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Wave Dynamics of Orthotropic Elastic Media under the Action of Impulse Forces and Lacuna

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***Abstract*— Here the model of anisotropic elastic medium is considered. Law of wave propagation for such mediums is more difficult than for isotropic medium and stress-strain state essentially depends from degree of its anisotropy. For the equations of motion of such media, fundamental solutions that correspond to the action of concentrated forces are constructed. The pictures of wave fronts and the amplitudes of displacements for orthotropic media under the action of a concentrated impulse force are presented. The existence of lacunae for strongly anisotropic media is shown.**

***Index Terms*—anisotropic medium, elasticity, Green's tensor, hyperbolic system**

I. Introduction

A study of the wave's propagation in continuous media refers to the actual problems of mechanics and mathematical physics. Such investigations are associated with the solution of boundary value problems for systems of equations of hyperbolic types. The solutions of these equations can have characteristic surfaces on which the solutions themselves, or their derivatives, are discontinuous [1]. In physical processes, they describe shock waves, on the fronts of which the studied characteristics of the process (velocities, stresses, pressure, temperature, etc.) can have jumps.

For this, various models are used to take into account the real properties of the medium. The most studied is the linearly elastic isotropic model. In this paper, we consider an anisotropic elastic medium. Such medium have the characteristics closest to the real environment, in particular rock massif. Wave propagation in such medium is subject to more complex laws than in an isotropic medium, and the stress-strain state of the medium depends strongly on the degree of anisotropy. For example, in medium with strong anisotropy of the elastic properties we have the lacunas (moving unperturbed regions bounded by the wave fronts and expanding over time), and the front of the wave is very different from the classic, has a complex non-smooth shape.

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II. System of Motion Equations

The motion equations of anisotropic media are described by the strictly hyperbolic system of equations with derivatives of the second order:

$$L_{ij}(\partial x, \partial t)u_j(x,t) + G_i(x,t) = 0, \quad (x,t) \in R^{N+1} \qquad (1)$$

$$\cdot L_{ij}(\partial x, \partial t) = C_{ij}^{ml}\partial_m\partial_l - \rho\delta_{ij}\partial_t^2, \quad i, j, m, l = \overline{1, N}. \qquad (2)$$

$$C_{ij}^{ml} = C_{ij}^{lm} = C_{ji}^{ml} = C_{ml}^{ij} \qquad \cdot \qquad (3)$$

where $\partial_x = (\partial_1, ..., \partial_N)$, $\partial_i = \partial / \partial x_i$, $\partial_t = \partial / \partial t$, ρ is the density of the medium, u_i are the components of displacement vector, δ_{ij} is Kronecker symbol. In physical problems $N = 2$ corresponds to a planar deformation, $N = 3$ corresponds to the spatial case. The matrix of elastic constants C_{ij}^{ml} has symmetry properties with respect to permutation of the indices (3) and satisfies the strict hyperbolicity condition: $W(n,v) = C_{ij}^{ml} n_m n_l v^i v^j > 0 \ \forall n \neq 0$, $v \neq 0$. For an isotropic elastic medium we have $C_{ij}^{ml} = \lambda\delta_{ij}\delta_{lm} + \mu(\delta_{im}\delta_{jl} + \delta_{jm}\delta_{il})$, where λ, μ are the elastic Lame constants. Assuming the summation over the repeated indexes in the above-mentioned limits of their variation in the product (similar to the tensor convolution), we omit the sum sign. In view of the positive definiteness of W, the characteristic equation of system (1)

$$\det\{C_{ij}^{ml} n_m n_l - \rho c^2 \delta_{ij}\} = 0,$$

has (taking into account multiplicity) the real $2N$ roots: $c = \pm c_k(n)$, $0 < c_k \leq c_{k+1}$, $k = \overline{1, N-1}$, having the meaning of the phase velocities for the harmonic analysis of the system (1) and depending on the direction of propagation of the wave in the general case.

Equations (1) are strictly hyperbolic. Solutions of such equations can have characteristic surfaces on which the solutions themselves or their derivatives are discontinuous. In physical problems, they describe shock waves, which is characteristic of external influences that have an impact character and are represented by discontinuous or singular functions.

