Algebraic Identification Method for Mass-Spring-Damper System

J. Becedas*, G. Mamani*, V. Feliu-Batlle * and H. Sira-Ramírez ** †‡

Abstract—**In this paper we describe a procedure for parameters identification using an algebraic identification method for a continuous time constant linear system. We make a specific application in the determination of the parameters mass-spring-damper system. The method is suitable for simultaneously identifying, both, the spring constant and the damping coefficient. It is found that the proposed method is computationally fast and robust with respect to noises. The identification algorithm has been verified by simulation results. The estimations are carried out on-line.**

Keywords: Algebraic Identification, System Identification, Mass-Spring-Damper System

1 INTRODUCTION

Parameter identification is used to obtain an accurate model of a real system, and the completed model provides a suitable platform for further developments of design, or control strategies investigation. On-line parameter identification schemes are actually used to estimate system parameters, monitor changes in parameters and characteristics of the system and for diagnostic purposes related to a variety of areas of technology. The identification schemes can be used to update the value of the design parameters specified by manufacturers.

In this article, we use an on-line algebraic method, of non-asymptotic nature for the estimation of the mass-spring-damper system. We simultaneously estimate the spring constant and the damping coefficient. The input variables to the estimator are the force input to the system and the displacement of the mass. The identification method is based on elementary algebraic manipulations. This method is based on the following mathematical tools: module theory, differential algebra and operational calculus. They were developed in [1]. A differential algebraic justification of this article follows similar lines to those encountered in [2], [3], [4] and also [5]. Let us recall that those techniques are not asymptotic.

Parameter estimation has been an important topic in system identification literature. The traditional theory is well developed, see [6] and [7]. The most well known technique for parameter estimation is the recursive least square algorithm. This paper basically focuses on an on-line identification method, of continuous-time nature, on a mechanical system. Our approach uses the model of the system, that is known almost most of times, the advantages are that it does not need any statistical knowledge of the noises corrupting the data; the estimation does not require initial conditions or dependence between the system input and output; and the algorithm is computed on-line.

We mention that the algebraic method has also been applied in [8] in the area of signal processing applications, and in [9] in flexible robots estimation with good results. In this last work it is also demonstrated the algebraic method independence to the input signal design.

Finally, this estimation method can be applied in a wide range of applications in which appears the mass-spring-damper model, such as vibration control [10] , impact dynamics [11], estimation of contact parameters [12], control in robotics [13] among others.

This paper is structured as follows: in section 2, the mass-spring-damper model is introduced and the algebraic identification method is presented. After the identification method is outlined, simulations results are presented in order to confirm the accuracy of the parameters estimation. This is accomplished in sections 3. Finally, section 4 is devoted to concluding remarks and suggestions for future research.

2 MASS-SPRING-DAMPER MODEL AND IDENTIFICATION PROCEDURE

This section is devoted to explain the linear model of the mass-spring-damper system and the algebraic identification method.

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2.1 Mass-Spring-Damper model

Although vibrational phenomena are complex, some basic principles can be recognized in a very simple linear model of a mass-spring-damper system. Such a system contains a mass m $[kg]$, a spring with spring constant k $[N/m]$ that serves to restore the mass to a neutral position, and a damping element which opposes the motion of the vibratory response with a force proportional to the velocity of the system, the constant of proportionality being the damping constant c $[Ns/m]$. An ideal mass-spring-damper system can be described with the following formula:

$$F_s = -kx \tag{1}$$

$$F_d = -cv = -c\dot{x} = -c\frac{dx}{dt} \tag{2}$$

This system of equation is derived by the Newton ´s law of motion which is

$$\sum F = ma = m\ddot{x} = m\frac{d^2x}{dt^2} \tag{3}$$

where a is the acceleration $[m/s^2]$ of the mass and x $[m]$ is the displacement of the mass relative to a fixed point of reference.The above equation combine to form the equation of motion, a second-order differential equation for displacement x as a function of time t[sec]:

$$m\ddot{x} + c\dot{x} + kx = F \tag{4}$$

Rearranging, we have

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F}{m} \tag{5}$$

An scheme of the system is depicted in Fig.1.

