$,
$\sum_{i=1}^{s} b_i\, a_{ij}\, c_j = 0.$ (12)

where value of $i \geq j \geq k$. The method also needs to satisfy the simplifying condition for hybrid method which is:

$$\sum_{i}^{s} a_{ij} = \frac{(c_i^2 + c_i)}{2}, \text{ for } i = 3, \dots, s. \quad (13)$$

In order to construct the method, we use the algebraic order conditions up to order four which involved equations (9) – (11), and simplifying condition (13), the equations are solved simultaneously using Maple package. We obtained the solution in terms of free parameters, a_{32}, a_{33}, and c_3 as follows:

$$a_{31} = -a_{32} - a_{33} + \frac{c_3}{2} + \frac{c_3^2}{2},$$
$$b_1 = \frac{1}{6(1+c_3)},$$
$$b_2 = \frac{6c_3 - 1}{6c_3}, \text{and} \, b_3 = \frac{1}{6c_3(1+c_3)},$$

Using minimization of the error constant in (8), we obtained the coefficients of $a_{32} = \frac{19}{24}$, $a_{33} = \frac{11}{600}$, and $c_3 = \frac{9}{10}$. The norm of the principal local truncation error coefficient for y_n is given by

$$\left\|\tau^{(5)}\right\|_2 = 1.88398 \times 10^{-2} ,$$

where $\left\|\tau^{(5)}\right\|_2$ is obtained from the error equations for the fifth order method. Hence, we have the three-stage fourth-order semi-implicit hybrid method denoted as NSIHM3(4) which is given below:

Table 2 : A new three-stage fourth-order semi-implicit hybrid method (NSIHM3(4))

-1	0		
0	0	0	
$\frac{9}{10}$	$\frac{9}{200}$	$\frac{19}{24}$	$\frac{11}{600}$
	$\frac{5}{57}$	$\frac{22}{27}$	$\frac{50}{513}$

III. Trigonometrically Fitted NSIHM3(4),

To apply the trigonometrically fitted properties to NSIHM3(4), we require equations (3) and (4) to integrate the linear combination of the function $\{\sin(vt), \cos(vt)\}$ for $v \in \mathcal{R}$. Which gives the following equations,

$$\cos(c_3 H) = 1 + c_3 - c_3 \cos(H) - H^2\{a_{31}\cos(H) + a_{32} + a_{33}\cos(c_3 H)\}, \quad (20)$$

$$\sin(c_3 H) = c_3 \sin(H) + H^2\{a_{31}\sin(H) - a_{33}\sin(c_3 H)\} \quad (21)$$

$$2\cos(H) = 2 - H^2\{b_1 \cos(H) + b_2 + b_3 \cos(c_3 H)\} \quad (22)$$

$$b_1 \sin(H) = b_3 \sin(c_3 H). \quad (23)$$

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where $H = vh$ as v and h are fitted frequency and step size respectively. By solving equations (20) and (21) simultaneously with the choice of coefficients $c_3 = \frac{9}{10}$ and $a_{32} = \frac{19}{24}$, we obtained and a_{33} in terms of H as below:

$$a_{31} = -\frac{1}{120}\Big\{28311552\cos\left(\frac{H}{10}\right)^{18} - 120324096\cos\left(\frac{H}{10}\right)^{16} + 212336640\cos\left(\frac{H}{10}\right)^{14} - 201277440\cos\left(\frac{H}{10}\right)^{12} + 110702592\cos\left(\frac{H}{10}\right)^{10} - 35641344\cos\left(\frac{H}{10}\right)^{8} + 24320\cos\left(\frac{H}{10}\right)^{8}H^2 + 6488832\cos\left(\frac{H}{10}\right)^{6} - 42560\cos\left(\frac{H}{10}\right)^{6}H^2 - 624960\cos\left(\frac{H}{10}\right)^{4} + 22800\cos\left(\frac{H}{10}\right)^{4}H^2 - 3800\cos\left(\frac{H}{10}\right)^{2}H^2 + 28560\cos\left(\frac{H}{10}\right)^{2} + 95H^2 - 336\Big\}/K, \text{ and}$$