III. Green's Tensor

Fundamental solutions of the system of equations (1) are its solutions, corresponding to the action of impulsive concentrated forces of the form $G_i(x,t) = \delta_i^k \delta(x,t)$ (the index k indicates the direction of the action of the force)

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described by the Dirac function $\delta(x,t)=\delta(x)\delta(t)$. Fundamental solutions are determined up to solutions of a homogeneous system of equations. A special place among them is occupied by the Green's tensor, which satisfies the conditions: $U_{jk}(x,t)=0 \;\; for \;\; t<0$, $\|x\| \ge c_{\max} \; t$.

To construct the Green's tensor, it is convenient to use the Fourier transform, which brings the system (1) to a system of linear algebraic equations of the form

$$L_{ij}(i\xi,i\omega)\overline{U}_j^k(\xi,\omega)+\delta_i^k=0 \qquad (4)$$

Here $(\xi,\omega)=(\xi_1,...,\xi_N,\omega)$ are the parameters of the Fourier transform corresponding to the variables (x,t), $L_{ij}(\xi,\omega)$ are homogeneous second-order polynomials corresponding to the differential operators (2). Solving the system (4), we obtain the transform of the Green's tensor, which, by virtue of the homogeneity of the differential polynomials, has the form

$$\overline{U}_{jk}(i\xi,i\omega)=-Q_{jk}(i\xi,i\omega)Q^{-1}(i\xi,i\omega)=Q_{jk}(\xi,\omega)Q^{-1}(\xi,\omega)$$

Where $Q_{jk}(\cdot)$ are the cofactor of the element with the index (k,j) of $\{L(i\xi,i\omega)\}$, $Q(\cdot)$ is the symbol of L (2): $Q(i\xi,i\omega)=(-1)^N \det\{L_{ij}(\xi,\omega)\}$.

In [2] it is shown that the construction of the Green tensor reduces to the calculation of integrals over the unit sphere. For odd N the above theorem allows to build only approach of the Green tensor. For even N to determine approach must be multi-dimensional integration of the surface integral over the unit sphere. However, in some cases, this procedure could be simplified.

The symmetry relations (3) allow the tensor C_{ij}^{ml} represented in the form of a square matrix $C_{\alpha\beta}$ $(\alpha,\beta=\overline{1,6})$. The correspondence between the pairs of indexes (ij), (ml) and the indexes α, β, established by the scheme $(11)\leftrightarrow 1$, $(22)\leftrightarrow 2$, $(33)\leftrightarrow 3$, $(23)=(32)\leftrightarrow 4$, $(31)=(13)\leftrightarrow 5$, $(12)=(21)\leftrightarrow 6$. Hooke's law for an anisotropic (orthotropic) elastic medium, which is under the conditions of plane deformation, will be written in the form $\sigma_{11}=C_{11}u_1,_1+C_{12}u_2,_2$, $\sigma_{12}=C_{66}u_1,_2$, $\sigma_{22}=C_{21}u_1,_1+C_{22}u_2,_2$

For such a medium, the Green's tensor is the sum of the residues of fractional-rational functions [3,4]:

$$U_j^k(x,t)=\frac{1}{\pi t}\operatorname{Im}\sum_{\substack{q=1\\ \operatorname{Im}\zeta_q>0}}^{2}\frac{Q_{jk}\left(\zeta_q,1,(x_1\zeta_q+x_2)/t\right)}{Q,_{\zeta}\left(\zeta_q,1,(x_1\zeta_q+x_2)/t\right)} \qquad (5)$$

Here $Q_{jj}(\cdot)=-L_{kk}(\cdot)$, $Q_{jk}(\cdot)=L_{jk}(\cdot)$ $j\neq k$ or $Q_{11}(\xi_1,\xi_2,\omega\cdot)=C_{66}\xi_1^2+C_{22}\xi_2^2+\rho\omega^2$, $Q_{12}(\xi_1,\xi_2,\omega\cdot)=-(C_{12}+C_{66})\xi_1\xi_2$, $Q_{22}(\xi_1,\xi_2,\omega\cdot)=C_{11}\xi_1^2+C_{66}\xi_2^2+\rho\omega^2$, ζ_q are the roots of equation $Q(\zeta,1,x_1\zeta+x_2)=0$, $Q=Q_{11}Q_{22}-Q_{12}^2$. In the expression (5), the residues of the fractional-rational functions in the upper half-plane are summed, which requires knowledge of the values of the roots of the polynomial Q:

