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Contraction on Some Fixed Point Theorem in $b_v(s)-$Metric Spaces

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***Abstract*—In this paper we established some fixed point theorems in the frame of a $b_v(s)-$metric space which is generalize of b-metric space and v-generalized metric space. Our results extended and improved the results of the classical results by Branchini and Sehgal.**

***Index Terms*— Fixed point theory, metric space, b-metric space, wt-distance**

I. Introduction

THE origin of fixed point theory known as the Banach contraction principle, which is the starting point of metric fixed point theory. The study of metric fixed point theory has been on the past two decades. Many fixed point theorems were obtained in the notion of metric space (or the standard metric space) and its generalization (see in [1], [3], [6], [8], [11] and [12]). One generalization of the metric space is b-metric space which was introduced by Bakhtin [2] and Czerwik [5]. Branciari [3] introduced the concept of $v-$generalized metric space. Furthermore, Mitrovic and Radenovic [10] introduced $b_v(s)$-metric spaces which generalizes the concept of b-metric spaces and v-generalize metric space. Applications of metric fixed point theory were given in different areas like variational and linear are inequalities and approximation theory. A number of researchers had defined contractive type mapping on a metric space. The classical by Banach, Kannan, Reich, Chetterea, and Sehgal gave a substantial new contractive mapping to prove the fixed point theorem. Hussan et. al. [7] introduced a new concept of wt-distance on b-metric space and proved some fixed point theorem by using wt-distance in a partially order b-metric space. Moreover, Some fixed point theorems on the setting of Banach and Reich contractions in $\mathrm{b_v(s)}$ -metric spaces were given by Mitrovic and Radenovic [10]. In this paper, we present some fixed point theorems in the setting of Sehgal contraction in $\mathrm{b_v(s)}$ -metric spaces.

II. Preliminaries

Definition 1 ([2]). Let X be a nonempty set and $s \geq 1$ a given reed number. A function $p: X \times X \to R^+[0,-\infty)$ is called $b-$metric if for all $x, y, z \in X$ the following condition are satisfied:

1) $p(x, y) = 0$ if and only if $x = y$;

2) $p(x, y) = p(y, x)$;

3) $p(x, z) \leq s[p(x, y) + p(y, z)]$

A triplet (X, p, s) is called a $b-$metric space with coefficient s. (in short bMS)

Definition 2 ([5]). Let X be a nonempty set and $s \geq 1$ a given real number. A mapping $p: X \times X \to [0,+\infty]$ is called a rectangular metric space is for all $x, y \in X$ satisfy.

1) $p(x, y) = 0$ *if and any if* $x = y$;

2) $p(x, y) = p(y, x)$

3) $p(x, y) \leq s[p(x,u) + p(u,v) + p(v, y)]$ for all distinct points $u, v \in X - \{x, y\}$.

The pair (X, p) *is called a rectangular b-metric space (in short RbMS)*

Definition 3 ([3]). Let X be a nonempty set and $s \geq 1$ a given read number, and $v \in N$. A function from $X \times X \to [0,\infty)$ is called $v-$generalized metric if for all $x, y \in X$ and for all distinct founts $t_1, t_2, ..., t_v \in X$, each of them different from x and y, the following conditions are satisfied:

1) $p_v(x, y) = 0$ if and only if $x = y$;

2) $p_v(x, y) = p_v(y, x)$;

3) there exists a real number $s \geq 1$ such that

$$p_v(x, y) \leq s[p_v(x,t_1) + p_v(t_1,t_2) + ... + p_v(t_v, y)].$$

Then (X, p_v) is called a $v-$generalized metric space.

Definition 4 ([11]). Let X be a nonempty set and $s \geq 1$ a given read number, and $v \in N$. A function p_v from $X \times X \to [0,\infty)$ is called $b_v(s)$-metric space if for all $x, y \in X$ and for all distinct founts $t_1, t_2, ..., t_v \in X$, each of them different from x and y, the following conditions are satisfied:

1) $p_v(x, y) = 0$ if and only if $x = y$;

2) $p_v(x, y) = p_v(y, x)$;

3) there exists a real number $s \geq 1$ such that

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$$p_v(x,y) \le s[p_v(x,t_1)+p_v(t_1,t_2)+...+p_v(t_v,y).$$

Definition 5 ([9]). Let (X,p_v) be a $b_v(s)-$metric space, $\{z_n\}$ be a sequence in X and $z \in X$. Then

1) The sequence $\{z_n\}$ is said to be convergent in (X,p_v) and converges to $z,$ if for every $\varepsilon>0$ there exists $k_0 \in \mathbb{N}$ such that $p(z_n,z)<\varepsilon$ for all $n>k_0$ and denote by $\lim_{n\to\infty} z_n = z.$

2) The sequence $\{z_n\}$ is said to Cauchy sequence in (X,p_v) if for every $\varepsilon>0$ there exists $k_0 \in \mathbb{N}$ such that $p(z_m,z_n)<\varepsilon$ for all $m,n \ge k_0$.

