An Estimated Replacement Approach for Stable Control of a Class of Nonlinear Systems with Unknown Functions of States

Nguyen Duy Hung and Nguyen Thi Huong Lan, *VIELINA*

***Abstract*—In this paper, we propose an approach for stable control of a class of nonlinear systems, which can be expressed in a state-feedback linearizable form with unknown nonlinear functions of states. The idea is to replace the unknown functions with estimated (not need to be accurate) functions and to use a universal approximator to compensate for the error caused by the replacement. For achieving a stable controller with a continuous control signal, a bisigmoid function based compensator is used and studied. In addition, the paper also deals with the control problem of input constraints and the way to examine this subject.**

***Index Terms*—nonlinear control, unknown functions, estimated replacement, universal approximators.**

I. Problem Formulation

Consider a SISO nonlinear system in its full state-feedback linearizable form [3]

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= f(\mathbf{x}) + g(\mathbf{x})u \\ y &= x_1 \end{aligned} \tag{1}$$

where $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T \in \Omega_{\mathbf{x}} \subset \Re^n$ is a state vector, $u(t) \in \Omega_u \subset \Re$ is an input ($\Omega_{\mathbf{x}}$, Ω_u are compact sets), $y(t) \in \Re$ is an output, and $f(\mathbf{x}) \in \Re$, $g(\mathbf{x}) \in \Re$ are unknown, but continuous and bounded functions. The control objective is to design a locally stable controller for tracking a reference trajectory $r(t) \in \Re$ with bounded error. Because $g(\mathbf{x})$ can not be zero, without loss of generality, we can assume that $g(\mathbf{x}) > 0$ for all $\mathbf{x} \in \Omega_{\mathbf{x}}$. Additionally, it also assumes that $\mathbf{x}$ are measurable whereas $r(t)$ and its derivatives up to the n-th one are bounded and known.

For the given control problem, many adaptive designs have been developed as shown in [7]-[12] and the references therein. In general, the unknown function(s) $g(\mathbf{x})$ or $g(\mathbf{x}), f(\mathbf{x})$ is/are approximated by adjustable function approximator(s) $\tilde{g}(\mathbf{x}, \boldsymbol{\theta}_g)$ or $\tilde{g}(\mathbf{x}, \boldsymbol{\theta}_g), \tilde{f}(\mathbf{x}, \boldsymbol{\theta}_f)$ respectively, where $\boldsymbol{\theta}_g, \boldsymbol{\theta}_f$ are weights or parameter vectors. As the aim is to design a stable adaptive controller with suitable adaptation law to reduce uncertainties in each case, $\tilde{g}$ must be other than zero on the domain $\Omega_{\mathbf{x}}$ to avoid singularities at $\tilde{g} = 0$ during adaptation. To deal with such a problem a parameter projection method is employed ([10], [11]), but this situation can also be avoided when using techniques presented in some schemes, such as a modified Lyapunov function ([7]) or a modified term ([8]).

Manuscript received July 6, 2007. VIELINA is the Vietnam Institute of Electronics, Informatics, and Automation. Address: 156A Quan Thanh St., Hanoi, Vietnam.

The authors are with the Center of Automation and Control, VIELINA (e-mail: ndhung@vielina.com).

II. An Estimated Replacement Approach

Suppose that, from a knowledge of the system we can find out continuous and bounded functions $\tilde{f}(\mathbf{x})$ and $\tilde{g}(\mathbf{x}) > 0$ such that if we replace $f(\mathbf{x})$, $g(\mathbf{x})$ in (1) with $\tilde{f}(\mathbf{x})$, $\tilde{g}(\mathbf{x})$ respectively, we can approximate $\dot{x}_n$ with bounded error, i.e., $\left|\Delta_{dxn}(\mathbf{x}, u)\right| \le W$ holds for all $\mathbf{x} \in \Omega_{\mathbf{x}}$, $u \in \Omega_u$ where

$$\begin{aligned} \Delta_{dxn}(\mathbf{x}, u) &= \tilde{f}(\mathbf{x}) - f(\mathbf{x}) + \left(\tilde{g}(\mathbf{x}) - g(\mathbf{x})\right)u \\ &= \Delta_f(\mathbf{x}) + \Delta_g(\mathbf{x})u \end{aligned}$$

and $W > 0$ is a bounded constant. Based on a method mainly derived from [3], let us define an error system

$$E(t, \mathbf{x}) = \mathbf{k}^T \mathbf{e} \tag{2}$$

where $\mathbf{e} = \mathbf{x} - \mathbf{r}$, $\mathbf{r}^T = \left[r, \dot{r}, \ldots, r^{(n-1)}\right]$, $\mathbf{k}^T = \left[k_1, \ldots, k_{n-1}, 1\right]$ with $s^{n-1} + k_{n-1}s^{n-2} + \ldots + k_1$ is a Hurwitz polynomial.