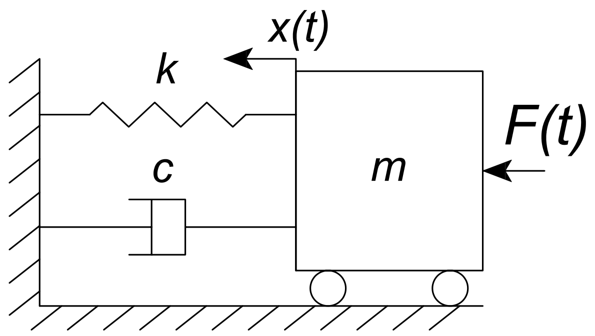

Figure 1: Mass-spring-damper system scheme.

Next, to simplify the equation, we define the following parameters: $B = \frac{c}{m}$, $K = \frac{k}{m}$ and $f = \frac{F}{m}$, and we obtain the second order system

$$\ddot{x} + B\dot{x} + Kx = f \tag{6}$$

The mass-spring-damper transfer function is then written as:

$$G(s) = \frac{x(s)}{f(s)} = \frac{1}{(s^2 + Bs + K)} \tag{7}$$

In our parameter identification scheme we will compute, in an algebraic form: B and K from linear identifiability. From these relation, and due to the fact that m is known, we have

$$k = Km \tag{8}$$

$$c = Bm \tag{9}$$

2.2 The Procedure of Algebraic Identification

Consider the second order system given in (6). c and k are unknown parameters and they are not linearly identifiable. Nevertheless, the parameter $\frac{c}{m}$ denoted by B and the parameter $\frac{k}{m}$ denoted by K are linearly identifiable.

We proceed to compute the unknown system parameters B and K as follows:

Taking Laplace transforms, of (6) yields,

$$s^2x(s) - sx(0) - \dot{x}(0) + B(sx(s) - x(0)) + Kx(s) = f(s) \tag{10}$$

Taking derivative with respect to the complex variable s,twice, we obtain independence of initial conditions. Then (10) results in an expression free of the initial conditions $\dot{x}(0)$ and $x(0)$.

$$\frac{d^2}{ds^2}\left[s^2x(s)\right] + B\frac{d^2}{ds^2}\left[s^2x(s)\right] = \frac{d^2}{ds^2}[f(s) - Kx(s)] \tag{11}$$

The terms of (11) can be developed as:

$$\frac{d^2}{ds^2}[s^2x(s)] = 2x + 4s\frac{dx}{ds} + s^2\frac{d^2x}{ds^2} \tag{12}$$

$$\frac{d^2}{ds^2}[sx(s)] = 2\frac{dx}{ds} + s\frac{d^2x}{ds^2} \tag{13}$$

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Recall that multiplication by s in the operational domain corresponds to derivation in the time domain. To avoid derivation, after replacing the expressions (12, 13, in equation (11), we multiply both sides of the resulting expression by s^{-2}. We obtain

$$B(2s^{-2}\frac{dx}{ds}+s^{-1}\frac{d^2x}{ds^2})+Ks^{-2}\frac{d^2x}{ds^2}= s^{-2}\frac{d^2f(s)}{ds^2}-2s^{-2}-4s^{-1} \quad (14)$$

In the time domain, one obtains the first equation for the unknown parameters B and K. It is a linear equation of the form

$$Bp_{11}(t)-Kp_{12}(t)=-q_1(t) \quad (15)$$

where p_{11}, p_{12} and q_1 are

$$p_{11}(t)=-2\int^{(2)}tx+\int t^2x \quad (16)$$

$$p_{12}(t)=\int^{(2)}t^2x \quad (17)$$

$$q_1(t)=\int^{(2)}(t^2f-2x)+4\int tx-t^2x \quad (18)$$

Notation[1]

The expression (14) is multiplied both sides by s^{-1} once more, leads to a second linear equation for the estimates B, and K.

