The Interaction between a Surface Nonuniform Foundation and a Regular System of Punches with Rough Bases

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Abstract—**Multibody contact problem for a nonuniform foundation and a regular system of rigid punches is under consideration. An appropriate boundary value problem is formulated and a system of 2D integral equations is obtained which contains integral operators of different type as well as prescribed rapid changing functions. An effective projection method for the solution of such a system is developed. A model problem is solved. Quantitative and qualitative conclusions are presented.**

Index Terms—**coating, multibody contact, nonuniform foundation, projection method, rapidly changing functions, regular system of punches, roughness**

Introduction

VARIOUS aspects of the contact interaction problems for foundations with coatings have been studied in good number of papers (see, e.g., [1]–[12]). This paper deals with foundations which contain nonuniform coatings fabricated using additive manufacturing technologies. Such foundations sometimes are called as surface nonuniform foundations. The strong material nonuniformity of foundations usually results from the process of layer by layer manufacturing of a coating (e.g., laser treatment, ion implantation, etc. [13], [14]) and its further processing. Papers [15], [16] consider the interaction and wear problems of a single punch with rough base and foundations with nonuniform coatings. Here we study the plane multibody contact problem for a surface nonuniform foundation and a regular system of rigid punches. A system of punches is said to be regular if the distances between neighboring punches as well as the widths of punches are equal to each other respectively and shapes of punches base are identical.

I. Statement of the Problem

We assume that viscoelastic aging layer of an arbitrary thickness H with a elastic coating of an thickness h lies on a rigid basis. We denote the moment of lower layer production by τ_2. We also assume that the coating rigidity $R(x)$ depend on the longitudinal coordinate. It is less than the rigidity of the lower layer or they are of the same order of magnitude. This rigidity described by periodic function with period L. There is smooth contact or perfect contact between layers and between the lower layer and the rigid base.

At time τ_0, the forces $P_i(t)$ with eccentricities $e_i(t)$ starts to indent system of n smooth rigid punches into the surface of such a foundation. We assume that the system of punches is regular, i.e. punch lengths $\ell = b_i - a_i$ are equal, the distances between neighbor punches are the same and equal to $L - \ell$, and punch base forms are similar, i.e. $g_i(x - a_i) \equiv g_j(x - a_j)$ $(i, j = 1, 2, \dots, n)$. Here a_i and b_i are left and right coordinates of ith punch. The lengths of lines of contact area are ℓ. The coating is assumed to be thin compared with the contact areas, i.e., its thickness satisfies the condition $h \ll \ell$.

The integral equation of this problem can be written in the form

$$\begin{aligned}
&\frac{q_i(x,t)h}{R(x)} + (\mathbf{I} - \mathbf{V})\frac{2(1-\nu_2^2)}{\pi E_2(t-\tau_2)} \\
&\quad\times \sum_{j=1}^{n} \int_{a_j}^{b_j} k_{\mathrm{pl}}\Big(\frac{x-\xi}{H}\Big) q_j(\xi,t)\, d\xi \\
&\quad= \delta_i(t) + \alpha_i(t)(x - \eta_i) - g_i(x), \qquad (1) \\
&\mathbf{V}f(x,t) = \int_{\tau_0}^{t} K_2(t-\tau_2, \tau-\tau_2) f(x,\tau)\, d\tau, \\
&K_2(t,\tau) = E_2(\tau)\tfrac{\partial}{\partial\tau}\big[E_2^{-1}(\tau) + C_2(t,\tau)\big], \\
&a_i \leqslant x \leqslant b_i, \quad t \geqslant \tau_0, \quad i = 1, 2, \dots, n,
\end{aligned}$$

where $\delta_i(t)$ are the punch settlements and $\alpha_i(t)$ are tilt angles, $\eta_i = \frac{1}{2}(a_i + b_i)$ are the punch midpoints; $E_2(t)$ and ν_2 are the Young modulus and Poisson's ratio of the lower layer; $\mathbf{I}$ is the identity operator; $\mathbf{V}$ are the Volterra integral operator with tensile creep kernel $K_2(t,\tau)$; $C_2(t,\tau)$ is the tensile creep function; contact rigidity $R(x)$ depend on the contact conditions between coating and lower layer; in the case of a smooth coating-layer contact, we have $R(x) = E_1(x)/[1 - \nu_1^2(x)]$, and in the case of an perfect contact, $R(x) = E_1(x)[1 - \nu_1(x)]/[1 - \nu_1(x) - 2\nu_1^2(x)]$, where $E_1(x)$ and $\nu_1(x)$ are the Young modulus and Poisson's ratio of the coating; $k_{\mathrm{pl}}[(x-\xi)/H]$ is known kernel of the plane contact problem, which has the form [17]:

$$k_{\mathrm{pl}}(s) = \int_0^\infty \frac{L(u)}{u}\cos(su)\, du, \qquad (2)$$

and, in the case of a smooth contact between the lower layer and the rigid base, $L(u) = (\cosh 2u - 1)/(\sinh 2u + 2u)$, and in the case of a perfect contact, $L(u) = (2\varkappa \sinh 2u - 4u)/(2\varkappa \cosh 2u + 4u^2 + 1 + \varkappa^2)$, $\varkappa = 3 - 4\nu_2$.

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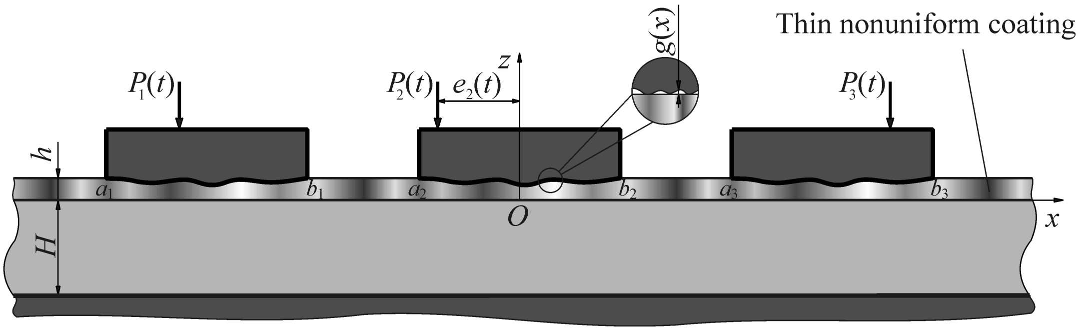

Fig. 1. Basic scheme of multiple contact interaction between surface nonuniform layer and system of rigid punches with rough bases.

We supplement Eq. (1) with the conditions of the punches equilibrium on the foundation

$$\int_{a_i}^{b_i} q_i(\xi,t)\,d\xi = P_i(t), \quad \int_{a_i}^{b_i} \xi q_i(\xi,t)\,d\xi = M_i(t). \tag{3}$$

Here $M_i(t) = e_i(t)P_i(t)$ denotes the moments of application of the forces $P_i(t)$.

Let us make the change of variables in (1) and (3) by the formulas

$$\begin{aligned}
&x^* = \frac{2(x-\eta_i)}{\ell}, \quad \xi^* = \frac{2(\xi-\eta_j)}{\ell}, \quad t^* = \frac{t}{\tau_0}, \quad \tau_2^* = \frac{\tau_2}{\tau_0},\\
&\lambda = \frac{2H}{\ell}, \quad \delta^{i*}(t^*) = \frac{2\delta_i(t)}{\ell}, \quad \alpha^{i*}(t^*) = \alpha_i(t),\\
&g^*(x^*) \equiv g^{i*}(x^*) = \frac{2g_i(x)}{\ell}, \quad c^*(t^*) = \frac{E_2(t-\tau_2)}{E_0},\\
&m(x^*) \equiv m^i(x^*) = \frac{E_0}{R(x)(1-\nu_2^2)}\frac{h}{\ell},\\
&q^{i*}(x^*,t^*) = \frac{2(1-\nu_2^2)q_i(x,t)}{E_2(t-\tau_2)},\\
&P^{i*}(t^*) = \frac{4P_i(t)(1-\nu_2^2)}{E_2(t-\tau_2)\ell}, \quad M^{i*}(t^*) = \frac{8M_i(t)(1-\nu_2^2)}{E_2(t-\tau_2)\ell^2},\\
&\mathbf{F}^{ij*}f(x^*) = \int_{-1}^{1} k^{ij}(x^*,\xi^*)f(\xi^*)d\xi^*,\\
&k^{ij}(x^*,\xi^*) = \frac{1}{\pi}k_{\mathrm{pl}}\left(\frac{x^*+\eta^{i*}-\xi^*-\eta^{j*}}{\lambda}\right) = \frac{1}{\pi}k_{\mathrm{pl}}\left(\frac{x-\xi}{H}\right),\\
&\mathbf{V}^* f(t^*) = \int_1^{t^*} K^{2*}(t^*,\tau^*)f(\tau^*)d\tau^*,\\
&K^{2*}(t^*,\tau^*) = K_2(t-\tau_2,\tau-\tau_2)\tau_0, \quad i,j = 1,2,\dots,n.
\end{aligned} \tag{4}$$