In the sense of performance analysis, the error system provides a quantitative measure of the closed-loop system performance. Hence, once the system dynamics are used with the definition of the error system to define the error dynamics, a Lyapunov candidate $V(E)$ is then used to provide a scalar measurement of the error system. In addition, in terms of boundedness, the error system and the Lyapunov candidate are also chosen such that bounding V will place bounds on the error system $|E|$ and the system states $|\mathbf{x}|$ too.

To focus on the main idea of this paper, we accept without

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proof that (2) satisfies the error system assumption (see Appendix A). Additionally, for the time being, we ignore the local stabilization case and do not take the state and input bounding conditions into consideration. Thus if we denote $\mathbf{k}_E^T = \left[k_1,\ldots,k_{n-1}\right]$ and $\mathbf{d}_E^T = \left[x_1 - r, x_2 - \dot{r},\ldots,x_{n-1} - r^{(n-2)}\right]$, the error system (2) can be rewritten as $E(t,\mathbf{x}) = \mathbf{k}_E^T \mathbf{d}_E + \left(x_n - r^{(n-1)}\right)$ and its time derivative (i.e., the error dynamic) becomes

$$\begin{aligned}\dot{E} &= \mathbf{k}_E^T \dot{\mathbf{d}}_E + \dot{x}_n - r^{(n)} \\ &= \mathbf{k}_E^T \dot{\mathbf{d}}_E - \Delta_{dxn} - r^{(n)} + \tilde{f} + \tilde{g}u.\end{aligned} \tag{3}$$

In terms of feedback linearization, use the control law

$$u = \tilde{u} = \tilde{g}^{-1}\left(-\mathbf{k}_E^T \dot{\mathbf{d}}_E + r^{(n)} - \tilde{f} - \eta E\right) \tag{4}$$

where $\eta > 0$ and consider the Lyapunov candidate $V(E) = \frac{1}{2}E^2$, then the time derivative of the Lyapunov function along the solution of the error dynamic (3) is bounded by

$$\begin{aligned}\dot{V} &= E\dot{E} = -\eta E^2 - \Delta_{dxn} E \\ &\le -\eta V - \tfrac{1}{2}\eta E^2 + W\left|E\right| \\ &= -\eta V + \frac{W^2}{2\eta} - \frac{1}{2\eta}\left(W - \eta\left|E\right|\right)^2 \\ &\le -\eta V + \frac{W^2}{2\eta}.\end{aligned}$$

Let V_0 and E_0 denote the V and E at $t = 0$, thus according to the lemma of ultimate bound (Appendix B) with $m_1 = \eta$ and $m_2 = \dfrac{W^2}{2\eta}$, we obtain

$$\begin{aligned}& V \le \frac{W^2}{2\eta^2}\left(1 - e^{-\eta t}\right) + V_0 e^{-\eta t} = V_H \\ & \Rightarrow \left|E\right| \le \sqrt{\frac{W^2}{\eta^2}\left(1 - e^{-\eta t}\right) + E_0^2 e^{-\eta t}} = E_H\end{aligned}$$

and $\lim_{t\to\infty} V_H = \dfrac{W^2}{2\eta^2} = V_\infty$, $\lim_{t\to\infty} E_H = \dfrac{W}{\eta} = E_\infty$.

Remark 1: If we denote $\boldsymbol{\eta} = \left[\eta k_1, k_1 + \eta k_2, \ldots, k_{n-1} + \eta\right]^T$ then $\mathbf{k}_E^T \dot{\mathbf{d}}_E + \eta E = \boldsymbol{\eta}^T \mathbf{e}$ and the control law (4) can be formulated as

$$u = \tilde{u} = \tilde{g}^{-1}\left(r^{(n)} - \boldsymbol{\eta}^T \mathbf{e} - \tilde{f}\right) \tag{5}$$

Remark 2: If $V_0 \le V\left(E_\infty\right)$ then $0 \le V \le V\left(E_\infty\right)$ for all $t \ge 0$ since V is positive definite so that it can not grow greater than $V\left(E_\infty\right)$. Furthermore, in the case of $V_0 > V\left(E_\infty\right)$ we have $\dot{V} \le 0$ until $V \le V\left(E_\infty\right)$, thus we find

$$0 \le V \le \max\left(V_0, V\left(E_\infty\right)\right) \Rightarrow \left|E\right| \le \max\left(\left|E_0\right|, E_\infty\right) \tag{6}$$

for all $t \ge 0$.

Remark 3: From (6), we see that if we choose $\left|E_0\right|$ small enough, the closed-loop system performance depends only on the error bound W in approximating $\dot{x}_n$ without considering about how large the individual approximation errors $\Delta_f(\mathbf{x})$ and $\Delta_g(\mathbf{x})$ in replacing the unknown functions are. This means that we can replace the unknown functions with preferred estimated functions at our convenience provided that the approximation error $\left|\Delta_{dxn}\right|$ is bounded by W.

Above results lead to the state of the following theorem.

