Numerical Method for Solving Heat Equation with Derivative Boundary Conditions

A. Cheniguel

Abstract— **In this paper Adomian decomposition method is studied and used for solving the non homogeneous heat equation, with derivative boundary conditions. The results obtained show that the numerical method based on the proposed technique gives us the exact solution. These results are more accurate and efficient in comparison to previous methods.**

Index Terms—**Adomian decomposition, method, derivative boundary conditions, inverse operator, exact solution, numerical methods for partial differential equations.**

I. Introduction

Many Authors have proposed numerical methods for solving nonlocal problems [6-11]. Later Akram[1] proposed an $O(h^3+t^3)L_0$-*stable* Parallel Algorithm for solving the problem. We consider here the one dimensional non homogeneous heat equation with derivative boundary conditions given by:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+q(x,t)\ 0<x<1, 0<t\leq T \quad (1)$$

Subject to

$$u(x,0)=g(x), 0\leq x\leq 1 \quad (2)$$

And

$$u_x(0,t)=f_1(t), 0<t\leq T \quad (3)$$

$$u_x(1,t)=f_2(t), 0<t\leq T \quad (4)$$

Where $g(x), f_1(t), f_2(t), q(x,t)$, are sufficiently smooth known functions and T is a given constant.

In this work, we propose a new technique based on the Adomian decomposition series solution [2,4]. The numerical examples show that results obtained coincide with the exact ones. The organization of this paper is as follows: in section 2, we give a brief definition of the method, in section 3, the accuracy and the efficiency of the Adomian decomposition method are investigated with numerical illustration.

II. Adomian Decomposition Method

A. Operator form

In this section we outline the steps to obtain a solution of problem (1)-(4) using Adomian decomposition method, which was initiated by G. Adomian [9-11]. For this purpose, it is convenient to rewrite the problem in the standard form:

Manuscript received January 05, 2011; revised March 11, 2011.
A. Cheniguel is with Department of Mathematics and Computer Science, Faculty of Sciences, Kasdi Merbah University Ouargla, Algeria (e-mail: cheniguelahmedl@yahoo.fr)

$$L_t(u)=L_{xx}(u)+q(x,t) \quad (5)$$

Where the differential operators $L_t(.)=\frac{\partial}{\partial t}(.)$ and $L_{xx}=\frac{\partial^2}{\partial x^2}$ assuming that the inverse L_t^{-1} exists and is defined as:

$$L_t^{-1}=\int_0^t(.)dt \quad (6)$$

Applying inverse operator on both the sides of (5) and using the initial condition, yields:

$$L_t^{-1}\big(L_t(u(x,t))\big)=L_t^{-1}\big(L_{xx}(u(x,t))\big)+L_t^{-1}\big(q(x,t)\big) \quad (7)$$

B. Application to the solution of the problem

From the above we obtain:

$$u(x,t)=u(x,0)+L_t^{-1}\big(L_{xx}(u(x,t))\big)+L_t^{-1}\big(q(x,t)\big) \quad (8)$$

Now, we decompose the unkown function *u(x,t)* as a sum of components defined by the series [12] :

$$u(x,t)=\sum_{k=0}^{\infty}u_k(x,t) \quad (9)$$

Where u_0 is identified as *u(x; 0)*. The components $u_k(x,t)$ are obtained by the recursive formula:

$$\sum_{k=0}^{\infty}u_k(x,t)=g(x)+L_t^{-1}\{L_{xx}(\sum_{k=0}^{\infty}u_k(x,t))\}+L_t^{-1}(q(x,t)) \quad (10)$$

