Proceedings of the World Congress on Engineering and Computer Science 2007
WCECS 2007, October 24-26, 2007, San Francisco, USA

Dependency of Fractional Fourier Span on Amplitude and Phase function of a Signal

Sukrit Shankar, Chetana Shanta Patsa, Jaydev Sharma

Abstract --- **Fractional Fourier Transform is a generalization of the classical Fourier Transform, which has found its most useful applications for the transient analysis of the signals. This paper studies the span of the Fractional Fourier Transform in relation with the amplitude and phase functions of the signal and provides a mathematical derivation for the generalized case. The derived expression is shown to be useful for calculating the optimal transform order to achieve minimal possible span for various signals. Based on the derived expression, we further establish the fact that the Fractional Fourier Transform (other than the transform order of $\pi/2$) is most effective for the analysis of chirp signals. We provide a comparison of our method for finding optimal transform order with previously given methods and show that it provides better results.**

Index Terms --- **Fractional Fourier Transform, Generalized Bandwidth Concept, Phase and Amplitude Function, Optimal Transform Order.**

I. INTRODUCTION TO FRACTIONAL FOURIER TRANSFORM

Fractional Fourier transform is a generalization of classical Fourier Transform. The idea of Fractional Fourier Transform was first introduced by Victor Namias in 1980[1]. After the establishment of the idea that transforms could be fractionalized by Namias in 1980, McBridge and Keer in 1987 refined the definition and mathematically described FrFT [2].

The traditional Fourier transform decomposes the signal in terms of sinusoids, which are perfectly localized in frequency, but are not at all localized in time. FrFT is a more generalized transform, in the sense that Fourier Transform forms a special case of FrFT. This transform expresses the signal in terms of an orthonormal basis formed by linear chirps. Linear chirps are the signals, whose instantaneous frequency varies linearly with time.

The Kernel for continuous Fractional Fourier Transform is given by

$$K_\alpha(t,u) = \sqrt{\frac{1 - j\cot\alpha}{2\pi}}\, e^{j\frac{t^2+u^2}{2}\cot\alpha - jut\,\mathrm{cosec}\,\alpha}$$

Using this kernel of FrFT, the FRFT of signal *x(t)* with transform order (α) is computed as

$$X_\alpha(u) = \int_{-\infty}^{\infty} x(t) K_\alpha(t,u)\,dt$$

And x(t) can be recovered from the following equation,

$$x(t) = \int_{-\infty}^{\infty} X_\alpha(u) K_{-\alpha}(u,t)\,du$$

II. MATHEMATICAL ANALYSIS OF FRACTIONAL FOURIER SPAN

The normalized energies of a given signal can be thought of as signal's energy density function. To quantitatively characterize the signal's behaviour, we use the moment concepts from the probability theory. Here we use the first moment to compute mean for the signal in "*t*" domain and "*u*" domain.

For signal *x(t),*

$$\langle t\rangle = \frac{1}{E}\int_{-\infty}^{\infty} t\left|x(t)\right|^2 dt$$

FrFT of *x(t)* is $X_\alpha(u)$

$$\langle u\rangle = \frac{1}{E}\int_{-\infty}^{\infty} u\left|X_\alpha(u)\right|^2 du$$

To expand the above equation, we take,

$$H_\alpha(u) = uX_\alpha(u)$$

So that,

Manuscript received July 22, 2007.

Sukrit Shankar is with the DSP Firmware for Broadband Media Processing Unit at Conexant Systems Private Limited, Hyderabad, INDIA.
Chetana Shanta Patsa is with IBM India Private Limited, INDIA.
Prof. Jaydev Sharma is with the Department of Electrical Engineering, IIT Roorkee, INDIA.

$$< u >= \frac{1}{E}\int_{-\infty}^{\infty} H_\alpha(u) X_\alpha^*(u) du$$

$$= \frac{1}{E}\int_{-\infty}^{\infty} h(t) x^*(t) dt$$

(By Parseval's Relation)

So,

We prove the differentiation property of FrFT, for expressing *h(t)* in terms of *x(t)*.

x(t) =

$$\int_{-\infty}^{\infty} X_\alpha(u) K_{-\alpha}(t,u) du$$

$$\frac{d}{dt} x(t) = \int_{-\infty}^{\infty} X_\alpha(u) . \frac{d}{dt}\{K_{-\alpha}(t,u)\} du$$