Then, omitting the asterisks, we obtain a system of mixed integral equations and additional conditions in the dimensionless form

$$\begin{aligned}
&c(t)m(x)q^i(x,t) + (\mathbf{I}-\mathbf{V})\sum_{j=1}^{n}\mathbf{F}^{ij}q^j(x,t)\\
&\qquad = \delta^i(t) + \alpha^i(t)x - g^i(x),\\
&\int_{-1}^{1} q^i(\xi,t)\,d\xi = P^i(t), \quad \int_{-1}^{1}\xi q^i(\xi,t)\,d\xi = M^i(t),\\
&-1 \leqslant x \leqslant 1, \quad t \geq 1, \quad i = 1,2,\dots,n.
\end{aligned} \tag{5}$$

There exist a lot of versions of mathematical statements for the contact problem for a system of punches in the plane case. It is easy to show that there is one of four types of conditions on each punch: the load force and moment are given, the tilt angle of the punch and the load force are given, the punch settlement and the load moment are given, or the settlement and tilt angle of the punch are given. Thus, there are 15 versions of mathematical statements for the plane contact problem for a system of punches.

In what follows, we construct the solution of the system of two-dimensional equations with the system of auxiliary conditions (5), which contains integral operators with constant as well as variable limits of integration and two different rapidly changing functions ($m(x)$ and $g(x) \equiv g^i(x)$, $i = 1,2,\dots,n$).

In what follows we will construct the solution of the case when all forces and moments are known.

II. The solution for known force and moment

Assuming that

$$\begin{aligned}
&\mathbf{q}(x,t) = q^i(x,t)\mathbf{i}^i, \quad \boldsymbol{\delta}(t) = \delta^i(t)\mathbf{i}^i, \quad \boldsymbol{\alpha}(t) = \alpha^i(t)\mathbf{i}^i,\\
&\mathbf{g}(x) = g^i(x)\mathbf{i}^i, \quad \mathbf{P}(t) = P^i(t)\mathbf{i}^i, \quad \mathbf{M}(t) = M^i(t)\mathbf{i}^i,\\
&\mathbf{k}(x,\xi) = k^{ij}(x,\xi)\mathbf{i}^i\mathbf{i}^j, \quad \mathbf{G}\mathbf{f}(x) = \int_{-1}^{1}\mathbf{k}(x,\xi)\cdot\mathbf{f}(\xi)\,d\xi,
\end{aligned} \tag{6}$$

we can represent system with additional conditions (5) as

$$\begin{aligned}
&c(t)m(x)\mathbf{q}(x,t) + (\mathbf{I}-\mathbf{V})\mathbf{G}\mathbf{q}(x,t) = \boldsymbol{\delta}(t) + \boldsymbol{\alpha}(t)x - \mathbf{g}(x),\\
&\int_{-1}^{1}\mathbf{q}(\xi,t)\,d\xi = \mathbf{P}(t), \quad \int_{-1}^{1}\xi\mathbf{q}(\xi,t)\,d\xi = \mathbf{M}(t),\\
&-1 \leqslant x \leqslant 1, \quad t \geqslant 1.
\end{aligned} \tag{7}$$

Hereinafter, it will be the summation over repeated upper indices i and j from 1 to n if the left side of the formula is independent of the index.

