Solution of Fractional Gas Dynamic Equation by Using Homotopy Perturbation with Natural Transform Method

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***Abstract*—In this paper, we describe the solution of fractional gas dynamic equation by using the perturbation method with natural transform method.**

***Index Terms*—The natural transform, Perturbation method, exact solution, fractional gas dynamic equation**

I. Introduction

THE preparation deal with dynamical systems modeled by differential, integro-differential and partial differential equations are important to solving physical problems. Closed-form analytical results are considered ideal and series solutions are considered approximation [1]. Most the solutions of the boundary value problems do not have closed-form, this has led to the developement of analytical techniques. The classical techniques for solving the model be given as Adomain decomposition method [1], the Homotopy Perturbation method (HPM) with Laplace transform [2]-[5], the Sumudu transform [2], and the Natural transform [6]-[10]. In this paper, we use the Homotopy Perturbation with the natural transform (NHPM) for solving the fractional gas dynamic equation.

II. Basic Idea

A. Definition of the Caputo derivative of fractional order α

The Caputo derivative of fractional order α of the function f(t) is defined as (see in [9]):

$$ {}^{C}D^{\alpha}[f(t)] = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t \dfrac{f^{(m)}(\tau)d\tau}{(t-\tau)^{\alpha-m+1}}, & m-1<\alpha<m; \\ \dfrac{d^m f(t)}{dt^m}, & \alpha = m. \end{cases} \quad (1)$$

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B. The Natural Transform and Its Inverse

The set of functions:

$$A = \left\{ f(t) : \exists M, \tau_1, \tau_2 > 0, \left| f(t) \right| < Me^{|t|/\tau_j}, if\ t \in (-1)^j \times [0,\infty) \right\}.$$

where M is a constant of finite number, τ_1 and τ_2 may be finite or infinite.

The natural transform V(s, u) of the function f(t) for all $t \geq 0$, is defined as:

$$N^+[f(t)] = V(s,u) = \int_0^\infty f(ut)e^{st}dt, u>0, s>0, \quad (2)$$

where $N^+[f(t)]$ is the Natural transformation of the function of f(t) in the set A, u and s are the Natural transform variables.

The inverse of the Natural transform is defined by

$$N^{-1}[V(s,u)] = f(t) = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} e^{st/u}V(s,u)ds, \quad (3)$$

where γ is real constant and the integral is taken along $s=\gamma$ in complex plane $s = x+iy$. The real number γ is choosen so that $s=\gamma$ lies on right of all (finite or countable infinite) singularities (see in [8]).

C. The Natural Transform of Fractional Derivative

If $N^+[f(t)]$ is the Natural transform of the function f(t), then the Natural transform of fractional derivative of order α of the function f(t), denoted by $N^+[f^{\alpha}(t)]$, is defined as:

$$N^+[f^{\alpha}(t)] = \frac{s^{\alpha}}{u^{\alpha}}V(s,u) - \sum_{k=0}^{n-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}f^{(k)}(0), \quad (4)$$

where u and s are the natural transform variables (see in [5]).

D. Method of Homotopy Perturbation with the Natural transformation (HPM-Natural Trans.)

We consider a general fractional nonlinear non-homogenues partial equation:

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$$D_t^{\alpha}V(x,t)+KV(x,t)+NV(x,t)=g(x,t) \tag{5}$$

with the initial condition as

$$V(x,0)=f(x), \tag{6}$$

where $D_t^{\alpha}V(x,t)$ is the Caputo fractional derivative of $V(x,t)$, K is the linear differential operator, N is the nonlinear differential operator, $g(x,t)$ is the nonzero given term, and $f(x)$ is the function of x. Applying the Natural transform on both sides of the equation (5), we have

$$\begin{aligned} N^{+}\left[D_t^{\alpha}V(x,t)\right]+N^{+}\left[KV(x,t)\right] \\ +N^{+}\left[NV(x,t)\right]=N^{+}\left[g(x,t)\right]. \end{aligned} \tag{7}$$

Now we calculate the equation (7) by using the definition of the Natural transform with some its properties and and the given initial condition (6). Then we apply the inverse of the Natural transform. The next step, we consider terms of the linear and nonlinear form. We apply the Homotopy perturbation (HPM) as:

$$V(x,t)=\sum_{n=0}^{\infty}q^nV_n(x,t), \tag{8}$$

and the nonlinear term

$$N[V(x,t)]=\sum_{n=0}^{\infty}q^nH_n(V), \tag{9}$$

where $H_n(V)$ is some He's polynomials [37] that given by

$$H_n(V_0,V_1,...,V_n)=\frac{1}{n!}\frac{\partial^n}{\partial q^n}\left[N\left(\sum_{j=0}^{\infty}q^jV_j\right)\right]_{p=0}, \quad n=0,1,2,... \tag{10}$$

Using the coefficient comparing, we have

$$q^0: V_0(x,t)=G(x,t);$$

$$q^1: V_1(x,t)=-N^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left[K(V(x,t)+H_0(V)\right]\right];$$

$$q^2: V_2(x,t)=-N^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left[K(V(x,t)+H_1(V)\right]\right]$$