This linear system can be represented in matrix form as:

$$PX=Q \quad (19)$$

where P is a matrix whose coefficients are time dependant, X is the column vector of the unknown parameters, and Q is a column vector whose coefficients are also time dependant. It is in general form,

$$\begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix}\begin{bmatrix} B \\ K \end{bmatrix}=\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} \quad (20)$$

$p_{21}(t)=\int p_{11}(t)$, $p_{22}(t)=\int p_{12}(t)$, $q_2(t)=\int q_1(t)$ being,

$$p_{21}(t)=-2\int^{(3)}tx+\int^{(2)}t^2x \quad (21)$$

$$p_{12}(t)=\int^{(3)}t^2x \quad (22)$$

$$q_2(t)=\int^{(3)}(t^2f-2x)+4\int^{(2)}tx-\int t^2x) \quad (23)$$

[1] $\int^{(n)}\phi(t)$ representing the iterated integral $\int_0^t\int_0^{\sigma_1}...\int_0^{\sigma_{n-1}}\phi(\sigma_n)d\sigma_n...d\sigma_1$ with $(\int\phi_{(t)}) = (\int^{(1)}\phi_{(t)})=\int_0^t\phi(\sigma)d\sigma$

The estimates of the parameters K and B can be readily obtained by solving the following linear equation

$$K=\frac{[-p_{21}(t)q_1(t)+p_{11}(t)q_2(t)]}{p_{11}(t)p_{22}(t)-p_{12}(t)p_{21}(t)} \quad (24)$$

$$B=\frac{[p_{22}(t)q_1(t)-p_{12}(t)q_2(t)]}{p_{11}(t)p_{22}(t)-p_{12}(t)p_{21}(t)} \quad (25)$$

The time realization of the elements of matrixes P and Q can be written in a State Space framework via time-variant linear (unstable) filters in order to make the physical implementation of the estimator easier in the real time platform:

$$\begin{aligned} p_{11}(t)&=x_1 \\ \dot{x_1}&=-t^3\theta_m(t)+x_2 \\ \dot{x_2}&=6t^2\theta_m(t)+x_3 \\ \dot{x_3}&=-6t\theta_m(t) \end{aligned} \quad (26)$$

$$\begin{aligned} p_{12}(t)&=y_1 \\ \dot{y_1}&=y_2 \\ \dot{y_2}&=t^3V(t)+y_3 \\ \dot{y_3}&=-3t^2V(t) \end{aligned} \quad (27)$$

$$\begin{aligned} q_{11}(t)&=t^3\theta_m(t)z_1 \\ \dot{z_1}&=-9t^2\theta_m(t)+z_2 \\ \dot{z_2}&=18t\theta_m(t)+z_3 \\ \dot{x_3}&=-6\theta_m(t) \end{aligned} \quad (28)$$

$$\begin{aligned} p_{21}(t)&=\xi_1 \\ \dot{\xi_1}&=\xi_2 \\ \dot{\xi_2}&=-t^3\theta_m(t)+\xi_3 \\ \dot{\xi_3}&=6t^2\theta_m(t)+\xi_4 \\ \dot{\xi_4}&=-6t\theta_m(t) \end{aligned} \quad (29)$$

where $\xi_2=p_{11}$

$$\begin{aligned} p_{22}(t)&=\rho_1 \\ \dot{\rho_1}&=\rho_2 \\ \dot{\rho_2}&=\rho_3 \\ \dot{\rho_3}&=t^3V(t)+\rho_4 \\ \dot{\rho_4}&=-3t^2V(t) \end{aligned} \quad (30)$$

where $\rho_2 = p_{12}$

$$\begin{aligned} q_{21}(t) &= \psi_1 \\ \dot{\psi}_1 &= t^3\theta_m(t) + \psi_2 \\ \dot{\psi}_2 &= -9t^2\theta_m(t) + \psi_3 \\ \dot{\psi}_3 &= 18t\theta_m(t) + \psi_4 \\ \dot{\psi}_4 &= -6\theta_m(t) \end{aligned} \tag{31}$$

where $\psi_2 = q_{11}$.