3) (X,p_v) is said to be a complete $b_v(s)-$metric space if every Cauchy sequence in X converges to some $z \in X$.

Definition 6 ([9]). Let p_v a real value function on a $b_v(s)-$metric is said to be $s-$lower semi-continuous at a point z in X if $\liminf_{z_n\to z} p(z_n)=\infty$ or $p(z) \le \liminf_{z_n\to z} sp(z_n),$ when ever $z_n \in X$ for each $n \in N$ and $\lim_{n\to\infty} z_n = z$.

Definition 7 ([11]). Let (X,p_v) be a $b_v(s)-$metric space with constant $s \ge 1$. Then a function $p_v : XxX \to [0,\infty)$ is called a $wt-$distance on X if the following conditions are satisfied:

1) $p(x,y) \le s\left[p(x,t_1)+p(t_1,t_2)+...+p(t_{v-1},t_v)+p(t_v,y\right]$ for every v distinct $t_1,t_2,t_3,...,t_v \in X\setminus\{x,y\}$;

2) for any $x \in X$, $p_v(x,\cdot): X \to [0,\infty)$ is $s-$lower semi-continuous;

3) for any $\varepsilon>0$ there exists $\delta>0$ such that $p(z,x)\le\delta$ and $p_v(z,y)\le\delta$ imply $p(x,y)\le\varepsilon.$

Lemma 1 ([11]). Let (X,p_v) be a $b_v(s)-$metric space with constant $s\ge1$ and let q be a $wt-$distance on X. let $\{z_n\}$ and $\{u_n\}$ be sequences in X. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to $0,$ and let $k,u,t \in X$. Then the following hold:

1) If $q(z_n,u)\le\alpha_n$ and $q(z_n,t)\le\beta_n$ for any $n\in\mathbb{N}$, then $u=t$;

2) If $q(z_n,u_n)\le\alpha_n$ and $q(z_n,t)\le\beta_n$ for any $n\in\mathbb{N}$, then $\{u_n\}$ converges to t;

3) If $q(z_n,z_m)\le\alpha_n$ for any $n,m\in\mathbb{N}$ with $m>n$, then is a Cauchy sequence;

4) If $q(u,z_n)\le\alpha_n$ for any $n\in\mathbb{N}$, then $\{z_n\}$ is a Cauchy sequence.

III. Main Results

Theorem 1. Let q be a $wt-$ distance on a $b_v(s)-$metric space (X, p) with constant $s\ge1$. Let T_1 and T_2 be mappings from X into itself. Suppose that there exists $r\in[0,\frac{1}{s})$ such that

1) $\max\left\{p(T_1(x),T_2T_1(x)),\, p(T_2(x),T_1T_2(x))\right\}$
$\le \delta\min\left\{p(x,T_1(x)),p(x,T_2(x))\right\}$

for every $x\in X$ and

2) $\inf\left\{p(x,v)+\min\left\{p(x,T_1x),\,p(x,T_2x)\right\}: x\in X\right\}>0$

for every $v\in X$ with v is not a common fixed point in X. Moreover, if $z=T_1z=T_{2z,}$ then $p(z,z)=0.$

Proof. Let $z_0\in X$ be arbitrary and defined a sequence $\{z_n\}$ by

$$z_n=\begin{cases} T_1z_{n-1}, if\ n\ \ is\ \ odd \\ T_2z_{n-1}, if\ n\ \ is\ \ even.\end{cases}$$

If $n\in\mathbb{N}$ is odd, then by using 1), we have

$$\begin{aligned} &p(z_n,z_{n+1})\\ &=p(T_1z_{n-1},T_2z_n)\\ &=p(T_1z_{n-1},T_2T_1z_{n-1})\\ &\le\max\left\{p(T_1z_{n-1},T_2T_1z_{n-1}),\,p(T_2z_{n-1},T_1T_2z_{n-1})\right\}\\ &\le\delta\min\left\{p(z_{n-1},T_1z_{n-1}),\,p(z_{n-1},T_2z_{n-1})\right\}\\ &\le\delta p(z_{n-1},T_1z_{n-1})\\ &=\delta p(z_{n-1},z_n).\end{aligned}$$