Theorem 1: The state-feedback control law (5) ensures that the solution of the error dynamic (3) is uniformly ultimately bounded by (6) if there exist continuous and bounded functions $\tilde{f}(\mathbf{x})$ and $\tilde{g}(\mathbf{x}) > 0$ such that $\left|\Delta_{dxn}(\mathbf{x},u)\right| \le W$ holds for all $\mathbf{x} \in \Omega_{\mathbf{x}}$, $u \in \Omega_u$ where $W > 0$ is a known bounded constant.

Proof: According to a theorem of condition for uniform ultimate boundedness ([2]), in proving Theorem 1 we wish to find some $\gamma_1(\left|E\right|)$, $\gamma_2(\left|E\right|) \in K_\infty$ and $\gamma_3(\left|E\right|) \in K$ defined on $\left[0,\infty\right)$ such that

$$\begin{aligned}& \gamma_1(\left|E\right|) \le V(E) \le \gamma_2(\left|E\right|) \\ & \dot{V}(E) = \frac{\partial V}{\partial E}\dot{E} \le -\gamma_3(\left|E\right|)\end{aligned} \tag{7}$$

for $\forall\left|E\right| \ge R$ and $t \ge 0$ with knowing that $V(E)$ is continuously differentiable on $\left|E\right| \ge R$.

Choosing $\gamma_1(\left|E\right|) = \gamma_2(\left|E\right|) = V(E) = \frac{1}{2}\left|E\right|^2$ we have

$$\begin{aligned}\dot{V}(E) &\le -\eta V + \frac{W^2}{2\eta} \\ &= -\varepsilon\eta\gamma_1(\left|E\right|) - (1-\varepsilon)\eta V + \frac{W^2}{2\eta}\end{aligned}$$

for ε satisfying $0 < \varepsilon < 1$. Let $\gamma_3(\left|E\right|) = \varepsilon\eta\gamma_1(\left|E\right|)$ we see that $\dot{V}(E) \le -\gamma_3(\left|E\right|)$ if and only if $V \ge \dfrac{W^2}{2(1-\varepsilon)\eta^2}$, or equivalently, $\left|E\right| \ge \dfrac{W}{\sqrt{1-\varepsilon}\;\eta} = R$. As the chosen functions fulfill requirement (7), Theorem 1 is thus proved.

Theorem 1 shows that it is possible to define (static) stabilizing controllers by applying the method of estimated replacement if we could find substitution functions satisfying the bounding condition over a valid region. But a problem arises when W is large, since though the error system bound may be decreased by choosing η large, the control signal may increase in amplitude and may start to oscillate. To dealing with such a problem, the usual approach is to compensate for error effects caused by the replacement. For this purpose, a number of techniques, such as nonlinear damping and dynamic normalization ([3]) may be used. In this sense, here we propose a method which comes from the notion that if we can approximate $\Delta_{dxn}(\mathbf{x},u)$ with sufficiently small error, it is possible to include an additional stabilizing component to increase the robustness of the closed-loop system.

Because $\Delta_{dxn}(\mathbf{x},u)$ is a continuous and bounded function defined on compact sets, it can be approximated by a universal approximator (such as a fuzzy system or a neural network) with arbitrary accuracy. Therefore by assumption that there are data available for tuning of an approximator to match certain condition, we can use it as a compensation component to form a robust state-feedback control law. This subject will be studied in more detail later in this paper. Now, before turning to developing a stable controller for making the closed-loop system more robust to system uncertainties, we will investigate some mathematical base.

III. Mathematical Base

Define a real-valued scalar function

$$\mu_E(\rho,\kappa,E)=|E|\left(\rho-\operatorname{sgn}(E)\operatorname{bsig}(\kappa,E)\right) \tag{8}$$

where $0<\rho\le 1$, $\kappa>0$ are parameters, $E\in\Re$ is a variable, $\operatorname{sgn}(E)$ is the sign function, and $\operatorname{bsig}(\kappa,E)=2/(1+e^{-\kappa E})-1$ is bisigmoidal.

Lemma 1. The function (8) reaches its positive maximum value of $\mu_{E\_\max}(\rho,\kappa)=\mu_E(\rho,\kappa,\pm E_m)$ at $\pm E_m$ where $x=\kappa E_m$ is the unique solution of the equation

$$\mu_x(\rho,x)=(\rho+1)e^{-2x}+2(\rho-x)e^{-x}+\rho-1=0 \tag{9}$$

Proof: Because (8) is an even function, thus we can take only the case $E\ge 0$, i.e., $\mu_{E+}(\rho,\kappa,E)=E\left(\rho-\operatorname{bsig}(\kappa,E)\right)$ into account. It follows that the derivative of μ_{E+} with respect to E can be calculated as

$$\frac{d}{dE}\mu_{E+}=\frac{(\rho+1)e^{-2x}+2(\rho-x)e^{-x}+\rho-1}{\left(1+e^{-x}\right)^2}=\frac{\mu_x(\rho,x)}{\left(1+e^{-x}\right)^2}$$

where $x=\kappa E\ge 0$. Obviously, μ_{E+} has its extremum at $x_m=\kappa E_m$ if satisfies $\mu_x(\rho,\kappa E_m)=0$. Next we will show that, $x=x_m$ is the unique solution of (9) and $\mu_{E\_\max}(\rho,\kappa)$ is a positive maximum.