Or:

$$u_0(x,t)=g(x)+L_t^{-1}(q(x,t)) \quad (11)$$

And:

$$u_{k+1}(x,t)=L_t^{-1}\Big(L_{xx}\big(u_k(x,t)\big)\Big),\ k\geq 0 \quad (12)$$

We note that the recursive relationship is constructed on the basis that the component $u_0(x,t)$ is defined by all terms that arise from the initial condition and from integrating the source term. The remaining components $u_k(x,t), k\geq 1$, can be completely determined recursively. Accordingly, considering the first few terms, relations (11) and (12) give:

$$\begin{aligned} u_0&=g(x)+L_t^{-1}(x,t)\\ u_1&=L_t^{-1}(L_{xx}(u_0(x,t)))\\ u_2&=L_t^{-1}(L_{xx}\big(u_{1(x,t)}\big))\\ u_3&=L_t^{-1}(L_{xx}(u_2(x,t)))\end{aligned} \quad (13)$$

and so on. As a result, the components u_0 ,$u_1, u_2, \ldots$ are identified and the series solution thus entirely determined. However, in many cases the exact solution in a closed form may be obtained for numerical purposes, we can use the approximation:

$$u(x,t)=\lim_{m\to\infty}\psi_m. \quad (14)$$

Where:

$$\psi_m=\sum_{k=0}^{m-1}u_k(x,t) \quad (15)$$

Evaluating more components of *u(x; t)* we obtain a more accurate solution. Noting that the convergence of this method has been proved in [2,4], Adomian and Rach [3]

ISBN: 978-988-19251-7-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

and Wazwaz [5] have investigated the phenomena of the self-canceling "noise" terms where sum of components vanishes in the limit. We observe that "noise" terms appear for non homogeneous cases only.

III. EXAMPLES

A. Example 1

We consider the nonhomogeneous heat equation:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+q(x,t)\ 0<x<1, 0<t\leq T$$
$$q(x,t)=-2e^{x-t},\qquad 0<x<1,\qquad 0<t\leq 1$$
$$u(x,0)=g(x)=e^x, 0\leq x\leq 1 \tag{16}$$
$$u_x(0,t)=f_1(t)=e^{-t}, 0<t\leq T$$
$$u_x(1,t)=f_2(t)=e^{1-t}, 0<t\leq T$$

We rewrite this problem in an operator form. Applying the above development, we obtain the first element as:

$$u_0(x,t)=g(x)+L_t^{-1}(q(x,t)) \tag{17}$$

After calculation, this yields:

$$u_0(x,t)=e^x+L_t^{-1}(-2e^{x-t})=e^x(-1+2e^{-t}) \tag{18}$$

The remaining elements are obtained through:

$$u_{k+1}(x,t)=L_t^{-1}\left(L_{xx}\big(u_k(x,t)\big)\right), k>0 \tag{19}$$

Applying this formula to the first three components yields:

$$u_1(x,t)=e^x(2-t-2e^{-t}) \tag{20}$$
$$u_2(x,t)=e^x(-2+2t-\tfrac{t^2}{2}+2e^{-t}) \tag{21}$$
$$u_3(x,t)=e^x(2-2t+t^2-\tfrac{t^3}{3!}-2e^{-t}) \tag{22}$$

Then the solution in the series form is given by:

$$u(x,t)=\sum_{k=0}^{\infty}u_k(x,t)$$

With the above results:

$$u(x,t)=e^x\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\cdots\right)=e^{x-t} \tag{23}$$

This solution is the exact one.

B. Example 2

In this second example, we consider the problem with the following conditions

$$q(x,t)=xt^2,\ 0<x<1,\ 0<t\leq 1$$
$$u(x,0)=sin(x)=0\leq x\leq 1 \tag{24}$$
$$u_x(0,t)=1\ 0<t\leq T$$
$$u_x(1,t)=sin(t)\ 0<t\leq T$$

Applying equation (11), we obtain the zeroth element as:

$$u_0(x,t)=\sin(x)+L_t^{-1}(xt^2)=\sin(x)+x(\tfrac{t^3}{3}) \tag{25}$$

Again the remaining components are obtained through:

$$u_{k+1}(x,t)=L_t^{-1}\left(L_{xx}\big(u_k(x,t)\big)\right), k>0$$

After calculations, we obtain the first three components:

$$u_1(x,t)=\int_0^t -\sin(x)\,dt=-tsin(x) \tag{26}$$
$$u_2(x,t)=\int_0^t tsin(x)dt=\frac{t^2}{2!}\ \sin(x) \tag{27}$$
$$u_3(x,t)=\int_0^t -\frac{t^2}{2}sin(x)dt=-\frac{t^3}{3!}\sin(x) \tag{28}$$

