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Robust Design Guidelines for Model Reference Adaptive Control

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Abstract–**In this paper a robust design process is introduced for a scalar model reference adaptive control (MRAC) algorithm. Three different types of MRAC control rules are reviewed and analysed in the frequency domain. A design process for MRAC is given within the range of plant settling times 0.01-100 seconds which is relevant for a wide range of mechanical systems. By using this design method the MRAC system stability can be made robust in the presence of a range of disturbances or plant uncertainties. An example of applying this method to hydraulic shaking table model with unmodelled oil column resonance is given at the end.**

Keywords: Adaptive control, Nonlinear dynamics, Gain wind-up, Robustness, Stability, Tuning method

1 Introduction

In this paper a robust design process is introduced for a scalar MRAC algorithm, which can ensure a robust control system within a large frequency range. Our intention is to provide a series of control design guidelines for the scalar MRAC system so that robust design can be achieved for a range of mechanical plants. Detailed discussions on MRAC are given by [1, 2, 3]. This work builds on the previous work on modifying the MRAC algorithm to improve robustness [4, 5]. The formulations of MRAC will be reviewed in section 2. Five different MRAC adaptive gain rules will be studied in section 3. In section 4 the robust design process will be explained step by step, and finally an example by applying this design process will be given in section 5.

2 MRAC brief review

In this section a brief review of the MRAC method is given for a single input single output (SISO) system. For more detailed discussions of MRAC see [1, 2, 3].

The system studied in this paper is based on a first-order linear plant approximation given by

$$\dot{x}(t) = -ax(t) + bu(t), \tag{1}$$

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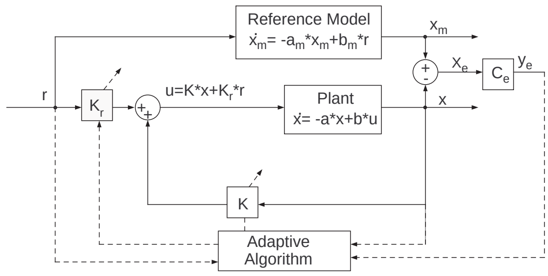

Figure 1: Schematic block diagram of the model reference adaptive control system. K and K_r are the gains from adaptive algorithm.

where $x(t)$ is the plant state, $u(t)$ is the control signal and a and b are the plant parameters. The control signal is generated from both the state variable and the reference (or demand) signal $r(t)$, multiplied by the adaptive control gains K and K_r, such that

$$u(t) = K(t)x(t) + K_r(t)r(t), \tag{2}$$

where, $K(t)$ is the feedback adaptive gain and $K_r(t)$ the feed forward adaptive gain. The plant is controlled to follow the output from a reference model

$$\dot{x}_m(t) = -a_m x_m(t) + b_m r(t), \tag{3}$$

where x_m is the state of the reference model and a_m and b_m are the reference model parameters which are specified by the controller designer. The block diagram of MRAC is illustrated by Fig. 1.

The object of the MRAC algorithm is for $x_e \to 0$ as $t \to \infty$, where $x_e = x_m - x$ is the error signal. The dynamics of the system can be rewritten in terms of the error such that

$$\dot{x}_e = (-a + bK)x_e + b(K^E - K)x_m + b(K_r^E - K_r)r, \tag{4}$$

where K^E and K_r^E are Erzberger gains. The Erzberger gains are defined as the linear gains which results in the plant response matching the reference model response [6]:

$$K^E = \frac{a - a_m}{b}, \quad K_r^E = \frac{b_m}{b}. \tag{5}$$

3 Different MRAC adaptive gain rules

In this section three different MRAC adaptive gain rules, MIT, Hyperstability and $\rho\ \phi$ modified MRAC version, i.e., will be reviewed, and all the control rules will be illustrated and explained based on their Bode plots.

3.1 MRAC by using the MIT rule

A basic method of MRAC adaptive gain is known as MIT rule [1], which is using an integral control in the adaptive gains, and designed by using Lyapunov stability theory. The formulation can be described as

$$\begin{aligned} K(t) &= \alpha \int_0^t C_e x_e x(\tau)\mathrm{d}\tau + K_0, \\ K_r(t) &= \alpha \int_0^t C_e x_e r(\tau)\mathrm{d}\tau + K_{r0}, \end{aligned} \tag{6}$$

where α is adaptive control weight representing the adaptive effort and K_0 and K_{r0} are the initial gain values and C_e can be chosen to ensure the stability of the feed forward block [2]. In the case of a first-order implementation, C_e is a scalar and therefore may be incorporated into the α adaptive control weight.