The matrix $P(t)$ is not invertible at time $t = 0$. This means that no estimation of the parameters is done at this time. But it is certainly invertible after an arbitrarily small time $t = \epsilon > 0$, then accuracy estimation of the motor parameters is obtained in a very short period of time. In practice we initialize the estimator after the small arbitrary time interval $t = \epsilon$ in order to assure that the estimator obtain good estimates. From $t = 0$ to $t = \epsilon$ there exist many singularities because of the divisions by the value zero (see equations (24) and (25), such singularities occur when $p_{11}(t)p_{22}(t) - p_{12}(t)p_{21}(t) = 0$ in the K estimator and $p_{11}(t)p_{22}(t) - p_{12}(t)p_{21}(t) = 0$ in the B estimator.

Since the available signals θ_m and V are noisy the estimation precision yielded by the estimator in (24)-(25) will depend on the Signal to Noise Ratio (SNR). We assume that θ_m and V are perturbed by an added noise with unknown statistical properties. In order to enhance the SNR, we simultaneously filter the numerator and denominator by the same low-pass filter. Taking advantage of the estimator rational form, the quotient will not be affected by the filters. This invariance is emphasized with the use of different notations in frequency and time domain:

$$K = \frac{F(s)\left[-p_{21}(t)q_1(t) + p_{11}(t)q_2(t)\right]}{F(s)(p_{11}(t)p_{22}(t) - p_{12}(t)p_{21}(t))} \tag{32}$$

$$B = \frac{F(s)\left[p_{22}(t)q_1(t) - p_{12}(t)q_2(t)\right]}{F(s)(p_{11}(t)p_{22}(t) - p_{12}(t)p_{21}(t))} \tag{33}$$

Remark 2.1 *Invariant low-pass filtering is based on pure integrations of the form* $F(s) = 1/s^p$, $p \geq 1$. *We assumed high frequency noises. This hypothesis were motivated by recent developments in Non-standard Analysis, towards a new non stochastic noise theory. More details in [14].*

3 SIMULATION RESULTS

This section is devoted to demonstrate the good performance of the theoretical algorithm previously explained. On the one hand, signals without any sort of noise are used in the implementation to show the time in which an ideal estimation is obtained. On the other hand, the input signals to the estimator are corrupted with a stochastic noise $n(t)$ with zero mean and standard deviation 10^{-2} in the measure of the position. Fig.3 depicts the implementation scheme of the estimator. Note that in estimations without noise the input $n(t)$ is zero. Independence with respect an specific design of the input to the system is also demonstrated by using two different inputs to the system: step input and sinusoidal input.

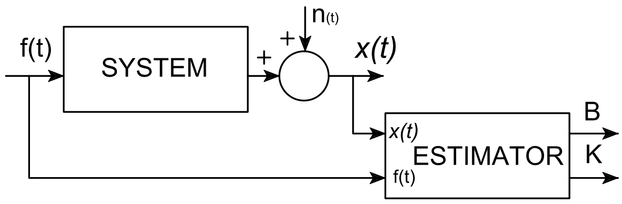

Figure 2: Implementation scheme of the estimator.

The parameters used in the estimation are depicted in Table 1.

Table 1: System parameters used in simulations

Parameter	Value
m	1 $[kg]$
c	2 [Ns/m]
k	3 [N/m]

The parameters $B = \frac{c}{m}$ and $K = \frac{k}{m}$ which will be estimated by the estimator will have values of 2 $[Ns/(mkg)]$ and 3 $[N/(mkg)]$ respectively.