Take the derivative of $\mu_x(\rho,x)$ with respect to x, we obtain

$$\frac{d}{dx}\mu_x(\rho,x)=2e^{-x}\left[x-(\rho+1)-(\rho+1)e^{-x}\right].$$

For studying $\mu_x(\rho,x)$, solve the equation $\frac{d}{dx}\mu_x(\rho,x)=0$ or equivalently

$$-x+(\rho+1)=-(\rho+1)e^{-x}. \tag{10}$$

This equation is in form of $x+b=ae^x$ where $a\ne 0$, thus according to [13] it has the single root, equal to $-b-\mathrm{w}(-ae^{-b})$ where $\mathrm{w}(x)$ is the Lambert w-function (Note that the Lambert w-function is the inverse function of $x=\mathrm{w}(x)e^{\mathrm{w}(x)}$). The substitution for $a=-(\rho+1)$ and $b=\rho+1$ leads to the solution of (10), afterward denote as $x_0=\rho+1+\mathrm{w}(\mathrm{p})$ where $\mathrm{p}(\rho)=(\rho+1)e^{-(\rho+1)}$.

Because of $\frac{d\,\mathrm{p}}{d\rho}=-\rho(2+\rho)e^{-(\rho+1)}<0$, p is decreasing for $\rho\in(0,1]$, therefore $\mathrm{p}(1)\le\mathrm{p}(\rho)<\mathrm{p}(0)$ or $\mathrm{p}\in\left[2/e^2,1/e\right)$. Consequently $\mu_x(\rho,x)$ has the unique extremum at x_0 and if we denote $\mu_0=\mu_x(\rho,x_0)$ then

$$\begin{aligned}\mu_0 &=(\rho+1)e^{-2x_0}+2(\rho-x_0)e^{-x_0}+\rho-1\\ &=(\rho+1)e^{-2(\rho+1)}\left(\frac{1}{e^{\mathrm{w}(\mathrm{p})}}\right)^2\\ &\quad+2\left[\rho-(\rho+1)-\mathrm{w}(\mathrm{p})\right]e^{-(\rho+1)}\frac{1}{e^{\mathrm{w}(\mathrm{p})}}+\rho-1\\ &=(\rho+1)e^{-2(\rho+1)}\left[\frac{\mathrm{w}(\mathrm{p})}{(\rho+1)e^{-(\rho+1)}}\right]^2\\ &\quad-2\left(1+\mathrm{w}(\mathrm{p})\right)e^{-(\rho+1)}\frac{\mathrm{w}(\mathrm{p})}{(\rho+1)e^{-(\rho+1)}}+\rho-1\\ &=\frac{\mathrm{w}^2(\mathrm{p})}{\rho+1}-2\left(1+\mathrm{w}(\mathrm{p})\right)\frac{\mathrm{w}(\mathrm{p})}{\rho+1}+\rho-1\\ &=\frac{\rho^2-\left(\mathrm{w}(\mathrm{p})+1\right)^2}{\rho+1}.\end{aligned}$$

Since the Lambert w-function is strictly increasing on $\left[-1/\mathrm{e},\infty\right)$ we get $\mathrm{w}(2/e^2)\le\mathrm{w}(\mathrm{p})<\mathrm{w}(1/e)$, thus $x_0>1$ and $\mu_0<0$ for all $\rho\in(0,1]$. In addition $\mu_x(\rho,0)=4\rho>0$ and $\mu_x(\rho,\infty)=\rho-1\le 0$ so that the graph of $\mu_x(\rho,x)$ cuts the x-axis only at $x_m\in(0,x_0)$ as well as the extremum μ_0 is the minimum of $\mu_x(\rho,x)$.

Note that $\frac{d}{dE}\mu_{E+}=\frac{\mu_x(\rho,x)}{\left(1+e^{-x}\right)^2}$, we can infer that μ_{E+} reaches its maximum value of $\mu_{E+}(\rho,\kappa,E_m)=\mu_{E\_\max}(\rho,\kappa)$ at $E_m=\frac{x_m}{\kappa}>0$ and as $\mu_{E+}(\rho,\kappa,0)=0$, the unique maximum is positive. This proves Lemma 1.

For a better understanding of Lemma 1, Fig. 1 shows graphs of (8) in cases of $\kappa = 5$ and $\kappa = 10$ with $\rho = 0.5, 0.9, 1$ in each example whereas Fig. 2 illustrates the graph of $E_m(\rho,\kappa)$ and $\mu_{E\_\max}(\rho,\kappa)$ with respect to ρ and κ in the case of $\kappa = 1$ and $\rho = 1$ respectively.