In general the initial gain values, K_0 and K_{r0}, can be set to zero such that the controller requires no knowledge of the plant parameters a and b [7] — although this generally leads to slower convergence to final gain values, or the Erzberger gains estimated based on a first-order system identification of the plant. The adaptive weight, α need to be selected in advance, and clearly have a significant influence on the rate of adaptation.

By using Laplace transform given zero initial conditions, Eq. 6 becomes,

$$\begin{aligned} (K(s) - K_0)/P_1(s) &= \alpha/s, \\ (K_r(s) - K_{r0})/P_2(s) &= \alpha/s, \end{aligned} \tag{7}$$

where s is the Laplace transform variable and

$$\begin{aligned} P_1(s) &= C_e X_e(s) X(s), \\ P_2(s) &= C_e X_e(s) R(s). \end{aligned} \tag{8}$$

If P_1 or P_2 is considered as overall input signal, Eq. 8 describes the relationship between output, the adaptive gains, and input P_1 and P_2. As we can see the transfer function on the right hand side are identical. A point to be noted from the transfer function is that when $s = \alpha$ the $output/input = 1$. In the Bode plot since s represents frequency ($s = jw$), at frequency of $\omega = \alpha$ rad/sec the magnitude is $0dB$. We also noticed there is a zero pole, which makes the transfer function marginally stable.

Transfer function Eq. 8 is illustrated as a Bode plot by Fig. 2. An example of $\alpha = 1$ is given in this figure. As we can see at frequency of $\alpha = 1$ rad/sec the magnitude is $0dB$, and the input with frequency higher than 1 rad/sec is minimised in the output signal — the adaptive control gains. This $0dB$ frequency can be adjusted by choosing different α values.

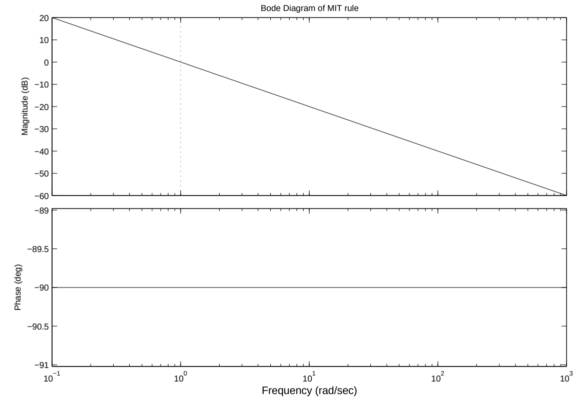

Figure 2: Transfer function of adaptive control gain by using MIT rule (integral control), adaptive weighting $\alpha = 1$. The vertical dot line indicates at α rad/sec the magnitude is 0 dB.

3.2 MRAC by using the hyperstability rule

In most of MRAC applications, the adaptive gains defined by using Hyperstability rule are considered as a standard case. The Hyperstability rule defines the adaptive gains with a proportional plus integral formulation as

$$\begin{aligned} K(t) &= \alpha \int_0^t Cex_e x(\tau)\mathrm{d}\tau + \beta C_e x_e x(t) + K_0, \\ K_r(t) &= \alpha \int_0^t C_e x_e r(\tau)\mathrm{d}\tau + \beta C_e x_e r(t) + K_{r0}, \end{aligned} \tag{9}$$

where α and β are adaptive weights. In a first-order implementation C_e can be incorporated into the α and β. The adaptive gain $K(t)$ can be written in the s-plane by using Laplace transform as

$$\frac{K(s) - K_0}{P_1(s)} = \frac{\beta(s + \alpha/\beta)}{s}. \tag{10}$$

The Laplace transform result of $K_r(t)$ is identical as the right hand side of Eq. 10. To keep the description simple we only analyse the adaptive gain K. In Eq. 10 the break frequency $\omega = \alpha/\beta$ rad/sec need to be noted, since a steady-state β exists in the frequency higher than this value.