$$a_{33} = -\frac{1}{60}\Big\{-60 + 15728640\cos\left(\frac{H}{10}\right)^{18} - 66846720\cos\left(\frac{H}{10}\right)^{16} + 117964800\cos\left(\frac{H}{10}\right)^{14} - 111820800\cos\left(\frac{H}{10}\right)^{12} + 61501440\cos\left(\frac{H}{10}\right)^{10} - 58368\cos\left(\frac{H}{10}\right)^{9} + 24320\cos\left(\frac{H}{10}\right)^{9} - 19768320\cos\left(\frac{H}{10}\right)^{8} + 116736\cos\left(\frac{H}{10}\right)^{7} - 48640\cos\left(\frac{H}{10}\right)^{7}H^2 + 3548160\cos\left(\frac{H}{10}\right)^{6} + 31920\cos\left(\frac{H}{10}\right)^{5}H^2 - 76608\cos\left(\frac{H}{10}\right)^{5} - 316800\cos\left(\frac{H}{10}\right)^{4} + 18240\cos\left(\frac{H}{10}\right)^{3} - 7600\cos\left(\frac{H}{10}\right)^{3}H^2 + 10800\cos\left(\frac{H}{10}\right)^{2} - 1140\cos\left(\frac{H}{10}\right) + 475\left(\frac{H}{10}\right)H^2\Big\}/K,$$

Where

$$K = H^2\Big\{-1 + 262144\cos\left(\frac{H}{10}\right)^{18} - 1114112\cos\left(\frac{H}{10}\right)^{16} + 1966080\cos\left(\frac{H}{10}\right)^{14} - 1863680\cos\left(\frac{H}{10}\right)^{12} + 1025024\cos\left(\frac{H}{10}\right)^{10} - 329472\cos\left(\frac{H}{10}\right)^{8} + 59136\cos\left(\frac{H}{10}\right)^{6} - 5280\cos\left(\frac{H}{10}\right)^{4} + 180\cos\left(\frac{H}{10}\right)^{2}\Big\}$$

Then, to get b-values, we solve linear system (22)-(23) with an additional order condition (9) for three-stage method which is

$$b_1 + b_2 + b_3 = 1.$$

The choice of coefficient $c_3 = \frac{9}{10}$, which gives the values for b as follows:

$$b_1 = -\frac{1}{2}\Big\{16384\cos\left(\frac{H}{10}\right)^{14} - 8192\cos\left(\frac{H}{10}\right)^{13} - 53248\cos\left(\frac{H}{10}\right)^{12} + 24576\cos\left(\frac{H}{10}\right)^{11} + 67584\cos\left(\frac{H}{10}\right)^{10} - 28160\cos\left(\frac{H}{10}\right)^{9} - 42240\cos\left(\frac{H}{10}\right)^{8} + 15360\cos\left(\frac{H}{10}\right)^{7} + 13440\cos\left(\frac{H}{10}\right)^{6} - 4000\cos\left(\frac{H}{10}\right)^{5} + 16\cos\left(\frac{H}{10}\right)^{4}H^2 - 2064\cos\left(\frac{H}{10}\right)^{4} + 432\cos\left(\frac{H}{10}\right)^{3} - 8\cos\left(\frac{H}{10}\right)^{3}H^2 + 148\cos\left(\frac{H}{10}\right)^{2} - 12\cos\left(\frac{H}{10}\right)^{2}H^2 - 16\cos\left(\frac{H}{10}\right) + 4\cos\left(\frac{H}{10}\right)H^2 - 4 + H^2\Big\}/M,$$

$$b_2 = \frac{1}{2}\Big\{1024\cos\left(\frac{H}{10}\right)^{10} + 512\cos\left(\frac{H}{10}\right)^{9} - 2560\cos\left(\frac{H}{10}\right)^{8} - 256\cos\left(\frac{H}{10}\right)^{8}H^2 - 1024\cos\left(\frac{H}{10}\right)^{7}H^2 + 2240\cos\left(\frac{H}{10}\right)^{6} + 448\cos\left(\frac{H}{10}\right)^{6}H^2 + 672\cos\left(\frac{H}{10}\right)^{5} - 800\cos\left(\frac{H}{10}\right)^{4} - 240\cos\left(\frac{H}{10}\right)^{4}H^2 - 160\cos\left(\frac{H}{10}\right)^{3}H^2 + 100\cos\left(\frac{H}{10}\right)^{2} + 40\cos\left(\frac{H}{10}\right)^{2}H^2 + 10\cos\left(\frac{H}{10}\right)H^2 - 4 - H^2\Big\}/N,$$

$$b_3 = -\Big\{16384\cos\left(\frac{H}{10}\right)^{14} - 61440\cos\left(\frac{H}{10}\right)^{12} + 92160\cos\left(\frac{H}{10}\right)^{10} - 70400\cos\left(\frac{H}{10}\right)^{8} + 28800\cos\left(\frac{H}{10}\right)^{6} + 16\cos\left(\frac{H}{10}\right)^{4}H^2 - 6064\cos\left(\frac{H}{10}\right)^{4} + 580\cos\left(\frac{H}{10}\right)^{2} - 20\cos\left(\frac{H}{10}\right)^{2}H^2 - 20 + 5H^2\Big\}/P,$$