Fig. 1. Graphs of $\mu_E(\rho,\kappa,E)$

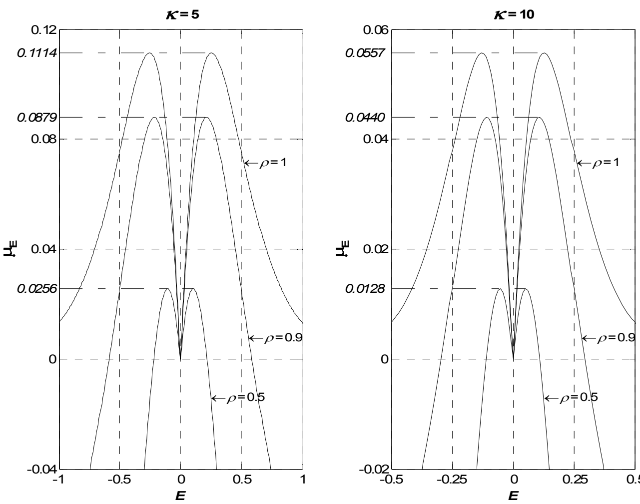
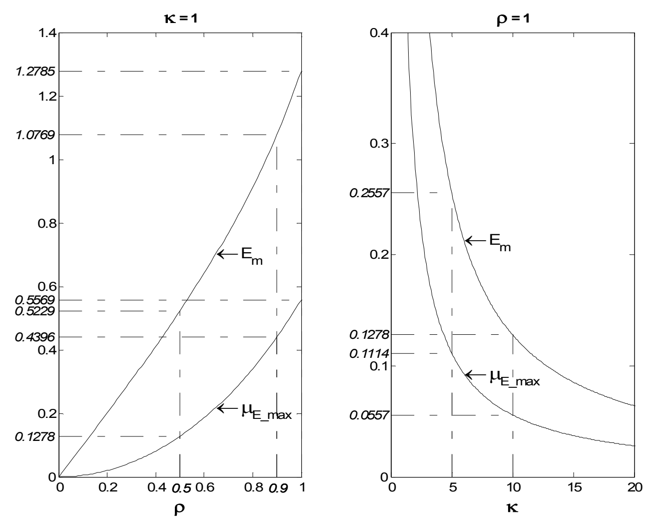


Fig. 2. Graphs of $E_m(\rho,\kappa)$ and $\mu_{E\_\max}(\rho,\kappa)$



IV. Controller Design

Recall from previous studies that we are going to develop a stable controller in the proposed approach called estimated replacement. The main concept in this approach is to seek estimated functions fitting the bounding requirement and to use a compensation technique to make the controller robust to uncertainties. The later problem can be considered in this section as follows.

Suppose that we have to design a controller for the tracking problem with the aim to keep the error system bounded by $E_\infty = \dfrac{W}{\eta}$ (it is assumed that $|E_0|$ can be selected small enough). However the estimated functions available for use only guarantee that

$$\left|\Delta_{dxn}(\mathbf{x},u)\right| = \left|\tilde{f}(\mathbf{x}) - f(\mathbf{x}) + \left(\tilde{g}(\mathbf{x}) - g(\mathbf{x})\right)u\right| \le \tilde{W}$$

for all $\mathbf{x} \in \Omega_{\mathbf{x}}$, $u \in \Omega_u$ where $\tilde{W} > W$. We will search for a solution to cope with this problem.

As mentioned above, the error function Δ_{dxn} can be approximated by the universal approximator within a compact set, which hereafter we denote as $F_\Delta(\mathbf{x},u,\boldsymbol{\theta})$ where $\boldsymbol{\theta} \in \Re^p$ is an adjustable parameter vector and $F_\Delta(\mathbf{x},u,\boldsymbol{\theta}) \in \Re$. Right now let $F_\Delta(\mathbf{x},u,\boldsymbol{\theta})$ represent a neural network or fuzzy system with tunable parameters $\boldsymbol{\theta}$. Assume that $W_\Delta > 0$ be the known approximation error bounding constant, which satisfies

$$\left|F_\Delta(\mathbf{x},u,\tilde{\boldsymbol{\theta}}) - \Delta_{dxn}(\mathbf{x},u)\right| \le W_\Delta \tag{11}$$

for all $\mathbf{x} \in \Omega_{\mathbf{x}}$, $u \in \Omega_u$ and $\tilde{\boldsymbol{\theta}} \in \Re^p$ is the best known parameter vector available from adjusting the parameters of the approximator. Therefore the problem for approximating $\dot{x}_n$ with error bound W can be considered as the problem for approximating Δ_{dxn} with error bound W_Δ. Thus, we can avoid the difficulty of dealing with choosing estimated functions correctly by working with a proper approximator to compensate for the effect of the replacement error. But one must determine how small W_Δ must be to achieve the desired closed-loop system performance.