3.1 Estimation without noise in the measure

In this subsection estimation of the system parameters are obtained by using ideal input signals to the estimator. In Fig.3(a) the step input to the system is shown. The response of the system to this input is depicted in Fig.3(b). The estimator has such signals as input. The estimation of the parameters B and K are almost immediately obtained (see Fig.4). At time $t = 0.04$ $[s]$ we get good estimates of the parameters with null error with respect the ideal values $B = 2$ $[Ns/(mkg)]$ and $K = 3$ $[N/(mkg)]$. Until time $t = 0.01$ $[s]$ the estimator has been initialized to zero value. With this estimates and bearing in mind the mass value $m = 1$ $[kg]$ from equations 8 and 9 the real values of $k = 3$ $[N/m]$ and $c = 2$ $[Ns/m]$ can be obtained respectively.

In the simulation with sinusoidal signal as input to the system are obtained the same results. The sinusoidal signal has amplitude value of 1 $[N]$ and frequency value 1 $[rad/s]$. Fig.5(a) depicts the sinusoidal input to the system, whereas Fig.5(b) depicts the response of the system to such an input. The estimation of the values $B = 2$ $[Ns/(mkg)]$ and $K = 3$ $[N/(mkg)]$ is shown in

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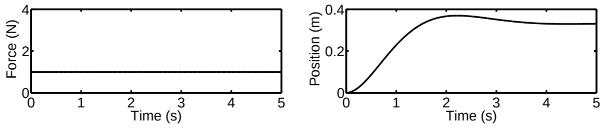


(a) (b)

Figure 3: (a) Step input to the system. (b) Response of the system to step input.

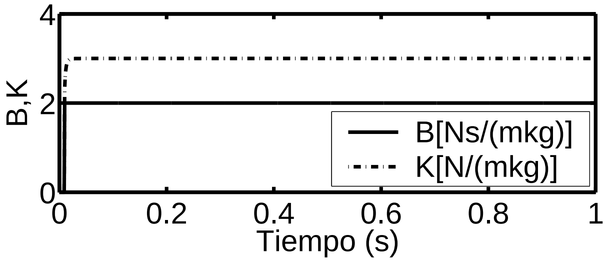
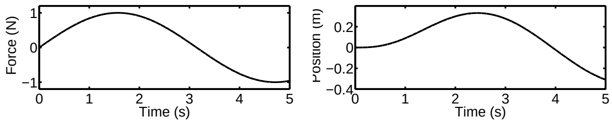


Figure 4: Estimation of B and K with step input.

Fig.6. In this case, we have initialized the estimator to value zero within 0.03 $[s]$. The estimates are obtained at time $t = 0.06$ $[s]$ and the values are maintained until the end of the experiment (see Fig.6).



(a) (b)

Figure 5: (a) Sinusoidal input to the system. (b) Response of the system to sinusoidal input.

3.2 Estimation with noise in the measure

In this case the response signals of the system $x(t)$ is corrupted by stochastic noise with zero mean and 10^{-2} standard deviation. We consider noise in this signal because is the only variable to measure in an experimental platform by means of sensors such as accelerometers, vision system,... The step and sinusoidal inputs to the system are the same that the used in Section 3.1 (see Fig.3(a) and Fig.5(a)). The input $x(t)$ corrupted by noise to the estimator is depicted in Fig.7(a) in the case in which the input to the system is the step signal, and in Fig.7(b) in the case in which the input to the system is the sinusoidal signal. Note that the noise strongly affects the signals.

The estimation of the parameters B and K when the input $f(t)$ is an step are depicted in Fig.8. The estimator has been initialized to zero until time $t = 1.1$ $[s]$ and at time $t = 2.5$ $[s]$ the values of $B = 2$ $[Ns/(mkg)]$ and $K = 3$ $[N/(mkg)]$ are obtained. When the input $f(t)$ is sinusoidal the results are that depicted in Fig.9 with the same results.