=

$$\int_{-\infty}^{\infty} \{-jt \cot\alpha + ju \cos ec\alpha\} X_\alpha(u) K_{-\alpha}(t,u) du$$

or,

$$F_\alpha(\frac{d}{dt} x(t)) = \{-jt\cot\alpha + ju\cos ec\alpha\} X_\alpha(u)$$

So $F_\alpha^{-1}[uX_\alpha(u)] = \sin\alpha[\frac{1}{j}\frac{d}{dt}x(t) + F_\alpha^{-1}\{t\cot\alpha X_\alpha(u)\}]$

We now calculate $F_\alpha^{-1}[t\cot\alpha X_\alpha(u)]$

$$F_\alpha^{-1}[t\cot\alpha X_\alpha(u)]$$

$$= \int_{-\infty}^{\infty} t\cot\alpha . X_\alpha(u) K_{-\alpha}(t,u) du$$

$$= t\cot\alpha . x(t)$$

$F_\alpha^{-1}[uX_\alpha(u)] = h(t)$

Thus,

<u>=

$$\frac{1}{E}\int_{-\infty}^{\infty} \sin\alpha[\frac{1}{j}\frac{d}{dt}x(t) + t\cot\alpha x(t)] . x^*(t) dt$$

Putting $x(t) = A(t)\, e^{j\phi(t)}$,

Where *A(t)* and $\phi(t)$ are real.

$x^*(t) = A(t)\, e^{-j\phi(t)}$

and

$\frac{d}{dt}x(t) = A(t)\, j\phi'(t) e^{j\phi(t)} + A'(t)\, e^{j\phi(t)}$

Solving further and with the fact that <*u*> is always real, we get

$$<u> = \frac{\sin\alpha}{E}\int_{-\infty}^{\infty} [\phi'(t) + t\cot\alpha] |x(t)|^2 dt$$

Now, the expressions for the span of the signal in time and Fractional Fourier domain about the respective mean values can be given as follows:

$$\Delta_t^2 = \frac{1}{E}\int (t - < t >)^2 |x(t)|^2 dt$$

$$= \frac{1}{E}\int (t)^2 |x(t)|^2 dt - < t >^2$$

$$\Delta_u^2 = \frac{1}{E}\int (u - < u >)^2 |X(u)|^2 du$$

$$= \frac{1}{E}\int (u)^2 |X(u)|^2 du - < u >^2$$

Now based on Parseval's theorem

$$\Delta_u^2 = \frac{1}{E}\int (u - < u >)^2 |X(u)|^2 du$$

$$= \frac{1}{E}\int H(u) H^*(u) du$$

$$= \frac{1}{E}\int h(t) h^*(t) dt$$

Where,

H(u) = (u-<u>)X(u), and $h(t) = F_\alpha^{-1}[H(u)]$

So,

$h(t) = F_\alpha^{-1}[(u - < u >) X(u)]$

$$= \int_{-\infty}^{\infty} (u - < u >) X(u) . K_{-\alpha}(t,u) du$$

$$= \int_{-\infty}^{\infty} uX(u) . K_{-\alpha}(t,u) du - <u> x(t)$$

$$= F_\alpha^{-1}[u.X(u)] - <u> x(t)$$

We have already calculated that

$$F_\alpha^{-1}[u.X(u)]$$

$$= \sin\alpha[-j\frac{d}{dt}x(t) + t\cot\alpha . x(t)]$$

Therefore,

h(t)=

$$\sin\alpha[-j\frac{d}{dt}x(t) + t\cot\alpha . x(t)] - < u > . x(t)$$

=

$$\sin\alpha[-j\frac{d}{dt}A(t)e^{j\phi(t)} + t\cot\alpha . A(t)e^{j\phi(t)}] - < u > A(t)e^{j\phi(t)} =$$

$$e^{j\phi(t)}[A(t)\{\sin\alpha\phi'(t) + t\cos\alpha . - < u >\} - j\sin\alpha A'(t)]$$

Now,

$h(t)h^*(t) =$

$$e^{j\phi(t)}[A(t)\{\sin\alpha . \phi'(t) + t\cos\alpha - < u >\} - j\sin\alpha A'(t)] \; * \; e^{-j\phi(t)}[A(t)\{\sin\alpha . \phi'(t) + t\cos\alpha - < u >\} + j\sin\alpha A'(t)]$$

(Using the fact that $<u>^* = <u>$)

Expanding the above expression, we finally obtain,

$$\Delta_u^2 = \frac{1}{E}\int h(t)h^*(t)dt =$$
$$\frac{1}{E}\int \{[A(t)\{\sin\alpha.\phi'(t) + t\cos\alpha - <u>\}]^2 + [\sin\alpha A'(t)]^2\}dt$$