In order to solve this problem, now we introduce the compensation component defined as

$$u_c = -\frac{F_\Delta(\mathbf{x},\tilde{u},\tilde{\boldsymbol{\theta}})}{\rho\tilde{g}(\mathbf{x})}\operatorname{bsig}\left(\kappa, EF_\Delta(\mathbf{x},\tilde{u},\tilde{\boldsymbol{\theta}})\right) \tag{12}$$

where ρ, κ are constants satisfying $0 < \rho \le 1$, $\kappa > 0$ and $\tilde{u}$ is specified by (5). Then adding the component (12) together with the state-feedback control law (5) forms the new control law

$$u = \tilde{u} + u_c \tag{13}$$

and consequently the following theorem is the extension of Theorem 1 to this case.

Theorem 2: If there exist an approximator $F_\Delta(\mathbf{x},u,\boldsymbol{\theta})$ and a parameter vector $\tilde{\boldsymbol{\theta}}$ such that $F_\Delta(\mathbf{x},u,\tilde{\boldsymbol{\theta}})$ can approximate $\Delta_{dxn}(\mathbf{x},u)$ with error bounded by W_Δ satisfying

$$0 < W_\Delta \le \sqrt{W^2 - \frac{2\eta}{\rho}\mu_{E\_\max}(\rho,\kappa)} \tag{14}$$

for all $\mathbf{x} \in \Omega_{\mathbf{x}}$, $u \in \Omega_u$ where $0 < \rho \le 1$, $\kappa > 0$ and $\eta > 0$ then the state-feedback control law (13) ensures that the solution of the error dynamic (3) is uniformly ultimately bounded by (6).

Proof:

For simplicity, denote $F_\Delta = F_\Delta(\mathbf{x},\tilde{u},\boldsymbol{\theta})$, $\tilde{F}_\Delta = F_\Delta(\mathbf{x},\tilde{u},\tilde{\boldsymbol{\theta}})$

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then from $\dot{E} = -\eta E + \tilde{g}(\mathbf{x})u_c - \Delta_{dxn}(\mathbf{x}, \tilde{u})$ we have

$$\begin{aligned}\dot{V} &= -\eta E^2 + E\tilde{g}(\mathbf{x})u_c - E\Delta_{dxn}(\mathbf{x}, \tilde{u}) \\ &\le -\eta E^2 - \frac{1}{\rho} E\tilde{F}_\Delta \,\mathrm{bsig}(\kappa, E\tilde{F}_\Delta) + |E||\Delta_{dxn}| \\ &\le -\eta E^2 + \frac{1}{\rho}|E|\left(\rho|\Delta_{dxn}| - F_\Delta \,\mathrm{sign}(E)\,\mathrm{bsig}(\kappa, E\tilde{F}_\Delta)\right).\end{aligned}$$

Since $\mathrm{sgn}(E)\,\mathrm{sgn}(\tilde{F}_\Delta) = \mathrm{sgn}(E\tilde{F}_\Delta)$ and $|\Delta_{dxn}| \le |\tilde{F}_\Delta| + W_\Delta$ so we obtain

$$\begin{aligned}\dot{V} &\le -\eta E^2 + |E|W_\Delta + \frac{|E|}{\rho}\left(\rho|\tilde{F}_\Delta| - \tilde{F}_\Delta \,\mathrm{sgn}(E)\,\mathrm{bsig}(\kappa, E\tilde{F}_\Delta)\right) \\ &= -\eta E^2 + |E|W_\Delta + \frac{|E\tilde{F}_\Delta|}{\rho}\left(\rho - \mathrm{sgn}(E\tilde{F}_\Delta)\,\mathrm{bsig}(\kappa, E\tilde{F}_\Delta)\right) \\ &= -\eta V + \frac{W_\Delta^2}{2\eta} - \frac{1}{2\eta}\left(W_\Delta - \eta|E|\right)^2 + \frac{1}{\rho}\mu_E(\kappa, E\tilde{F}_\Delta) \\ &\le -\eta V + \frac{W_\Delta^2}{2\eta} + \frac{1}{\rho}\mu_{E\_\max}(\rho, \kappa)\end{aligned}$$

where $\mu_E(\rho, \kappa, E)$ is defined as in (8) and $\mu_E(\rho, \kappa, E) \le \mu_{E\_\max}(\rho, \kappa)$ for all $0 < \rho \le 1$, $\kappa > 0$ and $E \in \Re$ as stated in Lemma 1.

Clearly, to have the error system bounded by (6), we need $\dot{V} \le -\eta V + \frac{W^2}{2\eta}$, hence it follows that the requirement (14) holds. Notice that because $W_\Delta > 0$, we must choose ρ, κ and η such that $\frac{2\eta}{\rho}\mu_{E\_\max}(\rho, \kappa) < W^2$.

Then similar to the proof of Theorem 1, we come to that the new control law (13) makes the solution of the error system (3) uniformly ultimately bounded by (6). This proves Theorem 2.

V. Input Constraints Analysis

Up to this point we have not taken a state boundedness and input constraints into account. However, for state boundedness, we can examine it using the error system boundedness. In this section, we only consider the case of input constraints. Notice that the original work on stabilization and tracking of feedback linearizable systems under input constraints in which we have utilized its concepts can be reviewed in [6].