$$Q(\zeta,1,x_1\zeta+x_2)=0$$

The roots of this equation of the fourth degree are complex conjugate; therefore, we always have two roots satisfying the condition $Im\,\zeta\ge 0$. In the case of an isotropic medium $C_{ij}^{ml}=\lambda\delta_{ij}\delta_{lm}+\mu(\delta_{im}\delta_{jl}+\delta_{jm}\delta_{il})$ (λ, μ are elastic constants of Lame) we have $\zeta_1=-\left(x_1x_2+c_1t\sqrt{r-c_1^2t^2}\right)/\left(x_1^2-c_1^2t^2\right)$, $\zeta_2=-\left(x_1x_2+c_2t\sqrt{r-c_2^2t^2}\right)/\left(x_2^2-c_2^2t^2\right)$ where $r=\sqrt{x_ix_i}$, $c_1=\sqrt{\lambda+2\mu/\rho}$, $c_1=\sqrt{\mu/\rho}$.

The Green's tensor generates a fundamental stress tensor, the components of which, according to Hooke's law are

$$S_{ij}^k(x,t)=\frac{H(t)}{\pi t}C_{ij}^{ml}\operatorname{Im}\sum_{\substack{q=1\\ \operatorname{Im}\zeta_q>0}}^{2}\frac{Q_{mk,x_l}Q,_{\zeta}-Q_{mk}\left(Q,_{\zeta}\right)_{,x_l}}{\left(Q,_{\zeta}\right)^2}$$

IV. Waves from Impulse Sources

An investigation of the process of propagation of nonstationary waves in anisotropic media shows that the stress-strain state of a medium essentially depends on the degree of its anisotropy. S5, in the case of an isotropic medium, the front of the wave from the pulsed source is concentric circles (spheres) expanding with the corresponding propagation velocities of the volume and shear waves. In media with weak anisotropy of elastic properties, the wave propagation pattern is similar to the wave propagation pattern in an isotropic medium, but wave fronts representing closed smooth curves differ slightly from concentric circles. In environments with a strong anisotropy of elastic properties, lacunae arise. The coordinates of such regions satisfy the conditions $\operatorname{Im}\zeta_q(x_1,x_2,t)=0,\ q=1{,}2$. This phenomenon is associated with the waveguide properties of a highly anisotropic medium, which are sharply expressed in directions with predominant rigidity and are weakened in those where the rigidity is small.

The existence of lacunae for hyperbolic equations with constant coefficients was discovered by the IG. Petrovsky [5]. They are given necessary and sufficient conditions for the existence of lacuna. Lacuna is components of the addition to the surface of the wave front, in which fundamental solutions (strong lacunae) vanish. An example of strong lacunas gives, in particular, the system of equations (1) in an even-dimensional space. Lacunas whose coordinates satisfy the conditions $\operatorname{Im}\zeta_q(x_1,x_2,t)=0,\ q=1{,}2$ arise for certain constants of equations (1) corresponding to strongly anisotropic media. For such media, the wave front patterns differ sharply from the classical front as in the case of isotropic media and have a complex non-smooth form.

Below are the wave fronts and displacement amplitudes under the action of a concentrated impulse force. Calculations were carried out for crystals of siltstone ($C_{11}=6{,}75$, $C_{12}=1{,}6875$, $C_{22}=6{,}75$, $C_{66}=2{,}5312*10^{10}$ n/m^2), aragonite ($C_{11}=16$, $C_{12}=3{,}73$, $C_{22}=8{,}67$, $C_{66}=4{,}27$), zinc (Zn) ($C_{11}=4{,}219$, $C_{12}=0{,}59$,