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Then we have the analytical solution

$$V(x,t)=\lim_{N\to\infty}\sum_{n=0}^{N}V_n(x,t).$$

III. Solution of Gas Dynamic Equation

We consider Fractional gas dynamic equation

$$D_t^{\alpha}V+\frac{1}{2}(V^2)_x-V(1-V)=0, \tag{11}$$

$0<\alpha\le 1$, with initial condition

$$V(x,0)=e^{-x} \tag{12}$$

Applying The Natural transform method on both sides of (11), and using the initial condition (12), we obtain

$$N^{+}\left[D_t^{\alpha}V\right]+\frac{1}{2}N^{+}\left[(V^2)_x\right]-N^{+}\left[V(1-V)\right]=N^{+}\left[0\right].$$

$$\frac{s^{\alpha}}{u^{\alpha}}V(x,s)-\frac{s^{\alpha-1}}{u^{\alpha}}V(x,0)=-\frac{1}{2}N^{+}\left[V_x^{\,2}\right]+N^{+}\left[V\right]-N^{+}\left[V^2\right].$$

$$\frac{s^{\alpha}}{u^{\alpha}}V(x,s)=\frac{s^{\alpha-1}}{u^{\alpha}}V(x,0)-\frac{1}{2}N^{+}\left[V_x^{\,2}\right]+N^{+}\left[V\right]-N^{+}\left[V^2\right].$$

By using the initial condition(12), we have

$$\begin{aligned} V(x,s)=\frac{1}{s}e^{-x}-\frac{1}{2}\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left[\left(V^2\right)_x\right]+\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left[V\right] \\ -\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left[V^2\right]. \end{aligned} \tag{13}$$

Taking the inverse of the Natural transform method on both sides of the equation (13), we have

$$\begin{aligned} N^{-1}\left[V(x,s)\right]=N^{-1}\left[\frac{1}{s}e^{-x}\right]-\frac{1}{2}N^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left[\left(V^2\right)_x\right]\right] \\ +N^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left[V\right]\right]-N^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left[V^2\right]\right]. \end{aligned} \tag{14}$$

Now we have

$$V(x,t)=e^{-x}-N^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left[\frac{1}{2}\left(V^2\right)_x-V-V^2\right]\right]. \tag{15}$$

Applying the HPM, we let

$$\begin{aligned} V(x,t)&=\sum_{n=0}^{\infty}q^nV_n(x,t) \\ &=e^{-x}-\left(N^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left[\frac{1}{2}\sum_{n=0}^{\infty}q^nH_n(V)-\sum_{n=0}^{\infty}q^nV_n(x,t)\right.\right.\right. \\ &\quad \left.\left.\left.+\sum_{n=0}^{\infty}q^nH_n^{*}\left(V^2\right)\right]\right]\right). \end{aligned}$$

By coefficient comparing of the power of q, we have

$$q^0: V_0(x,t)=e^{-x};$$

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$$q^1 : V_1(x,t) = -N^{-1}\left[\frac{u^\alpha}{s^\alpha} N^+\left[\frac{1}{2}H_0(V) - V_0 + H_0^*(V)\right]\right]$$

$$= -N^{-1}\left[\frac{u^\alpha}{s^\alpha} N^+\left[\frac{1}{2}(e^{-2x})_x - e^{-x} + e^{-2x}\right]\right]$$

$$= e^{-x}\frac{t^2}{\Gamma(\alpha+1)};$$

$$q^2 : V_2(x,t) = -N^{-1}\left[\frac{u^\alpha}{s^\alpha} N^+\left[\frac{1}{2}H_1(V) - V_1 + H_0^*(V)\right]\right]$$

$$= -N^{-1}\left[\frac{u^\alpha}{s^\alpha} N^+\left[\frac{1}{2}(2V_0V_1)_x - V_1 + 2V_0V_1\right]\right]$$

$$= e^{-x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)};$$

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Then the solution is

$$V(x,t) = e^{-x}\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots\right).$$

The approximate solution for $\alpha = 1$ is

$$V(x,t) = e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots\right).$$

We can write the exact solution in the form

$$V(x,t) = e^{t-x}.$$

TABLE I
COMPARISON BETWEEN THE EXACT SOLUTION AND HPM-NATURAL TRANS. WITH $\alpha = 1$

x	t	V(Exact)	V(HPM-Natural.Trans.)
0	0.1	1.10517091807565	1.105170918055560
0.1	0.1	1	0.999999999981820
0.2	0.1	0.904837418035960	0.904837418019509
0.3	0.1	0.818730753077982	0.818730753063097
0.4	0.1	0.740818220681718	0.740818220668250
0.5	0.1	0.670320046035639	0.670320046023453
0.6	0.1	0.606530659712633	0.606530659701607
0.7	0.1	0.548811636094026	0.548811636084049
0.8	0.1	0.496585303791409	0.496585303782381
0.9	0.1	0.449328964117222	0.449328964109053
1.0	0.1	0.406569659740599	0.406569659733208

It is observed that the values of the approximate solutions at $\alpha = 1$ which obtained by the Homotopy Perturbation with the Natural transform is close to the value of the exact solution at the sixth-term.

IV. CONCLUSION

Our solution of gas dynamic by the Homotopy perturbation with the Natural transform method is a general solution. If we replace u=1, it is the solution by using the Laplace transform. And if we replace s=1, it is the solution by using the Sumudu transform method.

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