The problem of input constraints can be stated here as how to select parameters (if they exist) for the control design so that the control input (13) always remains in a valid region Ω_u, which is defined as

$$\Omega_u = \{u \in \Re : |u| \le u_M\} \qquad (15)$$

where u_M is positive bounded constant. Additionally, it assumes $0 < g_L \le \tilde{g}(\mathbf{x})$ and $\tilde{f}(\mathbf{x})$, $\tilde{g}(\mathbf{x})$ can be chosen so that they are locally Lipschitz in $\mathbf{x}$.

Theorem 3: The state-feedback control law (13) ensures that the system error is uniformly ultimately bounded by (6) while satisfying input constraints $u \in \Omega_u$ where Ω_u is defined as (15) if $u_M > \frac{\tilde{W} + W_\Delta}{\rho g_L}$ and the condition (17) holds.

Proof: By assumption, the estimated functions $\tilde{f}(\mathbf{x})$, $\tilde{g}(\mathbf{x})$ are locally Lipschitz continuous, therefore we can find constants K_f, K_g such that

$$|\tilde{f}(\mathbf{x}) - \tilde{f}(\tilde{\mathbf{x}})| \le K_f\,|\mathbf{x} - \tilde{\mathbf{x}}|$$
$$|\tilde{g}(\mathbf{x}) - \tilde{g}(\tilde{\mathbf{x}})| \le K_g\,|\mathbf{x} - \tilde{\mathbf{x}}|$$

for $\forall \mathbf{x}, \tilde{\mathbf{x}} \in \Omega_{\mathbf{x}}$. From (13) and note that $\mathbf{x} = \mathbf{e} + \mathbf{r}$ we have

$$\begin{aligned}|u| &= \left|\frac{r^{(n)} - \boldsymbol{\eta}^T\mathbf{e} - \tilde{f}(\mathbf{x}) - \frac{1}{\rho}\tilde{F}_\Delta \,\mathrm{bsig}\left(\kappa, E\tilde{F}_\Delta\right)}{\tilde{g}(\mathbf{x})}\right| \\ &= \left|\frac{r^{(n)} - \boldsymbol{\eta}^T\mathbf{e} - \tilde{f}(\mathbf{e}+\mathbf{r}) + \tilde{f}(\mathbf{r}) - \tilde{f}(\mathbf{r})}{\tilde{g}(\mathbf{r})} - \frac{\tilde{F}_\Delta \,\mathrm{bsig}(\kappa, E\tilde{F}_\Delta)}{\rho\tilde{g}(\mathbf{r})}\right| \\ &\quad \times \left|\frac{\tilde{g}(\mathbf{r})}{\tilde{g}(\mathbf{e}+\mathbf{r})}\right|\end{aligned}$$

Since $|\tilde{F}_\Delta| \le \tilde{W} + W_\Delta$ and recall that $\tilde{W} > W$, we get

$$\begin{aligned}|u| &\le \left(\left|\frac{r^{(n)} - \tilde{f}(\mathbf{r})}{\tilde{g}(\mathbf{r})}\right| + \left|\frac{\tilde{f}(\mathbf{r}) - \tilde{f}(\mathbf{e}+\mathbf{r})}{\tilde{g}(\mathbf{r})}\right| + \left|\frac{\boldsymbol{\eta}^T\mathbf{e}}{\tilde{g}(\mathbf{r})}\right| \right. \\ &\quad \left. + \left|\frac{\tilde{F}_\Delta \,\mathrm{bsig}(\kappa, E\tilde{F}_\Delta)}{\rho\tilde{g}(\mathbf{r})}\right|\right) \times \left|1 - \frac{\tilde{g}(\mathbf{e}+\mathbf{r}) - \tilde{g}(\mathbf{r})}{\tilde{g}(\mathbf{e}+\mathbf{r})}\right| \\ &\le \left(\left|\frac{r^{(n)} - \tilde{f}(\mathbf{r})}{\tilde{g}(\mathbf{r})}\right| + \frac{K_f\,|\mathbf{e}|}{|\tilde{g}(\mathbf{r})|} + \frac{|\boldsymbol{\eta}||\mathbf{e}|}{|\tilde{g}(\mathbf{r})|} + \frac{|\tilde{F}_\Delta|}{\rho|\tilde{g}(\mathbf{r})|}\right)\left(1 + \frac{K_g\,|\mathbf{e}|}{|\tilde{g}(\mathbf{e}+\mathbf{r})|}\right) \\ &\le \left(\left|\frac{r^{(n)} - \tilde{f}(\mathbf{r})}{\tilde{g}(\mathbf{r})}\right| + \left(|\boldsymbol{\eta}| + K_f\right)\frac{|\mathbf{e}|}{g_L} + \frac{\tilde{W} + W_\Delta}{\rho g_L}\right)\left(1 + K_g\,\frac{|\mathbf{e}|}{g_L}\right)\end{aligned}$$