Hence,

$$\Delta_u^2 = \frac{1}{E}\int \{\sin\alpha\phi'(t) + t\cos\alpha - <u>\}^2 A^2(t)dt + \frac{1}{E}\int \sin^2\alpha (A'(t))^2 dt$$

III. FINDING OPTIMAL ORDER FOR MINIMUM SPAN

There exist various types of signals for which a compact transform is many a time desired. The classical Fourier Transform may not find the optimal representation for all types of signals. With Fractional Fourier Transform also, the representation varies with the transform order and hence, one can obtain the most compact representation with some optimal value of the order. The optimal value of the order often needs to be found out. As already known that the Fractional Fourier transform represents a signal in the form of linear chirps, the FrFT of a linear chirp at a suitable order is the simplest computation and is essentially an impulse. The situation is similar to the fact that the Fourier transform of a sinusoid is an impulse, i.e. the most compact representation through the transform is of the signal in whose terms the transform represents the signal.

A simple formula based on the geometric relationship of the time frequency representation of the chirp is given by [3] which provides the most accurate results for simple chirps. However, with the derived expression, we shall show that the calculation of an optimal order is possible for any type of signal. Also, for variants of linear chirp signals, the order calculated here provides better compact spans in the Fractional Fourier domain as compared with that calculated with the method in [3].

We take the following three signals in consideration and find their optimal order for the minimum possible span.

$$x_1(t) = \sin(\pi/3t)\,, x_2(t) = (1/\pi)^{1/4}.e^{-1/2 t^2}$$
$$x_3(t) = x_2(t).e^{j20t}$$

The first signal is the simple sinusoid, the second one is the normalized Gaussian signal while the third one is the frequency modulated normalized Gaussian signal. For all the three signals, we find the optimal orders for the minimal span through a MATLAB program based on the mathematical expression relating the Fractional Fourier span with the amplitude and phase functions of the signal. The results are given below.

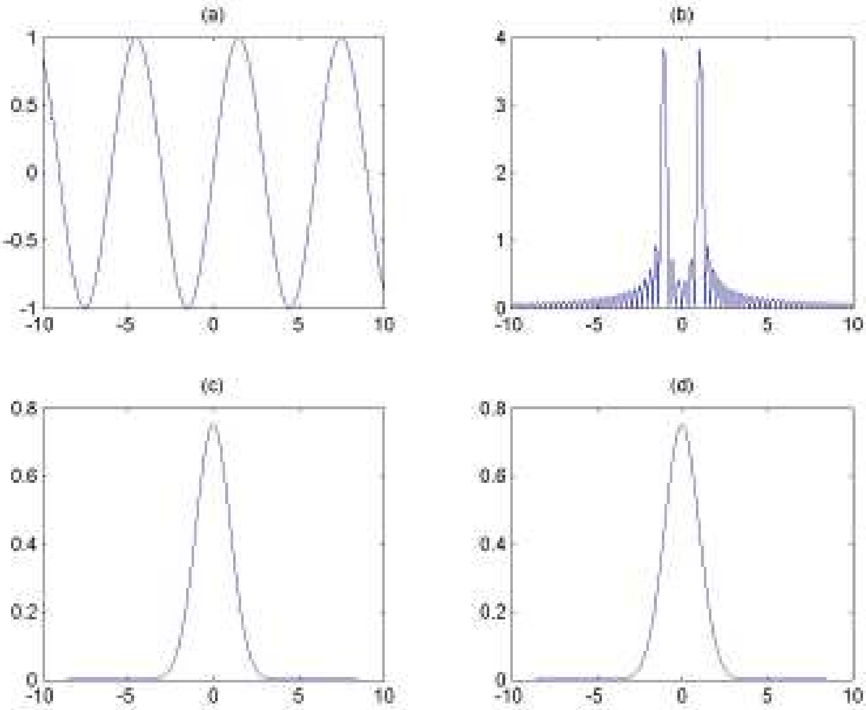

Fig.1 (a) $x_1(t) = \sin(\frac{\pi}{3}t)$ **(b) |FrFT(x₁(t))| at optimal transform order of 1.5708**

(c) $x_2(t) = (\frac{1}{\pi})^{\frac{1}{4}}.e^{-\frac{1}{2}t^2}$ **(d) |FrFT(x₂(t))| at optimal transform order of 1.5708**

Both the real sinusoid and the normalized Gaussian signal give the minimum span representation for the order of 1.5708 which corresponds to π/2, for which FrFT is same as the classical Fourier Transform. The first result is in synchronization with the fact that sinusoid's frequency spectrum is an impulse, which is the most compact possible representation of any signal. For the second signal also, the optimal order is 1.5708 and the transform looks same as the original signal (Fig.1). This result satisfies the fact that the Fourier transform of a Gaussian signal is also Gaussian and that it provides the lower bound for uncertainty relation. The optimal order for the frequency modulated Gaussian signal is also found to be 1.5708 and the plots are given in Fig. 2.