Now we introduce the notation

$$\begin{aligned}
&\mathbf{Q}(x,t) = \sqrt{m(x)}\left[\mathbf{q}(x,t) + \frac{\mathbf{g}(x)}{c(t)m(x)}\right],\\
&\mathbf{K}(x,\xi) = \frac{\mathbf{k}(x,\xi)}{\sqrt{m(x)m(\xi)}}, \quad \mathbf{F}\mathbf{f}(x) = \int_{-1}^{1}\mathbf{K}(x,\xi)\cdot\mathbf{f}(\xi)\,d\xi.
\end{aligned}$$

Then operator equation and auxiliary conditions (7) can be reduced to the following integral equation with the kernel $\mathbf{K}(x,\xi)$:

$$\begin{aligned}
&c(t)\mathbf{Q}(x,t) + (\mathbf{I}-\mathbf{V})\mathbf{F}\mathbf{Q}(x,t) = \frac{\boldsymbol{\delta}(t) + \boldsymbol{\alpha}(t) + \tilde{c}(t)\tilde{\mathbf{g}}(x)}{\sqrt{m(x)}},\\
&\int_{-1}^{1}\frac{\mathbf{Q}(\xi,t)}{\sqrt{m(\xi)}}\,d\xi = \tilde{\mathbf{P}}(t), \quad \int_{-1}^{1}\frac{\xi\mathbf{q}(\xi,t)}{\sqrt{m(\xi)}}\,d\xi = \tilde{\mathbf{M}}(t),\\
&-1 \leqslant x \leqslant 1, \quad t \geqslant 1,
\end{aligned} \tag{8}$$

where

$$\tilde{\mathbf{g}}(x)=\int_{-1}^{1}\frac{\mathbf{k}(x,\xi)\cdot\mathbf{g}(\xi)}{m(\xi)}d\xi,\quad \tilde{c}(t)=(\mathbf{I}-\mathbf{V})\frac{1}{c(t)},$$
$$\tilde{\mathbf{P}}(t)=\mathbf{P}(t)+\frac{1}{c(t)}\int_{-1}^{1}\frac{\mathbf{g}(\xi)}{m(\xi)}d\xi, \tag{9}$$
$$\tilde{\mathbf{M}}(t)=\mathbf{M}(t)+\frac{1}{c(t)}\int_{-1}^{1}\frac{\xi\mathbf{g}(\xi)}{m(\xi)}d\xi.$$

We seek the solution of Eq. (8) in the class of vector functions continuous in time t in the Hilbert space $L_2([-1,1],V)$. To this end, we at first construct an orthonormal system of vector functions in $L_2([-1,1],V)$ which contains the factor $1/\sqrt{m(x)}$ and remaining basis functions can be written as the products of vector functions depending on x and weight function $1/\sqrt{m(x)}$. The system of vector functions which satisfies the above conditions can be obtained by the following formulas:

$$\mathbf{p}_k^i(x)=\frac{\mathbf{p}_k^{i*}(x)}{\sqrt{m(x)}},\quad \mathbf{p}_k^{i*}(x)=p_k^*(x)\mathbf{i}^i,\quad d_{-1}=1,$$
$$J_k=\int_{-1}^{1}\frac{\xi^k}{m(\xi)}d\xi,\quad d_k=\begin{vmatrix}J_0 & \cdots & J_k\\ \vdots & \ddots & \vdots\\ J_k & \cdots & J_{2k}\end{vmatrix}. \tag{10}$$
$$p_k^*(x)=\frac{1}{\sqrt{d_{k-1}d_k}}\begin{vmatrix}J_0 & J_1 & \cdots & J_k\\ J_1 & J_2 & \cdots & J_{k+1}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & x & \cdots & x^k\end{vmatrix}.$$

Thus, $\{\mathbf{p}_k^i(x)\}$ ($i=1,2,\ldots,n$, $k=0,1,2,\ldots$) is a basis in $L_2([-1,1],V)$.