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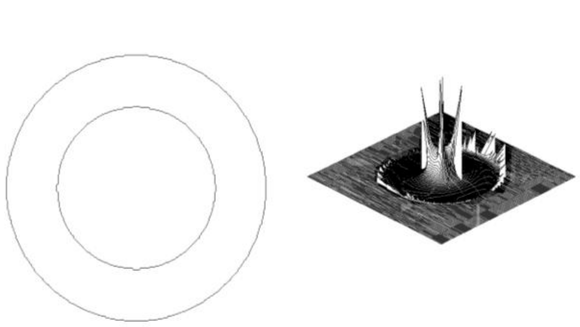
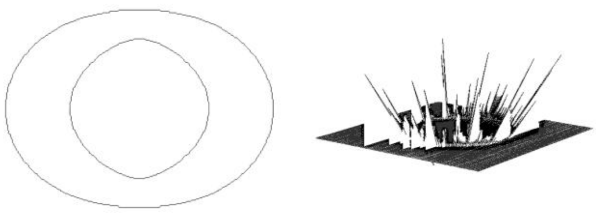
$C_{22} = 1{,}645$, $C_{66} = 1{,}0$), topaz ($C_{11} = 28{,}2$, $C_{12} = 13{,}1$, $C_{22} = 34{,}9$, $C_{66} = 12{,}6$) and potassium pentaborate ($C_{11} = 5{,}82$, $C_{12} = 2{,}29$, $C_{22} = 3{,}59$, $C_{66} = 0{,}57$).



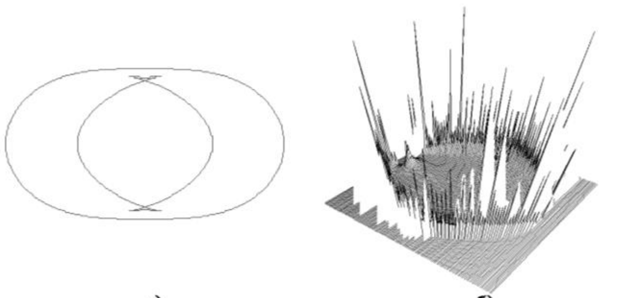
a) б)

Fig. 1 Picture wave fronts (a) and the amplitude of movements (b) for siltstone under the action of a concentrated force



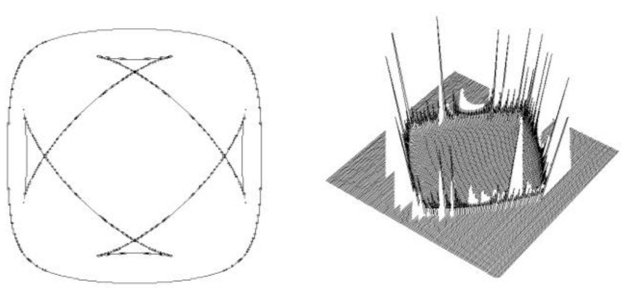
a) б)

Fig. 2 Picture wave fronts (a) and the amplitude of movements (b) for aragonite under the action of a concentrated force



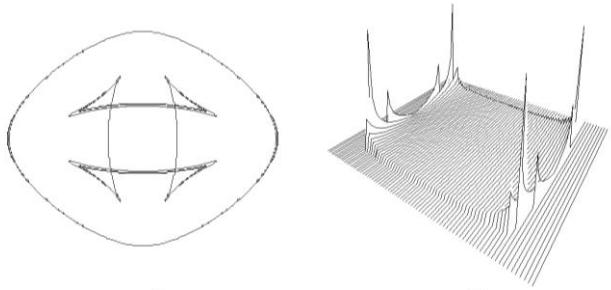
a) б)

Fig. 3 Picture wave fronts (**a**) and the amplitude of movements (**b**) for Zn under the action of a concentrated force



a) б)

Fig.4 Picture wave fronts (**a**) and the amplitude of movements (**b**) for topaz under the action of a concentrated force



a) б)

Fig. 5 Picture wave fronts (a) and the amplitude of movements (b) for potassium pentaborate under the action of a concentrated force

It can be seen from the figures that unlike isotropic siltstone (Fig. 1) and weakly anisotropic aragonite (Fig. 2), there are lacunae (represented by triangular regions) for orthotropic zinc, topaz and potassium-pentaborate, which are strongly anisotropic media.

Studies show that the location of lacuna depends on the matrix of constants and the degree of anisotropy is determined (in the planar case) by the value of the coefficients

$$A_1 = (C_{11} - C_{66})(C_{22} - C_{66}) - (C_{12} + C_{66})^2$$
$$A_2 = (C_{11} - C_{66})C_{22} - (C_{12} + C_{66})^2$$
$$A_3 = (C_{22} - C_{66})C_{11} - (C_{12} + C_{66})^2$$

in the following way:

if $A_1 = 0$, $A_2 = A_3 > 0$ we have an isotropic medium, the wave front from a pulsed source is a concentric circle;

if $A_1 < 0$, $A_2, A_3 > 0$ we have the case of weak anisotropy, the fronts of the quasilongitudinal and quasi-transverse waves have the form of convex closed curves different from the circles with the center at the source.is an isotropic medium, the wave front from a pulsed source is a concentric circle.