Fig. 3. The left signal is $x_3(t) = x_2(t).e^{j20t}$ and the right is the FrFT at α = π/2.

All the above three signals did not contain a signal like a chirp. Finding the optimal order for the signals that do not contain transients, one finds that the optimal order is always near 1.5708 which means that the Fourier Transform nearly provides the minimum bandwidth representation for such signals. However, the case for the signals with even a single transient is different. We show this by finding the optimal order for a linear chirp signal with a normalized Gaussian envelope.

Consider the signal $x_4(t) = x_2(t).e^{j(\beta t^2 + \gamma t + \lambda)}$, which is a linear chirp with a Gaussian envelope. The optimal order for this signal with $\beta = \gamma = \lambda = 1$ is found out to be 0.6486 radians, which gives the minimal span representation for the signal. The representation is much more compact as compared to the Fourier domain representation of the chirp signal, as shown in Fig. 3 and Fig.4. Here we plot the magnitude of the transform on an expanded x-axis to clearly show the difference.

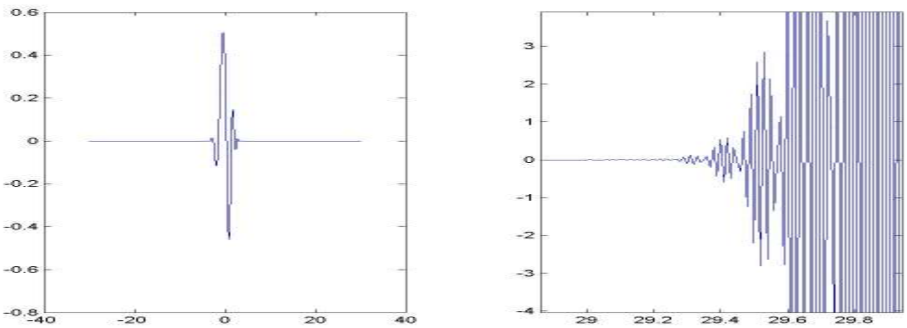

Fig.3. The left figure is the original signal *$x_4(t)$* while the right one is the expanded part of the most significant portion of the Fourier Transform of *$x_4(t)$*

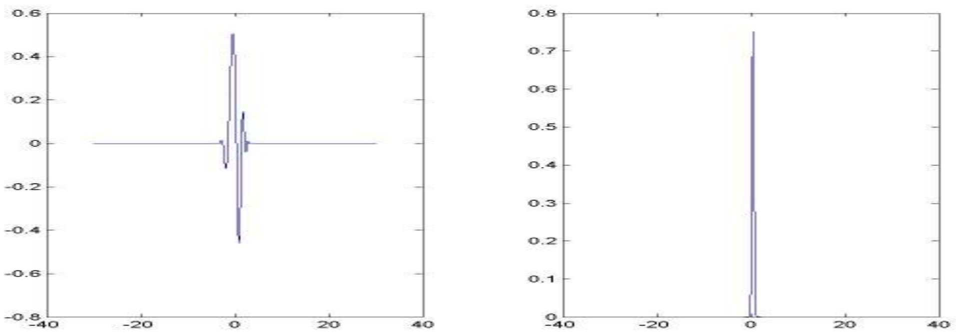

Fig.4. The left figure is the original signal $x_4(t)$ while the right one is the FrFT of $x_4(t)$ at the calculated optimal order of 0.6486

Hence, we establish that the Fourier transform is not suitable for the signals containing transients and for such cases, FrFT shall be used in general to provide a compact and a meaningful representation. Hence, FrFT is often useful for the transient analysis of the signals.

IV. COMPARISON OF THE PROPOSED METHOD WITH THE PREVIOUSLY ESTABLISHED METHODS.

The method for finding the optimum α is through a geometrical representation of the time frequency plane given in [3]. For a signal of the form $e^{j(\beta t^2+\gamma t+\lambda)}$, the optimum order as given by [3] is α = arctan (1/2β). However, as we shall show, the order found out in this manner is not always truly optimal for all type of transient signals. Fig. 5 and Fig. 6 depict that the optimal order found out for the simplest chirps with the proposed method in this paper provides a better minimum span

.