Eq. 10 is illustrated by Bode plot Fig. 3. An example of $\alpha = 1$, $\beta = 0.1$ is given in this figure. As we can see after the break frequency $\alpha/\beta = 10$ rad/sec the magnitude becomes a steady state gain of $20log_{10}(\beta) = -20$. Comparing with MIT rule, the Hyperstability rule allows the designer to adjust the break frequency by using α/β ratio, and set the $output/input$ to a fixed magnified ratio β in the frequency higher than α/β rad/sec.

3.3 MRAC with $\rho\ \phi$ modification

MRAC with $\rho\ \phi$ modification is based on Hyperstability rule, and a combination of ρ and ϕ modified MRAC algorithm. The purpose of developing this strategy is to

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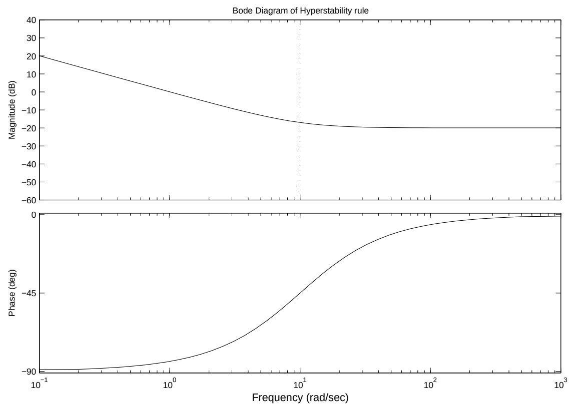

Figure 3: Transfer function of adaptive control gain by using Hyperstability rule (proportional plus integral). The vertical dot-line indicates the frequency α/β rad/sec. The adaptive weights are $\alpha = 1$, $\beta = 0.1$.

create an 'adaptive window' such that adaptive gain control power can be focused within this frequency window. It can eliminate the adaptive control gain wind-up problem caused by disturbance [8, 5] and instability problem caused by unexpected high frequency dynamics in control plant [4]. MRAC with ρ ϕ modification can be described by equations as

$$\begin{aligned} K_m(s) = \quad & \tfrac{\phi^2 s}{(s+\rho^2)(s+\phi^2)} K(s) + \tfrac{\rho^2}{s+\rho^2} K^*(s) \\ & + \tfrac{s^2}{(s+\rho^2)(s+\phi^2)} K^*(s), \\ K_{rm}(s) = \quad & \tfrac{\phi^2 s}{(s+\rho^2)(s+\phi^2)} K_r(s) + \tfrac{\rho^2}{s+\rho^2} K_r^*(s) \\ & + \tfrac{s^2}{(s+\rho^2)(s+\phi^2)} K_r^*(s), \end{aligned} \tag{11}$$

where $()_m$ denotes modified MRAC. ρ and ϕ are constants needed to be selected by the designer, and $K^*(s)$ and $K_r^*(s)$ are steady state gains, which ideally are equal to the Erzberger Gains. $K(t)$ and $K_r(t)$ are Hyperstability MRAC control gains described by Eq. 9.

It can be noticed from Eq. 11 that the first term on the right hand side with $K(s)$ is the adaptive gain control part, and the second and third terms are steady-state gain control based on $K^*(s)$. The whole structure can be illustrated by Fig 4. An adaptive window is created between ρ^2 and ϕ^2 rad/sec. The steady-state gain control is used outside the window. The ρ term cuts off the adaptive power to low frequency, and term ϕ cuts off the adaptive power to high frequency. The same situation can be found from modified gain K_{rm}. The following research is concerned with the adaptive gain control part. The transfer function of adaptive gain control part can be derived by substituting Eq. 10 into K_m in Eq. 11

$$\frac{K_m(s) - K_0}{P_1(s)} = \frac{\phi^2 \beta (s + \alpha/\beta)}{(s + \rho^2)(s + \phi^2)}. \tag{12}$$

It can be noticed from Eq. 12 that both poles $-\rho^2$ and $-\phi^2$ are negative given ρ ϕ are non-zero. Hence the adaptive gain is asymptotically stable rather than marginally stable in the Hyperstability MRAC. The MRAC with ρ ϕ modification has three break frequencies ρ^2, α/β and ϕ^2 rad/sec. Two stead-states α/ρ^2 and β are corresponding to the frequency ranges of (0, ρ^2) and (α/β, ϕ^2). The transfer function Eq. 12 can be illustrated by Bode plot Fig. 5. Fig. 5 shows the example of $\rho^2 = 1$, $\alpha/\beta = 10$ and $\phi^2 = 10^3$ rad/sec. In the frequency lower than $\rho^2 = 1$ rad/sec the adaptive gain control part can be set to a steady-state of $\alpha/\rho^2 = 1$(0dB), and in frequency between $\alpha/\beta = 10$ and $\phi^2 = 10^3$ rad/sec the adaptive gain control part can be set to the steady-state $\beta = 0.1$(-20dB). In frequency higher than $\phi^2 = 10^3$ rad/sec the adaptive power is cut off.