In order to have the control input remain in Ω_u, we need

$$\left|\frac{r^{(n)} - \tilde{f}(\mathbf{r})}{\tilde{g}(\mathbf{r})}\right| \le \bar{u} = \frac{u_M}{1 + K_g\,\frac{|\mathbf{e}|}{g_L}} - \left(|\boldsymbol{\eta}| + K_f\right)\frac{|\mathbf{e}|}{g_L} - \frac{\tilde{W} + W_\Delta}{\rho g_L}. \qquad (16)$$

In addition, as $|E| = |\mathbf{k}^T\mathbf{e}| \le \max\left(|E_0|, E_\infty\right)$ so we can write $|\mathbf{e}(t)| \le K \max\left(|E_0|, E_\infty\right) = e_M$ where $e_M > 0$. Let's define

$$M = \frac{u_M}{1 + K_g\,\frac{e_M}{g_L}} - \left(|\boldsymbol{\eta}| + K_f\right)\frac{e_M}{g_L} - \frac{\tilde{W} + W_\Delta}{\rho g_L}$$

then $M \le \bar{u}$ and we see that if $M > 0$ then (16) always holds. To have $M > 0$, it requires

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$$u_M > \left(1 + K_g \frac{e_M}{g_L}\right)\left(\left(|\boldsymbol{\eta}| + K_f\right)\frac{e_M}{g_L} + \frac{\tilde{W} + W_\Delta}{\rho g_L}\right)$$

$$\Leftrightarrow K_g\left(|\boldsymbol{\eta}| + K_f\right)\left(\frac{e_M}{g_L}\right)^2 + \left(|\boldsymbol{\eta}| + K_f + K_g \frac{\tilde{W} + W_\Delta}{\rho g_L}\right)\frac{e_M}{g_L}$$

$$+ \frac{\tilde{W} + W_\Delta}{\rho g_L} - u_M < 0$$

The above quadratic inequation is in the form of $Az^2 + Bz + C < 0$ where $z = \frac{e_M}{g_L} > 0$ and

$$A = K_g\left(|\boldsymbol{\eta}| + K_f\right) > 0$$

$$B = |\boldsymbol{\eta}| + K_f + K_g \frac{\tilde{W} + W_\Delta}{\rho g_L} > 0$$

$$C = \frac{\tilde{W} + W_\Delta}{\rho g_L} - u_M .$$

Let $z_1 < z_2$ are roots of the polynomial $Az^2 + Bz + C$ then the solution of the quadratic inequation is $z_1 < z < z_2$. Since if $C \geq 0$, the mentioned polinomial has non-positive roots so we need $C < 0$, i.e., $u_M > \frac{\tilde{W} + W_\Delta}{\rho g_L}$ so that it has a positive one. It follows that

$$0 < z = \frac{K}{g_L}\max\left(|E_0|, E_\infty\right) < z_2 = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

and therefore we must choose (if it exists)

$$\max\left(|E_0|, E_\infty\right) < \frac{g_L}{K} z_2 \text{ and } \left|\frac{r^{(n)} - \tilde{f}(\mathbf{r})}{\tilde{g}(\mathbf{r})}\right| \leq M \qquad (17)$$

for solving the problem of input constraints. (Q.E.D.)

VI. Conclusion

In summary, the proposed approach gives a new concept to design stable controllers for state-feedback linearizable systems with unknown functions of states. In this way we can also avoid the problem of singularities mentioned above because the estimated functions for replacement can be chosen at our intention and they are known in advance. However the controller we have developed in this paper is static, that is its parameters are not adjustable during operation and therefore it is "less robust" to uncertainties than an adaptive equivalent. Due to the scope of this topic, we will study adaptive schemes in another paper. Additionally, achieved results are intended to be used in real time control systems for industrial applications in the fields of control of chemical processes, water treatment control and robot control.

Appendix A
An Error System Assumption (Assumption 6.1 in [3])

Assume the error system $\mathbf{E}(t, \mathbf{x})$ is such that $\mathbf{E} = \mathbf{0}$ implies $y(t) \to r(t)$ and that the function $\mathbf{E}(t, \mathbf{x})$ satisfies $|\mathbf{x}| \leq \psi_{\mathbf{x}}(t, |\mathbf{E}|)$ for all t, where $\psi_{\mathbf{x}} : \Re^+ \times \Re^+ \to \Re$ is bounded for any bounded $\mathbf{E}$ and $\psi_{\mathbf{x}}(t, e)$ is nondecreasing with respect to $e \in \Re^+$ for each fixed t.

Appendix B
A Ultimate Bound Study (Lemma 2.1 in [3])

If $V(t, \mathbf{E}) : \Re^+ \times \Re^n \to \Re^+$ is positive definite and $\dot{V} \leq -m_1 V + m_2$ where $m_1 > 0$ and $m_2 \geq 0$ are bounded constants, then $V(t, \mathbf{E}) \leq \frac{m_2}{m_1} + \left(V(0) - \frac{m_2}{m_1}\right)e^{-m_1 t}$ for all t.

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