The Hilbert space $L_2([-1,1],V)$ can be presented as the direct sum of orthogonal subspaces $L_2([-1,1],V)=L_2^{(1)}([-1,1],V)\oplus L_2^{(2)}([-1,1],V)$, where $L_2^{(1)}([-1,1],V)$ is the Euclidean space with the basis $\{\mathbf{p}_0^i(x),\mathbf{p}_1^i(x)\}_{i=1,2,\ldots,n}$ and $L_2^{(2)}([-1,1],V)$ is the Hilbert space with the basis $\{\mathbf{p}_2^i(x),\mathbf{p}_3^i(x),\ldots\}_{i=1,2,\ldots,n}$. The integrand and the right-hand side of integral equation (6) can also be presented in the form of the algebraic sum of functions continuous in time t and ranging in $L_2^{(1)}([-1,1],V)$ and $L_2^{(2)}([-1,1],V)$, respectively, i.e.,

$$\begin{aligned}
&\mathbf{Q}(x,t)=\mathbf{Q}_1(x,t)+\mathbf{Q}_2(x,t),\\
&\frac{\boldsymbol{\delta}(t)+\boldsymbol{\alpha}(t)+\tilde{c}(t)\tilde{\mathbf{g}}(x)}{\sqrt{m(x)}}=\boldsymbol{\Delta}_1(x,t)+\boldsymbol{\Delta}_2(x,t),\\
&\mathbf{Q}_1(x,t)=z_0^i(t)\mathbf{p}_0^i(x)+z_1^i(t)\mathbf{p}_1^i(x),\\
&\boldsymbol{\Delta}_1(x,t)=\frac{\boldsymbol{\delta}(t)+\boldsymbol{\alpha}(t)x+\tilde{c}(t)\tilde{\mathbf{g}}_1(x)}{\sqrt{m(x)}}\\
&\quad=\left[\sqrt{J_0}\delta^i(t)+\frac{J_1}{\sqrt{J_0}}\alpha^i(t)+g_0^i\tilde{c}(t)\right]\mathbf{p}_0^i(x)\\
&\quad+\left[\sqrt{\frac{J_0J_2-J_1^2}{J_0}}\alpha^i(t)+g_1^i\tilde{c}(t)\right]\mathbf{p}_1^i(x),\\
&\boldsymbol{\Delta}_2(x,t)=\frac{\tilde{c}(t)\tilde{\mathbf{g}}_2(x)}{\sqrt{m(x)}}.
\end{aligned} \tag{11}$$

Here $\mathbf{Q}_1(x,t),\mathbf{f}_1(x,t)\in L_2^{(1)}([-1,1],V)$, $\mathbf{Q}_2(x,t),\mathbf{f}_2(x,t)\in L_2^{(2)}([-1,1],V)$, $\tilde{\mathbf{g}}(x)=\tilde{\mathbf{g}}_1(x)+\tilde{\mathbf{g}}_2(x)$, $\tilde{\mathbf{g}}_1(x)/\sqrt{m(x)}=g_0^i\mathbf{p}_0^i(x)+g_1^i\mathbf{p}_1^i(x)\in L_2^{(1)}([-1,1],V)$, $\tilde{\mathbf{g}}_2(x)/\sqrt{m(x)}\in L_2^{(2)}([-1,1],V)$. Coefficients g_0^i and g_1^i can be determined by formulas

$$g_k^i=\sum_{l=0}^{\infty}K_{kl}^{ij}\int_{-1}^{1}\frac{\mathbf{p}_l^j(\xi)\cdot\mathbf{g}(\xi)}{\sqrt{m(\xi)}}d\xi,\quad k=0,1,$$

where K_{mn}^{ij} are expansion coefficients of the kernel $\mathbf{k}(x,\xi)$:

$$\begin{aligned}
&\mathbf{K}(x,\xi)=\sum_{m,l=0}^{\infty}K_{ml}^{ij}\mathbf{p}_m^i(x)\mathbf{p}_l^j(\xi),\\
&K_{ml}^{ij}=\int_{-1}^{1}\int_{-1}^{1}\mathbf{p}_m^i(x)\cdot\mathbf{K}(x,\xi)\cdot\mathbf{p}_l^j(\xi)\,dx\,d\xi,\\
&m,l=0,1,\ldots,\quad i,j=1,2,\ldots,n
\end{aligned} \tag{12}$$

Note that the formula for $\mathbf{Q}(x,t)$ contains known term $\mathbf{Q}_1(x,t)$ which is determined by the first auxiliary condition (6)

$$z_0^i(t)=\frac{\tilde{P}^i(t)}{\sqrt{J_0}},\quad z_1^i(t)=\frac{J_0\tilde{P}^i(t)+J_1\tilde{M}^i(t)}{\sqrt{J_0(J_0J_2-J_1^2)}}, \tag{13}$$

and the term $\mathbf{Q}_2(x,t)$ must be found. Conversely, for the right-hand side, one should find $\boldsymbol{\Delta}_1(x,t)$, while $\boldsymbol{\Delta}_2(x,t)$ is known and determined be the function $\tilde{\mathbf{g}}_2(x)$. These peculiarities permit one to class the resulting problem as a specific case of the generalized projection problem stated in [18]–[20].