where

$$M = H^2\Big\{8192\cos\left(\frac{H}{10}\right)^{14} - 4096\cos\left(\frac{H}{10}\right)^{13} - 26624\cos\left(\frac{H}{10}\right)^{12} + 12288\cos\left(\frac{H}{10}\right)^{11} + 33792\cos\left(\frac{H}{10}\right)^{10} - 14080\cos\left(\frac{H}{10}\right)^{9} - 21120\cos\left(\frac{H}{10}\right)^{8} + 7680\cos\left(\frac{H}{10}\right)^{7} + 6720\cos\left(\frac{H}{10}\right)^{6} - 2016\cos\left(\frac{H}{10}\right)^{5} - 1016\cos\left(\frac{H}{10}\right)^{4} + 228\cos\left(\frac{H}{10}\right)^{3} + 62\cos\left(\frac{H}{10}\right)^{2} - 9\cos\left(\frac{H}{10}\right) - 1\Big\},$$

$$N = H^2\Big\{\Big(16\cos\left(\frac{H}{10}\right)^{5} - 8\cos\left(\frac{H}{10}\right)^{4} - 20\cos\left(\frac{H}{10}\right)^{3} + 8\cos\left(\frac{H}{10}\right)^{2} + 5\cos\left(\frac{H}{10}\right) - 1\Big)\Big(1 - 12\cos\left(\frac{H}{10}\right)^{2} + 16\cos\left(\frac{H}{10}\right)^{4}\Big)\Big\}, \text{and}$$

$$P = H^2\Big\{8192\cos\left(\frac{H}{10}\right)^{14} - 28672\cos\left(\frac{H}{10}\right)^{12} + 39424\cos\left(\frac{H}{10}\right)^{10} - 26880\cos\left(\frac{H}{10}\right)^{8} + 9408\cos\left(\frac{H}{10}\right)^{6} - 16\cos\left(\frac{H}{10}\right)^{5} - 1568\cos\left(\frac{H}{10}\right)^{4} +$$

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$$20\ \cos\left(\frac{H}{10}\right)^3 + 98\ \cos\left(\frac{H}{10}\right)^2 - 5\ \cos\left(\frac{H}{10}\right) - 1\Big\}.$$

We transform the above formulae into Taylor series expansions as

$$a_{31} = \frac{9}{200} - \frac{189}{12500}H^2 - \frac{19321}{2500000}H^4 - \frac{33394877}{11250000000}H^6 - \frac{739479023}{675000000000}H^8 + O(H^{10})$$

$$a_{32} = \frac{11}{600} - \frac{7917}{400000}H^2 - \frac{658851}{80000000}H^4 - \frac{60824490931}{20160000000000}H^6 - \frac{2661195574241}{2419200000000000}H^8 + O(H^{10}),$$

$$b_1 = \frac{5}{57} + \frac{23}{4560}H^2 + \frac{158653}{5774560000}H^4 + \frac{2003803}{114912000000}H^6 + O(H^8),$$

$$b_2 = \frac{22}{27} - \frac{49}{6480}H^2 - \frac{72973}{272160000}H^4 - \frac{1383449}{163296000000}H^6 + O(H^8),\text{ and}$$

$$b_3 = \frac{50}{513} + \frac{31}{12312}H^2 - \frac{4139}{517104000}H^4 - \frac{556343}{62052480000}H^6 + O(H^8).$$

The other coefficients of the method remain the same. This method is denoted as trigonometrically-fitted three-stage fourth-order semi-implicit hybrid method (TF-NSIHM3(4)).