If n is even, then by 1), we have

$$\begin{aligned} &p(z_n,z_{n+1})\\ &=p(T_2z_{n-1},T_1z_n)\\ &=p(T_2z_{n-1},T_1T_2z_{n-1})\\ &\le\max\left\{p(T_1z_{n-1},T_2T_1z_{n-1}),\,p(T_2z_{n-1},T_1T_2z_{n-1})\right\}\\ &\le\delta\min\left\{p(z_{n-1},T_1z_{n-1}),\,p(z_{n-1},T_2z_{n-1})\right\}\\ &\le\delta p(z_{n-1},T_2z_{n-1})\\ &=\delta p(z_{n-1},z_n).\end{aligned}$$

Thus for any positive integer $n,$ we obtain

$$p(z_n,zn+1)\le\delta p(z_{n-1},z_n). \quad (3)$$

$$p(z_n,z_{n+1})\le\delta^n p(z_0,z_1). \quad (4)$$

For $m,n\in\mathbb{N}$ with $m>n,$ we have by repeated of (4) that

$$\begin{aligned} p(z_n,z_m)\le s\Big[&p(z_n,z_{n+1})+p(z_{n+1},z_{n+2})+...+p(z_{n+v-3},z_{n+v-2})\\ &+p(z_{n+v-2},z_{n+n_0})+p(z_{n+n_0},z_{m+n_0})+p(z_{m+n_0},z_m).\end{aligned}$$

Then

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$$\begin{aligned} p(z_n,z_m) &\le s\left[\delta^n+\delta^{n+1}+\ldots+\delta^{n+v-3}\right]p(z_0,z_1) \\ &\quad +sp\left(z_{n+v-2},z_{n+n_0}\right)+p\left(z_{n+n_0},z_{m+n_0}\right)+p\left(z_{m+n_0},z_m\right) \\ &= s\left[\delta^n+\delta^{n+1}+\ldots+\delta^{n+v-3}\right]p(z_0,z_1) \\ &\quad +s\delta^n p\left(z_{v-2},z_{n_0}\right)+s\delta^{n_0}p\left(z_n,z_m\right) \\ &\quad +s\delta^m p(z_{n_0},z_0)+\left(1-s\delta^{n_0}\right)p\left(z_n,z_m\right). \end{aligned}$$

$$\begin{aligned} \left(1-s\delta^{n_0}\right)p\left(z_n,z_m\right) &\le s\left[\delta^n+\delta^{n+1}+\ldots+\delta^{n+v-3}\right]p(z_0,z_1) \\ &\quad +s\delta^n p\left(z_{v-2},z_{n_0}\right)+s\delta^m p\left(z_{n_0},z_0\right). \end{aligned}$$

We have that $\{z_n\}$ is Cauchy sequence in X. Since X is complete, $\{z_n\}$ converge to some point $z\in X$. Let $n\in\mathbb{N}$ be fixed. Then since $\lim_{n\to\infty} z_n = z$ and $p(z_n,\cdot)$ is $s-$lower semi continuous, we have

$$\begin{aligned} p(z_n,z) &\le \liminf_{n\to\infty} sp(z_n,z_m) \\ &\le \frac{s^2\delta^n}{1-s\delta^{n_0}}\left[1+\delta+\delta^2+\ldots+\delta^{v-3}\right]p(z_0,z_1) \\ &\quad +\frac{s^2\delta^n}{1-s\delta^{n_0}}p\left(z_{v-2},z_{n_0}\right)+\frac{s^2\delta^m}{1-s\delta^{n_0}}p\left(z_{n_0},z_0\right). \end{aligned}$$

Assume that v is not a common fixed point of T_1 and T_2. Then hypothesis 2), we have

$$\begin{aligned} 0 &< \inf\left\{p(z,v)+\min\left\{p(z,T_1z),p(z,T_2z)\right\}: z\in X\right\} \\ &\le \inf\left\{p(z_n,v)+\min\left\{p(z_n,T_1z_n),p(z_n,T_2z_n)\right\}: n\in\mathbb{N}\right\} \\ &\le \inf\left\{\frac{s^2\delta^n}{1-s\delta^{n_0}}p(z_0,z_1)\left[1+\delta+\ldots+\delta^{v-3}\right]\right. \\ &\quad +\frac{s^2\delta^n}{1-s\delta^{n_0}}p\left(z_{v-2},z_{n_0}\right) \\ &\quad \left.+\frac{s^2\delta_m}{1-s\delta^{n_0}}p\left[z_{n_0},z_0\right]+\delta^n p\left(z_n,z_{n+1}\right): n\in\mathbb{N}\right\} \\ &= 0, \end{aligned}$$

which is a contradiction. Therefore, $v=T_1v=T_2v.$ If $T_1v=v=T_2v$ for some $v\in X$, then