We can introduce the orthogonal projection operator mapping the space $L_2([-1,1],V)$ onto subspace $L_2^{(1)}([-1,1],V)$

$$\mathbf{P}_1\mathbf{f}(x)=\int_{-1}^{1}\mathbf{f}(\xi)\cdot[\mathbf{p}_0^i(\xi)\mathbf{p}_0^i(x)+\mathbf{p}_1^i(\xi)\mathbf{p}_1^i(x)]\,d\xi.$$

Obviously, the orthoprojector $\mathbf{P}_2=\mathbf{I}-\mathbf{P}_1$ maps the space $L_2([-1,1],V)$ onto $L_2^{(2)}([-1,1],V)$. It is clear, that $\mathbf{P}_k\boldsymbol{\Delta}(x,t)=\boldsymbol{\Delta}_k(x,t)$, $\mathbf{P}_k\mathbf{Q}(x,t)=\mathbf{Q}_k(x,t)$ ($k=1,2$).

Using [18], we apply the orthogonal projection operator $\mathbf{P}_2$ to equation (8). As a result, we obtain the equation for determining $\mathbf{Q}_2(x,t)$ with a known right-hand side

$$\begin{aligned}
&c(t)\mathbf{Q}_2(x,t)+(\mathbf{I}-\mathbf{V})\mathbf{P}_2\mathbf{F}\mathbf{Q}_2(x,t)\\
&\quad=-(\mathbf{I}-\mathbf{V})\mathbf{P}_2\mathbf{F}\mathbf{Q}_1(x,t)+\frac{\tilde{c}(t)\tilde{\mathbf{g}}_2(x)}{\sqrt{m(x)}}.
\end{aligned} \tag{14}$$

It is necessary to construct its solution in the form of an expansion in the eigenfunctions of the operator $\mathbf{P}_2\mathbf{F}$ which is a compact, strong positive, and self-adjoint operator from $L_2^{(2)}([-1,1],V)$ into $L_2^{(2)}([-1,1],V)$. The system of eigenfunctions of such an operator is a basis in the space $L_2^{(2)}([-1,1],V)$. The spectral problem for the operator $\mathbf{P}_2\mathbf{F}$ can be written in the form

$$\begin{aligned}
&\mathbf{P}_2\mathbf{F}\boldsymbol{\varphi}_k(x)=\gamma_k\boldsymbol{\varphi}_k(x),\\
&\boldsymbol{\varphi}_k(x)=\sum_{m=2}^{\infty}\psi_{km}^i\mathbf{p}_m^i(x),\quad k=2,3,\ldots.
\end{aligned} \tag{15}$$

This problem leads to the search for solutions of spectral problem about coefficients γ_k and ψ_{km}^i ($k,m=2,3,\ldots$, $i=1,2,\ldots,n$):

$$\sum_{l=2}^{\infty}K_{ml}^{ij}\psi_{kl}^j=\gamma_k\psi_{km}^i,\quad k,m=2,3,\ldots,$$

where coefficients K_{ml}^{ij} determined by (12).

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We expand the functions $\mathbf{Q}_2(x,t)$ and $\tilde{\mathbf{g}}_2(x)/\sqrt{m(x)}$ with respect to the new basis functions $\boldsymbol{\varphi}_k(x)$ $(k = 2, 3, \ldots)$ in $L_2^{(2)}([-1,1], V)$, i.e.,

$$\mathbf{Q}_2(x,t) = \sum_{k=2}^{\infty} z_k(t)\boldsymbol{\varphi}_k(x), \quad \frac{\tilde{\mathbf{g}}_2(x)}{\sqrt{m(x)}} = \sum_{k=2}^{\infty} g_k \boldsymbol{\varphi}_k(x),$$

where coefficients g_k defined by

$$g_k = \sum_{m=2}^{\infty} \psi_{km}^i \sum_{l=0}^{\infty} K_{ml}^{ij} \int_{-1}^{1} \frac{\mathbf{p}_l^j(\xi)\cdot \mathbf{g}(\xi)}{\sqrt{m(\xi)}} d\xi, \quad k = 2, 3, \ldots$$