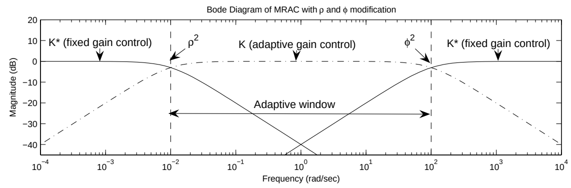

Figure 4: The structure diagram of MRAC with ρ ϕ modification. Dash-dot line illustrates the adaptive gain control part. Solid line illustrates the steady-state gain control part. Two vertical dash line illustrate ρ^2 and ϕ^2 separately.

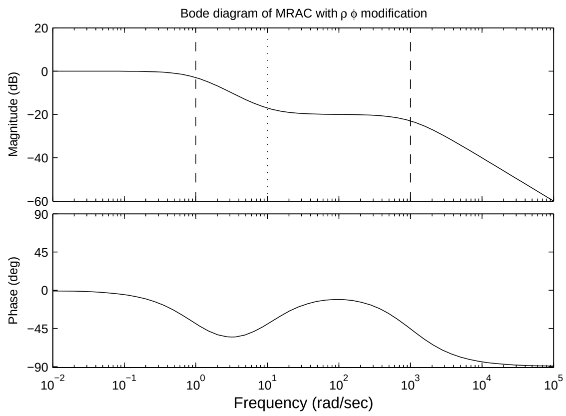

Figure 5: Transfer function of the adaptive gain control part of MRAC by using ρ and ϕ modification. Vertical dash line on left and right illustrate the frequency of ρ^2 and ϕ^2 rad/sec correspondingly, and vertical dot line is the frequency of α/β rad/sec. The adaptive weights are $\alpha = 1$, $\beta = 0.1$, $\rho^2 = 1$ rad/sec and $\phi^2 = 10^3$ rad/sec.

Comparing with the standard MRAC strategy, the ρ ϕ modified version gives the designer more options to specify the behaviour of the controller in the frequency domain. Depending on the desired performance, the designer can select the parameters α, β, ρ and ϕ to obtain a robust adaptive window — detailed below.

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4 MRAC robust design process

In this section a robust design process will be introduced by using MRAC strategy. The study here is mainly concerned on the plants that can be estimated by using first order transfer function, but could be combined with disturbance and unmodelled higher order dynamics. The design process can be taken in four steps, concerning on the plant parameter, the reference model, adaptive weights and adaptive window.

4.1 Step1: Find out the plant break frequency a by doing system ID within the operation frequency

The robust MRAC design process needs to know a limited amount knowledge of the plant. This is the normal trade off between robustness and *aprior* knowledge.

The break frequency a or settling time t_{sp}, i.e.,first order plant, needs to be known or estimated. (The relationship of $a = 4/t_{sp}$ can be used to convert t_{sp} to a.) Given the operation frequency range is known, a or t_{sp} can be found out by using chirp signal to do system identification. Then the model of the first order plant can be given as $X(s)/R(s) = b/(s + a)$.

4.2 Step2: Decide the reference model break frequency a_m

The key parameter in the reference model is the break frequency a_m or settling time t_s. It is possible to have reference model settling time t_s slightly faster than plant settling time t_{sp}, so that the control system can be forced to settle faster than it's open loop response. But if t_s is set too fast, the control system will become unstable [1]. The relationship between t_s and t_{sp} is studied against a range of plant settling time 0.01-100 seconds, or plant break frequency 0.04-400 rad/sec. To make the results clearer in frequency domain, the ratio between t_{sp} and t_s has been convert to the ratio of a_m and a, by using relationship $t_{sp}/t_s = a_m/a$.