For strong anisotropy, lacunae appear in the medium and the wave front differs sharply from the classical one, has a complex non-smooth form, and

if $A_1 < 0$, $A_2 < 0$, $A_3 > 0$ then lacunas are formed on the axis x_1 ,

if $A_1 < 0$, $A_2 > 0$, $A_3 < 0$ then lacunas are formed on the axis x_2 .

In these cases, the field of a quasilongitudinal wave is a three-connected region. In addition, lacunas can be located for

$A_i < 0$ $(i = 1,2,3)$ lacunas can be located on both axes simultaneously

$A_i > 0$ $(i = 1,2,3)$ lacunas can be located between the axes.

These are the cases of the five-region region of the field of quasilongitudinal perturbations.

It can be seen from the figures that in the case of an isotropic medium - siltstone the front of the wave from the pulsed source represents concentric circles (spheres) expanding with the corresponding velocities of propagation of the volume and shear waves (Fig. 1). So, for orthotropic zinc, topaz and potassium-pentaborate, which are strongly anisotropic media, there are lacunas (they are represented by triangular regions). The location of the lacunas is different: for zinc, it is on one axis (the quasilongitudinal wave field is a three-connected region) (Fig. 3), for topaz on both orthotropic axes (Fig. 4), for potassium pentaborate between the orthotropic axes (Fig. 5). For these media, the region of a quasilongitudinal wave is a five-connected region bounded by an external front and parts of the inner wave front that connect the return points to each other, and with nodal points. These sections of the inner front of the wave, forming closed piecewise smooth lines, are internal fronts of the quasilongitudinal wave. The front of a quasitransverse wave consists of piecewise smooth curves

Below are the results of calculations of the stress-strain state in the perturbed zone. The figures show the distribution pattern of displacements along the axis for the media under consideration in cases of concentrated force, dipole, co-torque, flat expansion center, rotation center.

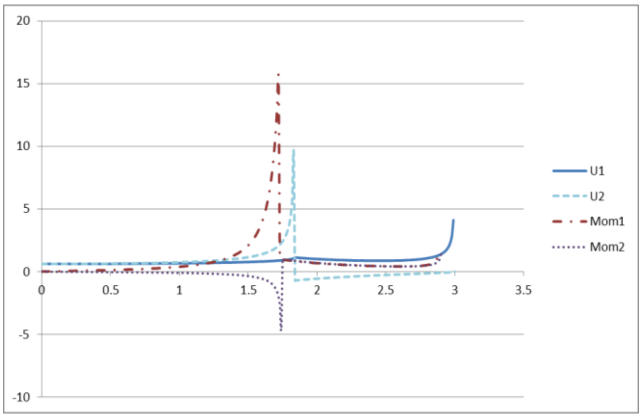


Fig.6 Components of the Green's tensor for siltstone under the action of concentrated forces and moments at t = 3

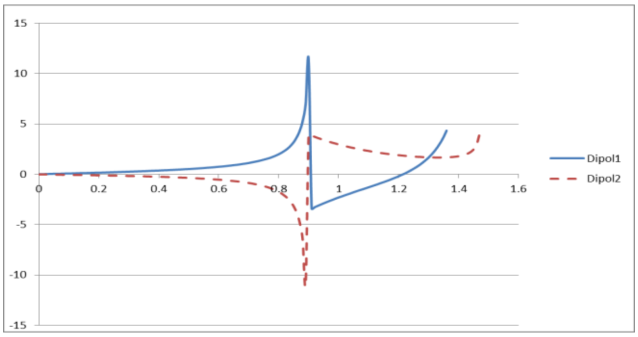


Fig.7 Components of the Green's tensor for siltstone under the action of the dipole for t = 1.5

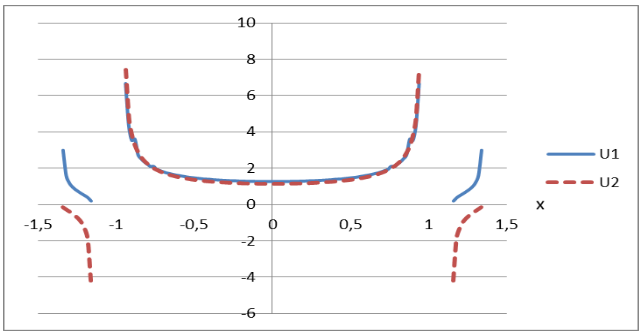


Fig. 2 Components of the Green's tensor for topaz under the action of a concentrated force