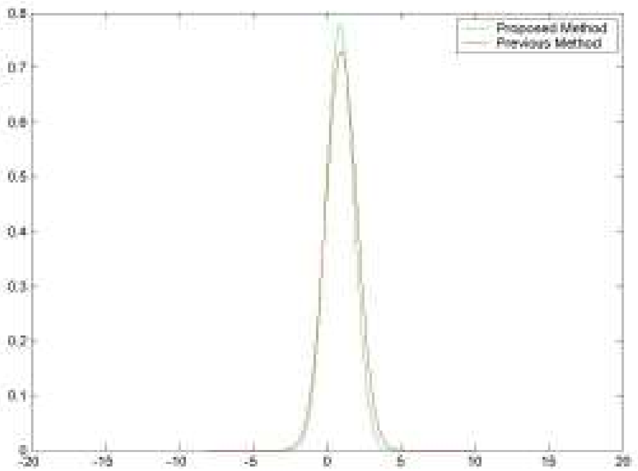

Fig.5. The spans for optimum α for x(t) = $x_2(t).e^{j(\beta t^2+\gamma t+\lambda)}$ with $\beta = 0.1, \gamma = 1, \lambda = 1$

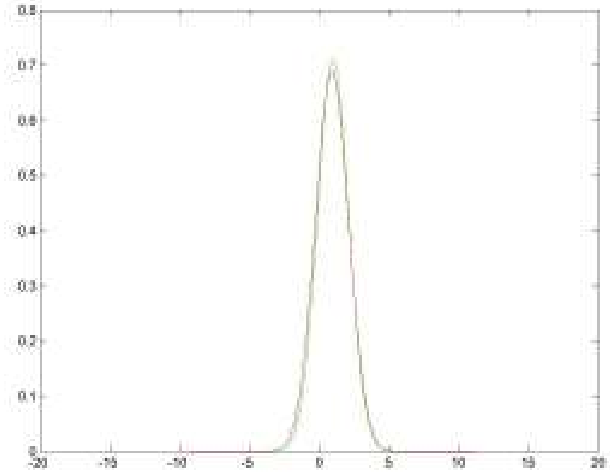

Fig.6. The spans for optimum α for x(t) = $x_2(t).e^{j(\beta t^2+\gamma t+\lambda)}$ **with** $\beta = 0.2, \gamma = 1, \lambda = 1$

It can be seen from the figures above that the spans found out with our proposed method are slightly lesser. This is the case for the simplest chirps. A more clear comparison for various chirp rates is given in Table 1.

	Optimum α of previous method	Optimum α of our method	Min Span with previous method	Min Span with our method
β = 0.1	1.3734	2.0659	0.5592	0.4534
β = 0.2	1.1903	1.4581	0.7186	0.624
β = 0.5	0.7854	1.991	1.7071	0.5255
β = 10	0.05	1.8341	220.475	181.743

Table 1. Comparison of the minimum possible spans for some transform order values with the proposed method and the previous method.

It can be seen from Table 1 that as the chirp rate increases or the signal starts having more transients, the proposed method gives lesser span as compared to that with the previous method. A real time signal generally contains all types of signals ranging from the simplest sinusoidal portions to the most complex transients for varying time slots and also some amount of white or pink noise. Such a signal requires a portion of the signal to be analysed in the most compact form as possible without any loss. The finding of optimum order for FrFT with the proposed method is hence extremely useful for the analysis of real time signals including many transients.

V. CONCLUSION

We have derived an expression showing the dependency of the Fractional Fourier span on the amplitude and phase functions of the given signal. Using the expression, we find the optimal orders for the minimal span representation of the signal. We consolidate the idea mathematically that the simplest signals give the most compact representation with the classical Fourier transform and that the Fractional Fourier transform in general is most useful for the analysis of the transient signals. We finally show that our method for finding the optimal order is more efficient than the previously used methods based on the geometrical representation of the time frequency plane.

REFERENCES

[1] *Namias* "The fractional Fourier Transform and its applications in quantum mechanics" J. Inst.Maths.Applicaions, vol.25, pp.241-265, 1980.

[2] *A.C.McBridge* and *F.H.Kerr* "On Namias' fractional Fourier Transforms" , IMA J. Appl. Math., vol.39., pp 159-175, 1987.

[3] *Yuri Rzhanov, Laurie Linnett,* "The Analysis of Multiple Linear Chirp Signals".

ISBN:978-988-98671-6-4