All tests in section 4 is processed using the conditions: 1) input frequency is equal to plant break frequency, 2) sampling time for control system is smaller than 10% plant settling time, 3) disturbance is included in the testing plant, i.e., 5% white Gaussion noise, 4) stability checked by control gains and error converge within two times plant settling time. Fig. 6 shows the relationship between a_m/a and a. It can be noticed from Fig. 6 that as plant break frequency a increases, a_m/a ratio for the stable system decrease. The recommended values in each frequency ranges are

- $a_m/a = 4.5e^{-log_{10}(a)}$ is recommended when $a \in$ (0.04 0.4)rad/sec or $t_{sp} \in$ (10 100)sec.

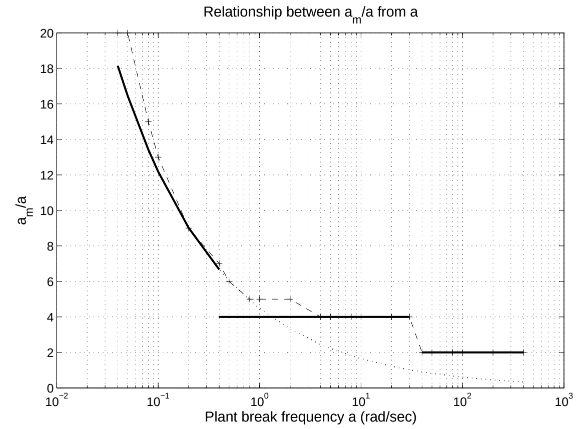

Figure 6: The relationship between a_m/a and plant break frequency a. The dash-line with + marks shows the original data of a_m/a for stable MRAC against a rad/sec, and area below it is stable. The dot-line illustrates the exponential curve fitting. The black lines illustrate the recommended values in each frequency ranges.

- $a_m/a = 4$ is recommended when $a \in$ (0.4 40)rad/sec or $t_{sp} \in$ (0.1 10)sec.
- $a_m/a = 2$ is recommended when $a \in$ (40 400)rad/sec or $t_{sp} \in$ (0.01 0.1)sec.

The Erzberger gains can be calculated now as by Eq. 5, and K^* and K_r^* are set to Erzberger gains.

4.3 Step3: Select adaptive weights α, β.

In this step we first select α/β value and then α. β value can be calculated by using α/β and α. The relationship between α/β and a is tested in the range of plant break frequency 0.04-400 rad/sec by fixing $\alpha = 1$ and checking α/β stability value. In different frequency ranges the corresponding recommended a_m values are used. Fig. 7 illustrates the results. As a increases in frequency α/β becomes large. Since α is fixed the β value decreases and the effective adaptive power decreases. It can to be noticed in Fig. 7 that the exponential equation fit well in the high frequency range of (40 400)rad/sec but not the low frequency part. Experiment results show that the following recommended values are effective to ensure the system robustness as well as small error.

- $\alpha/\beta = a$ is recommended when $a \in$ (0.04 40)rad/sec or $t_{sp} \in$ (0.1 100)sec.
- $\alpha/\beta = 0.2e^{2log_{10}(a)}$ is recommended when $a \in$ (40 400)rad/sec or $t_{sp} \in$ (0.01 0.1)sec.

Now we find the relationship between α and a by using the recommended α/β values. The result is illustrated by Fig. 8. Here the whole frequency range is divided

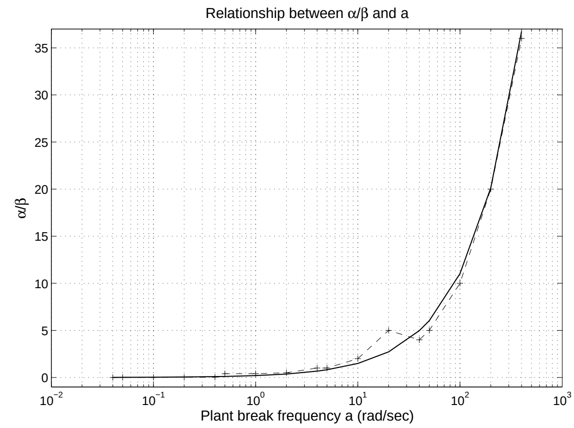

Figure 7: The relationship between α/β and plant break frequency a as $\alpha = 1$. The dash-line with + marks shows the original data of α/β for stable MRAC against a rad/sec, and area above it is stable. The solid line illustrates the exponential curve fitting with equation $\alpha/\beta = 0.2e^{2log_{10}(a)}$.