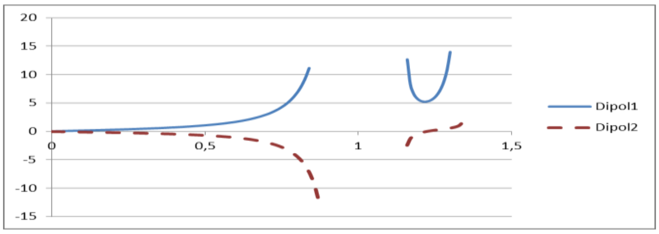


Fig. 3 Components of the Green's tensor for topaz under the action of a dipole

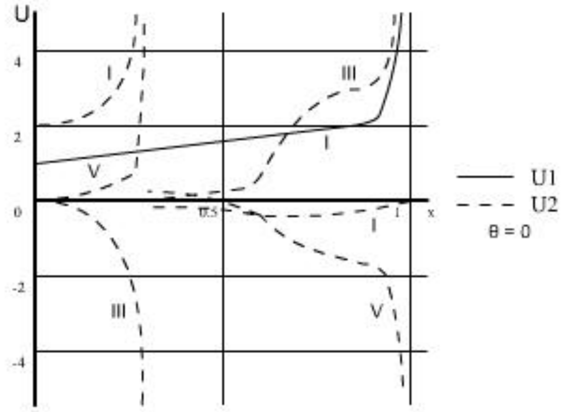


Fig.8 Components of the Green's tensor for potassium pentaborate under the action of concentrated forces

Figure 8 shows the distribution of the displacement tensor components along the axis (inclination angle) for potassium pentaborate under the action of the concentrated force (I), the concentrated moment (III), the center of rotation (V)).

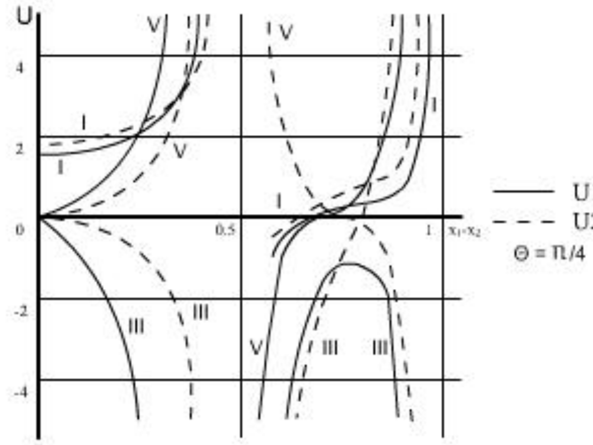


Fig.9 Components of the Green's tensor for potassium pentaborate under the action of concentrated forces

Figure 9 shows the distribution of the components of the displacement tensor along an axis located at an angle $\theta = \pi / 4$ to the axis x_1 for potassium pentaborate under the action of a concentrated force (I), a concentrated moment (III), the center of rotation (V).

In the calculations it was assumed that $D = 1$, $M = 1$, in the case of a dipole $e = (1,0)$, for a concentrated moment $G^0 = (1,0)$, $e = (0,1)$.

Calculations show that there are no displacements in the sections corresponding to the lacunas.

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V. Conclusion

The study of the processes of wave propagation from foci of earthquakes is associated with the study of the stress-strain state of the medium under the action of distributed mass forces. For regular $G_k(x,t)$ the components of the displacement field are the following integral representations: $\hat{u}_i(x,t) = \int_0^\infty d\tau \int_{R^N} U_{ik}(x-y,t-\tau)G_k(y,\tau)dV(y)$.

For a distant source of an earthquake, the distance to which substantially exceeds its dimensions, the models of concentrated sources in the form of singular generalized functions with point support (poles, dipoles, multipoles, etc.) are used [6]. The displacement field then has the form of a convolution U_{jk} with the corresponding G_k: $u_j = U_{jk} * G_k$, $j,k = \overline{1,N}$ which should be taken according to the rules of convolution definition in the theory of generalized functions.

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