$$\begin{aligned} p(v,v) &= \max\left\{p\left(T_1v,T_2T_1v\right),p\left(T_2v,T_1T_2v\right)\right\} \\ &\le \delta\min\left\{p\left(v,T_1v\right),p\left(v,T_2v\right)\right\} \\ &= \delta\min\left\{p(v,v),p(v,v)\right\} \\ &= \delta p(v,v) \end{aligned}$$

which give that $p(v,v)=0.$

Corollary 2. Let (X,p) be complete $b_v(s)-$metric space with constant $s\ge 1$ and $T:X\to X$. Let q be a $wt-$distance on X. Suppose there exists $\delta\in[0,\frac{1}{s})$ such that

$$q\left(Tx,T^2x\right)\le\delta q(x,Tx)$$

for every $x\in X$ and that

$$\inf\left\{p(x,y)+p(x,Tx): x\in X\right\}>0$$

for every $y\in X$ with $y\ne Ty$. Then T has a fixed point in X. Moreover, if $z=Tz$, then $p(z,z)=0.$

Proof. We have the desired result by taking $T_1=T_2=T$ in Theorem 1.

Theorem 3. Let (X,p) be complete $b_v(s)-$metric space with constant $s\ge 1$ and $T:X\to X$. Suppose there exists $\delta\in[0,\frac{1}{s})$ such that

$$p(Tx,Ty)\le\delta\max\left\{p(x,y),p(x,Tx),p(y,Ty),p(y,Tx)\right\} \quad (5)$$

for every $x,y\in X$. Then T has a unique fixed point in X.

Proof. We treat the $b_v(s)-$metric p as $wt-$distance on X. From (5), we have

$$\begin{aligned} p(Tx,T^2x) &= p(Tx,T(Tx)) \\ &\le \delta\max\left\{p(x,Tx),d(x,Tx),d(Tx,T^2x),p(Tx,Tx)\right\} \\ &= \delta\max\left\{p(x,Tx),p(Tx,T^2x)\right\}. \quad (6) \end{aligned}$$

Without loss of generality, we assume that $Tx\ne T^2x$. For, otherwise, T has a fixed point. Since $\delta<\frac{1}{s}$, we obtain from (6) that

$$p(Tx,T^2x)\le\delta p(x,Tx). \quad (7)$$

For every $x\in X$. Assume that there exists $v\in X$ with $v\ne Tv$ and $\inf\left\{p(x,v)+p(x,Tx): x\in X\right\}=0$. Then there exists a sequence $\{z_n\}$ in X such that

$$\lim_{n\to\infty}\left\{p(z_n,v)+p(z_n,Tz_n)\right\}=0.$$

So, we get $\lim_{n\to\infty}p(z_n,v)=0$ and $\lim_{n\to\infty}p(z,Tz_n)=0$. So we obtain from 1) in Lemma 1 that $\lim_{n\to\infty}Tz_n=v$. We also have

$$\begin{aligned} p(v,Tv) &\le s\left[p(v,Tz_n)+d(Tz_n,z_n)+d(z_n,z_{n+1})+p(z_{n+1},z_{n+2})\right. \\ &\quad \left.+\ldots+p(z_{n+v-3},z_{n+v-2})+p(z_{n+v-2},Tv)\right]. \end{aligned}$$

Then we have $p(v,Tv)\le sp(v,Tv)$. Therefore $p(v,Tv)=0$. Then $v=Tv$. This is a contradiction. Hence, if

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$v \neq Tv,$ then $\inf\{p(x,y)+p(x,Tx) : x \in X\} > 0.$ By applying Corollary 2, we obtain a fixed point of T in X. For uniqueness, let v and v^* are fixed points of T, that is $T(v) = v$ and $T(v^*) = v^*$. It follows from (5) that

$$\begin{aligned} p\left(v,v^*\right) &\leq \delta \max\left\{p(v,v^*), p(v,Tv), p(v^*,Tv^*), p(v^*,Tv)\right\} \\ &= \delta \max\left\{p(v,v^*), p(v,v), p(v^*,v^*), p(v^*,v)\right\} \\ &= \delta p(v,v^*). \end{aligned}$$

Therefore, we have $p(v,v^*) = 0,$ i.e., $v = v^*$.

Remark 1. If we put $s=1$ and $p(x,y)=0$ in Theorem 1. we have the fixed point result given by Branchini [4].

Remark 2. If we put $s=1$ and $v=1$ in Theorem 2 we have the fixed point result given by Sehgal [13].

IV. Conclusion

Our theorem 2 extended many theorems in the previous results given by many authors in the setting of metric space (standard metric pace), b-metric space, bMS, RbMS, and $v-$generalized metric space as in remark 1. and remark 2. Moreover, we shall extend this idea by using others contractions.

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