IV. Problems Tested and Numerical Results

In this section, The new semi-implicit hybrid method NSIHM3(4) and the trigonometrically fitted version of the method TF-NSIHM3(4) are used to solve a set of DDE test problems. These methods are tested over a large interval to indicate that the new TF-NSIHM3(4) is suitable for integrating oscillatory problems. In the implementation, divided difference interpolation of the same order as the methods is used to evaluate the delay term that is $y(t-\tau)$. A measure of accuracy is examine using the absolute error which is defined by

$$Absoluteerror = max\{|y(t_n) - y_n|\}$$

The efficiency curves of the log of max error versus the execution time in second for interval
$0 \le x \le 10{,}000$ are given in Figures 1-4.
The following notations are used to indicate the respective methods:

- **TF-NSIHM3(4)**: Trigonometrically-fitted Semi-implicit three-stage fourth-order hybrid method developed in this paper.
- **NSIHM3(4)**: Semi-implicit three-stage fourth-order hybrid method developed in this paper.
- **SIHM3(5)** : Semi-implicit three-stage fifth-order hybrid method developed in Ahmad et al.[20].
- **RKN3(4)**: Explicit three-stage fourth-order RKN method by Hairer et al. [22].
- **PFRKN4(4)**: Explicit four-stage fourth-order Phase-fitted RKN method by Papadopoulos et al.[10]
- **DIRKN3(4)**: Diagonally Implicit three-stage fourth-order Runge-KuttaNyström method derived in Senu et al.[7].

Below are the set DDEs used as test problems.

Problem 1 *(problem in Schmidt [23])*

$$y''(t) = -\frac{1}{2}\,y(t) + \frac{1}{2}y(t-\pi)\,, 0 \le t \le 8\pi\,, y_o = 0.$$

The fitted frequency is $v = 1$.
Exact solution is $y(t) = \sin(t)$.

Problem 2 *(problem in Schmidt [23])*

$$y''(t) - y(t) + \eta(t)y\left(\frac{t}{2}\right) = 0\,, 0 \le t \le 2\pi,\ \text{where}$$

$$\eta(t) = \frac{4\sin(t)}{\sqrt{(2-2\cos(t))}}\,, \eta(0) = 4.$$

Fitted frequency is $v = 2$. Exact solution is $y(t) = \sin(t)$.

Problem 3 *(problem in Ladas and Stavroulakis [24])*

$$y''(t) = y(t-\pi)\,, 0 \le t \le 8\pi\,, y_o = 0.$$

Fitted frequency is $v = 1$. Exact solution is $y(t) = \sin(t)$.

Problem 4 *(problem in Bhagat Singh [25])*

$$y''(t) = -\frac{\sin(t)}{2-\sin(t)}y(t-\pi)\,, 0 \le t \le 8\pi\,,\ y_o = 2.$$

Fitted frequency is $v = 1$. Exact solution is $y(t) = 2 + \sin(t)$.

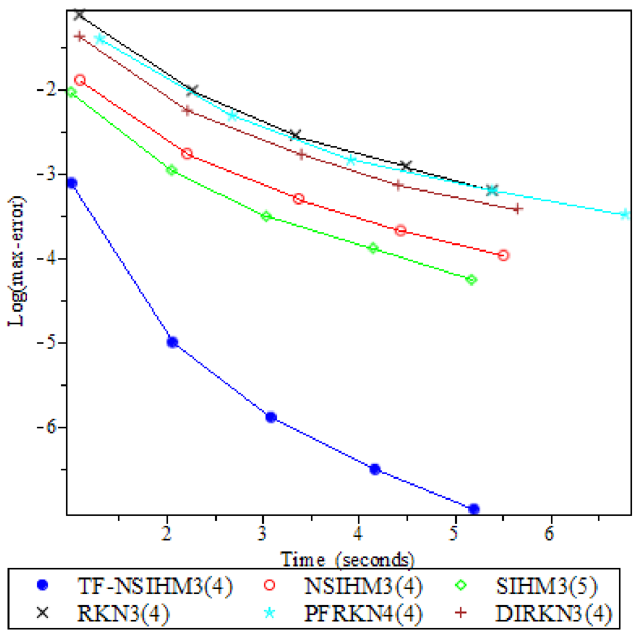

Fig. 1: The efficiency curves for TF-SIHM3(4) for problem 1 with $h = \frac{\pi}{4i}$, for $i = 1, \ldots, 5$.

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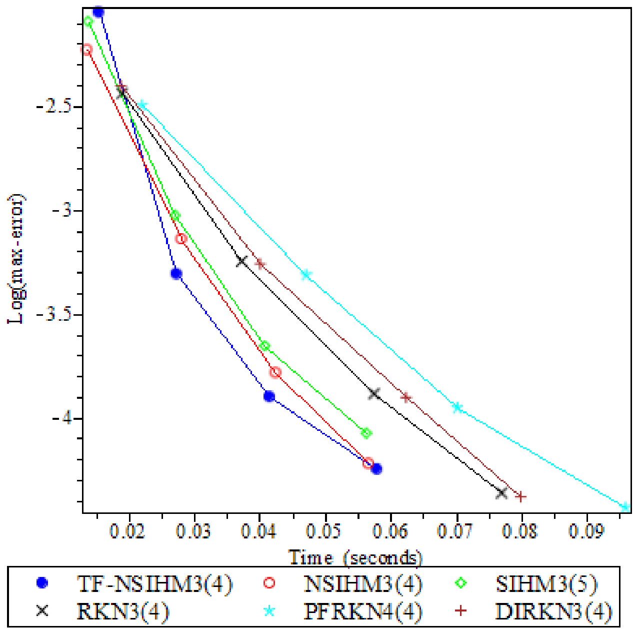