Substituting this equation into (14) and taking into account that the unknown expansion functions $z_k(t)$ $(k = 2, 3, \ldots)$ can be determined by the formula

$$\begin{aligned} z_k(t) &= (\mathbf{I} + \mathbf{W}_k)\big\{(\mathbf{I} - \mathbf{V})[g_k c^{-1}(t) \\ &\quad - K_{0k}^i z_0^i(t) - K_{1k}^i z_1^i(t)]\big\}/[c(t) + \gamma_k], \\ K_{0k}^i &= \sum_{m=2}^{\infty} K_{0m}^{ij}\psi_{km}^j, \quad K_{1k}^i = \sum_{m=2}^{\infty} K_{1m}^{ij}\psi_{km}^j, \qquad (16) \\ \mathbf{W}_k f(t) &= \int_1^t R_k^*(t,\tau) f(\tau)\, d\tau, \end{aligned}$$

where $R_k^*(t,\tau)$ $(k = 1, 2, \ldots)$ is the resolvent of the kernel $K_k^*(t,\tau) = \gamma_k K_2(t,\tau)/[c(t) + \gamma_k]$.

Note that the final solution has the following structure

$$\begin{aligned} q^i(x,t) &= \frac{1}{m(x)}\bigg[z_0^i(t)p_0^*(x) + z_1^i(t)p_1^*(x) \\ &\quad + \sum_{k=2}^{\infty} z_k(t)\sum_{m=2}^{\infty} \psi_{km}^i p_m^*(x)\bigg] - \frac{g(x)}{c(t)m(x)}, \end{aligned}$$

i.e., one can explicitly write out the weight functions $m(x)$ and $g(x)$ in the solution. Note that the coating nonuniformity is related to $m(x)$ and punches base forms function is related to $g(x)$ in the relations of change of variables (4), they can described by complicated and rapidly changing functions. The formulas obtained permit obtaining efficient solutions. Such a result can hardly be done by other known methods.

In order to find the unknown punch settlement and tilt angle we act integral equation (8) by operator $\mathbf{P}_1$

$$\begin{aligned} \alpha^i(t) &= \sqrt{\frac{J_0}{J_0 J_2 - J_1^2}}\bigg\{c(t)z_1^i(t) + (\mathbf{I} - \mathbf{V})\bigg[-\frac{g_1^i}{c(t)} \\ &\quad + K_{10}^{ij} z_0^j(t) + K_{11}^{ij} z_1^j(t) + \sum_{k=2}^{\infty} K_{1k}^i z_k(t)\bigg]\bigg\}, \end{aligned}$$

$$\begin{aligned} \delta^i(t) &= \frac{1}{\sqrt{J_0}}\bigg\{c(t)z_0^i(t) + (\mathbf{I} - \mathbf{V})\bigg[-\frac{g_0^i}{c(t)} \\ &\quad + K_{00}^{ij} z_0^j(t) + K_{01}^{ij} z_0^j(t) + \sum_{k=2}^{\infty} K_{0k}^i z_k(t)\bigg]\bigg\} - \alpha^i(t)\frac{J_1}{J_0}. \end{aligned}$$

III. Conclusions

• We stated and solved the plane problem of contact interaction between a viscoelastic aging foundation with a nonuniform coating and a regular system of rigid punches in the case when the punches base shape and the coating nonuniformity are described by rapidly changing functions. We stated and solved the plane problem of contact interaction between a viscoelastic aging foundation with a nonuniform coating and a regular system of rigid punches in the case when the punches base shape and the coating nonuniformity are described by rapidly changing functions.

• The solution of the problem is obtained analytically in the form of series. The expression for the contact stresses contains functions which describe the nonuniformity of the coating and the punches base shape explicitly. This allows one to perform computations for actual complex mechanical and geometrical characterictics of the multibody contact problem using small number of expansion terms.

• Other existing methods for the solution of this problem diverge with the increase of the time parameter or give an error for contact characteristics up to 100%.

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