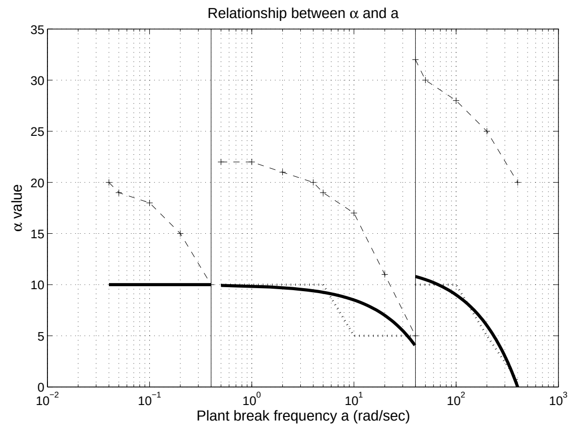

Figure 8: The relationship between α and plant break frequency a. Two vertical lines divide the whole plot into three frequency sections. In each sections different a_m/a and α/β are chosen by following the previous recommended values. The dash-line with + marks shows maximal α for stable MRAC. The dot-line illustrates the α values which gives minimal error. The dark-solid lines show the recommended α values.

into three sections, 0.04-0.4, 0.4-40 and 40-400 rad/sec, and in each section the corresponding recommended a_m and α/β values are used. In each section as a increases the maximal α for stable system decreases. In general as the adaptive weight α increases control system settles fast and error decreases, but too large α will cause the system instability. One way to choose the α value is to use the value which gives the minimal error, as recommended here

- $\alpha = 10$ is recommended when $a \in (0.04\ 0.4)$rad/sec or $t_{sp} \in (10\ 100)$sec.
- $\alpha = -0.15a + 10$ is recommended when $a \in (0.4\ 40)$rad/sec or $tsp \in (0.1\ 10)$sec.
- $\alpha = -0.03a + 12$ is recommended when $a \in (40\ 400)$rad/sec or $tsp \in (0.01\ 0.1)$sec.

Some other factors could also affect choosing the adaptive weights, the input frequency ω_r, i.e. With ω_r close to the plant break frequency a the maximum adaptive weights that can be chosen for a stable system increases. But the effect was not found to be significant.

4.4 Step4: Designing the adaptive window

In this step an adaptive window will be selected, and by using this window the designer can limit the adaptive power only to the interesting frequency range, and cut off the adaptive power in both low and high frequency disturbance or unexpected dynamics, and the robustness of overall control system is ensured [4]. The principal rule of setting the adaptive window is that a, a_m, α/β, and input frequency ω_r should be included in the window defined by ρ^2, ϕ^2. The key results of this design process are summarized in table 1.

5 Experiment by using MRAC design process

In this section an example will be given by applying the MRAC design process on a reconfigurable electrical circuit, a Quansar analog plant simulator. By using it an hydraulic shaking table model is created as

$$G(s) = \frac{2.108}{(s+1.147)}\frac{229.521}{(s^2+2.895s+231.516)}, \qquad (13)$$

where the nominal first order plant is $2.108/(s + 1.147)$, and high order dynamics part is $229.521/(s^2 + 2.895s + 231.516)$ which represents the oil column resonance in the hydraulic shaking table. By carrying out system identification to operation frequency range 0-10 Hz, the break frequency of the nominal plant $a = 1.147$ can be found. The input signal is sinusoid wave given as $r(t) = 0.3 + 1.85\sin(0.8t)$.

Since this plant has unmodelled high frequency dynamics and electronic noise, by using standard MRAC the control system become unstable within 15 sec, Fig. 9. Using the MRAC design process, since $a \in (0.4\ 40)$rad/sec the following parameters can be chosen by following the design process as

- $a_m = 4a = 4$,
- $\alpha/\beta = a = 1$,
- $\alpha = -0.15a + 10 = 9.85$,

Table 1: MRAC parameter design table

Parameters	Recommend values
a	system ID
$a \in (0.04\ 0.4)$rad/sec	
a_m	$4.5e^{-log_{10}(a)} * a$
α/β	a
α	10
$a \in (0.4\ 40)$rad/sec	
a_m	$4a$
α/β	a
α	$-0.15a + 10$
$a \in (40\ 400)$rad/sec	
a_m	$2a$
α/β	$0.2e^{2log_{10}(a)} * a$
α	$-0.03a + 12$
Steady-state gains	$K^* = \frac{a-a_m}{b}$, $K_r^* = \frac{b_m}{b}$
ρ^2,ϕ^2	ρ^2 <(a,a_m,r_w,α/β)< ϕ^2