Fig. 2: The efficiency curves for TF-SIHM3(4) for problem 2 with $h = \frac{\pi}{16i}$, for $i = 1, \dots, 4$.

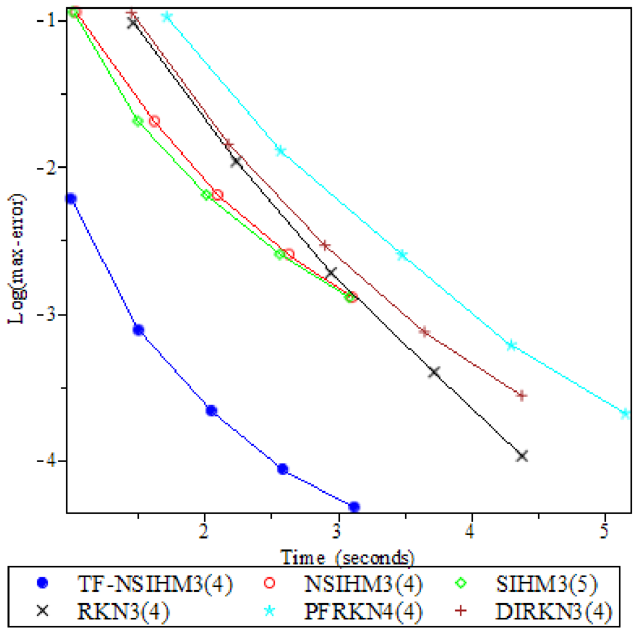

Fig. 4: The efficiency curves for TF-SIHM3(4) for problem 4 with $h = \frac{\pi}{2i}$, for $i = 2, \dots, 6$.

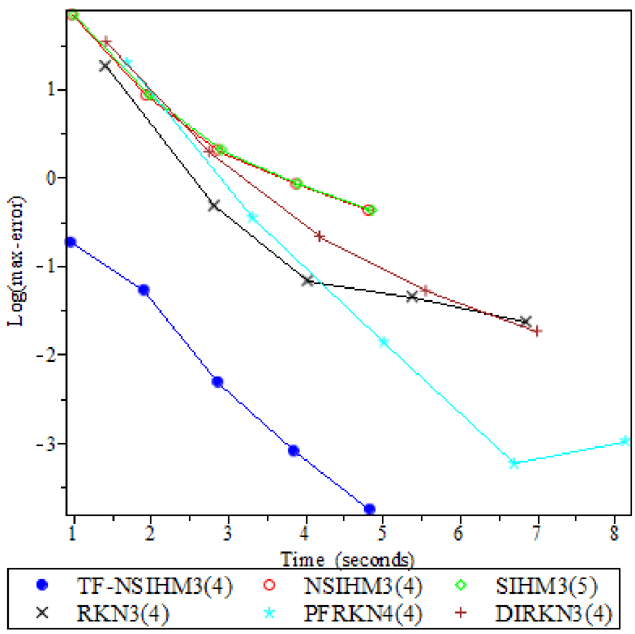

Fig. 3: The efficiency curves for TF-SIHM3(4) for problem 3 with $h = \frac{\pi}{4i}$, for $i = 1, \dots, 5$.

In analyzing the numerical results, methods of the same order are compared. We presented the efficiency curves where the logarithm of the maximum global error are plotted against the CPU time taken in second. From Figures 1-4, we observed that TF-NSIHM3(4) is the most efficient method for integrating second order DDEs which have oscillatory solutions. The method takes lesser time to compute the solutions compared to RKN and DIRKN methods.

V. Conclusion

In this paper, we derived semi implicit hybrid method of three-stage and fourth order (SIHM3(4)), then the method is trigonometrically fitted and denoted as TF-SIHM3(4). The method is then used to solve DDE problems which are oscillatory in nature. From the efficiency curves shown in Figures 1-4, we can conclude that the new TF-SIHM3(4) is very efficient compared to the original non-fitted method as well as other well-known existing methods of the same order in the scientific literature.

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