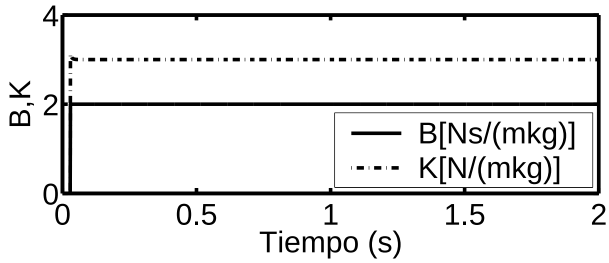
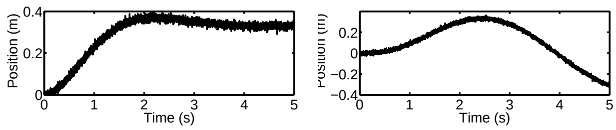


Figure 6: Estimation of B and K with sinusoidal input.



(a) (b)

Figure 7: (a) Response of the system to step input and noise in the measure of the mass position. (b) Response of the system to sinusoidal input and error in the measure of the mass position.

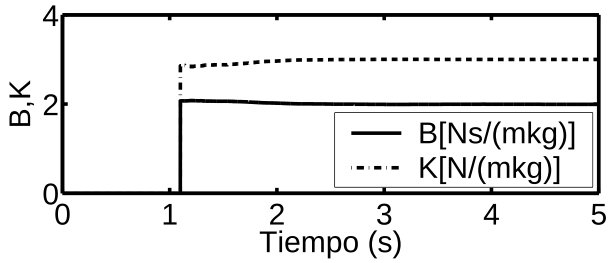


Figure 8: Estimation of B and K with step input and noise in the measure of the mass position.

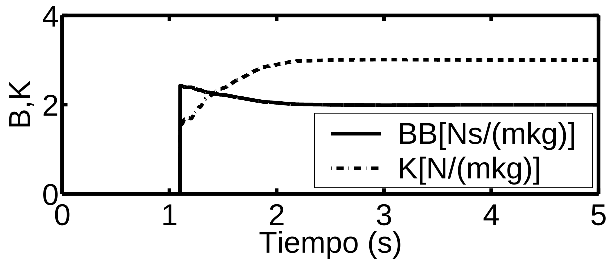


Figure 9: Estimation of B and K with sinusoidal input and noise in the measure of the mass position.

The difference between the ideal case and this case in which noise in the measure appears is that the time in the estimation is different. Whereas in the estimations without noise the values are obtained in 0.3 $[s]$ after the initialized zero value, the estimations with noise are carried out in 1.4 $[s]$ after such initialization. Nevertheless, no error appears every estimation. Further, the different input signals to the system do not affect the estimated values.

4 CONCLUSIONS

Parameter identification using an on-line, non asymptotic, algebraic identification method for continuous-time constant linear systems has been proposed for the estimation of unknown parameters of a mass-spring-damper model.

In this research, an algebraic technic in order to estimate the physical values of the spring and damper of the system is presented. This is based on a bunch of (unstable) filters that vary on time and which are combined with classical low pass filters. The resultant expressions are obtained from derivative algebraic operations, including the unknown constants elimination through derivations with respect the complex variable s.

The only input variables to the estimator are the input force to the system and the displacement of the mass. Among the advantages of this approach we find:it is independent of initial conditions; the methodology is also robust with respect to zero mean high frequency noises as seen from digital computer based simulations; the estimation is obtained in a very short period of time and good results are achieved; a direct estimation of the parameters is achieved without translation between discrete and continuous time domains; and the approach does not need a specific design of the inputs needed to estimate the parameters of the plant because exact formulas are proposed. Therefore its implementation in regulated closed loop systems is direct.

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