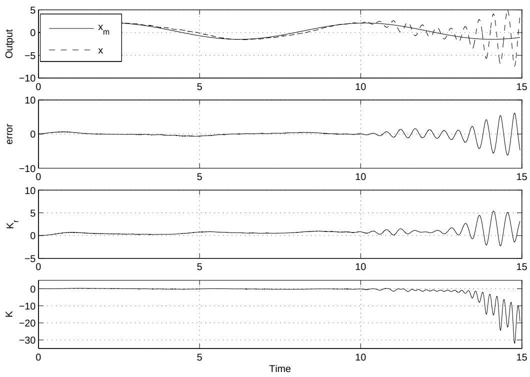

Figure 9: Plant with unmodelled high frequency dynamics, damping ratio 0.1, standard MRAC. Input signal r(t)=0.3+1.85sin(1t), $\alpha = \beta = 0.5$. System is unstable.

- $K^* = (a - a_m)/b = -0.879$, $K_r^* = b_m/b = 1.423$,
- $\rho^2 = 0.5$, $\phi^2 = 5$.

Fig. 10 shows the control result by following the design process. In this case the system is stable and robust, related examples are discussed by [4, 5].

6 Conclusions

In this paper we have introduced a robust design process for a scalar model reference adaptive control (MRAC) algorithm. Different types of MRAC control rules have been reviewed and analysed in the frequency domain by using Bode plot analysis. Then a design process for MRAC has been developed for systems with plant settling times in the range 0.01-100 seconds which is relevant to a wide range of mechanical systems. By using this design method a robust adaptive window is obtained meaning

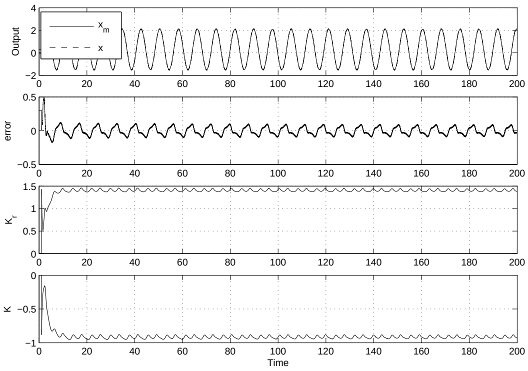

Figure 10: Plant with unmodelled high frequency dynamics, damping ratio 0.1, controlled by $\rho\ \phi$ modified MRAC. Input signal r(t)=0.3+1.85sin(0.8t), $\alpha = \beta = 9.85$, $\rho^2 = 0.5$, $\phi^2 = 5$. System is unstable, and error and gains settle within around 10 seconds.

that the system is robust in the presence of noise or unmodelled dynamics. An example of applying this method to hydraulic shaking table model with unmodelled oil column resonance is given to demonstrate the usefulness of the design process.

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References

[1] Karl J. Aström and Björn Wittenmark. *Adaptive control.* Addison-Wesley, second edition, 1995.

[2] Y. D. Landau. *Adaptive control:The model reference approach.* Marcel Dekker:New York, 1979.

[3] S. Sastry and M. Bodson. *Adaptive control:Stability, convergence and robustness.* Prentice-Hall:New Jersey, 1989.

[4] L. Yang, S.A. Neild, and D.J. Wagg. Modified model reference adaptive control for plants with unmodelled high frequency dynamics. In *ICINCO Int Conf Informatics in Control, Automation & Robotics*, pages 196 – 201, Angers, France., May 2007.

[5] L. Yang, S.A. Neild, D.J. Wagg, and D.W. Virden. Model reference adaptive control of a nonsmooth dynamical system. *Nonlinear Dynamics*, 46(3):323–335, 2006.

[6] H. K. Khalil. *Nonlinear Systems.* Macmillan:New York, 1992.

[7] D. P. Stoten and M. Di Bernardo. Application of the minimal control synthesis algorithm to the control and synchronization of chaotic systems. *International Journal of Control*, 65(6):925–938, 1996.

[8] L. Yang. A modified model reference adaptive control algorithm to improve system robustness. Master's thesis, University of Bristol, 2004.

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