And so on.
Then the solution in the series form is given by:

$$u(x,t)=\sum_{k=0}^{\infty}u_k(x,t)$$

That is:

$$u(x,t)=\frac{t^3}{3}x+\left(1-\frac{t}{1!}+\frac{t^2}{2!}-\frac{t^3}{3!}+\cdots\right)\sin(x) \tag{29}$$

Or:

$$u(x,t)==\frac{t^3}{3}x+e^{-t}\sin(x) \tag{30}$$

This is the exact solution.

C. Example 3

Here, we consider the problem with the following boundary and initial conditions:

$$q(x,t)=0,\qquad 0<x<1,\qquad 0<t\leq 1$$
$$u(x,0)=sin(\pi x)=0\leq x\leq 1 \tag{31}$$
$$u_x(0,t)=\pi e^{-\pi^{2t}}\ 0<t\leq T$$
$$u_x(1,t)=-\pi e^{\pi^{2t}}\ 0<t\leq T$$

After rewriting the problem in operator form, we obtain the zeroth component as

$$u_0(x,t)=\sin(\pi x) \tag{32}$$

And the first three components are obtained as:

$$u_1(x,t)=L_t^{-1}\left(L_{xx}\big(u_0(x,t)\big)\right)=-\pi^2tsin(\pi x) \tag{33}$$
$$u_2(x,t)=L_t^{-1}\left(L_{xx}\big(u_1(x,t)\big)\right)=\frac{t^2}{2!}\pi^4\sin(\pi x) \tag{34}$$
$$u_3(x,t)=L_t^{-1}\left(L_{xx}\big(u_2(x,t)\big)\right)=-\frac{t^3}{3!}\pi^6\sin(\pi x) \tag{35}$$

And so on.
Then the solution in the series form is given by:

$$u(x,t)=\sum_{k=0}^{\infty}u_k(x,t)$$

Using the above developments we obtain:

$$u(x,t)=\sin(\pi x)\left(1-\frac{\pi^2 t}{1!}+\frac{\pi^4t^2}{2!}-\frac{\pi^6t^3}{3!}+\cdots\right) \tag{36}$$

Which gives the exact solution

$$u(x,t)=\sin(\pi x)e^{-\pi^2 t} \tag{37}$$

IV. CONCLUSION

The results obtained in this paper compared to those obtained by Akram[1] show that the Adomian decomposition method is more accurate, the computation of the components of the solution are easy and take less time in comparison with other classical methods.

REFERENCES

[1] M. Akram, A parallel algorithm for the heat equation with derivative boundary conditions, Intern. Mathematical forum, vol. 2, n° 12, pp. 565 – 574, 2007.
[2] G.Adomian;, *Solving Frontier Problems of Physics : The Decomposition Method,* Kluver Academic Publishers, Dordrecht, 1994.
[3] G.Adomian and R.Rach, "Noise terms in decomposition solution series," Computers Math. Appl. vol. 24, n° 11, pp. 61 -64, 1992.
[4] G.Adomian, " A review of the decomposition method in applied mathematics," J.Math.Anal.Appl. vol. 135, pp. 501 -544, 1988.
[5] A.M.Wazwaz "Necessary conditions for the appearance of noise terms in decomposition solution series," J.Math,Anal.Appl, vol. 5, 1997, pp. 265-274.
[6] M. Deghan, "On the numerical solution of diffusion equation with a nonlocal boundary conditions," Math. Prob. In Eng. Vol. 2, pp.81 – 92, 2003.

ISBN: 978-988-19251-7-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

[7] M. A.Rehman and M. S.A.Taj, " Fourth -order Method for non – homogeneous Heat equation with nonlocal Boundary conditions," Appl Math Sciences, vol. 3, N°37, pp. 1811-1821, 2009.
[8] Z. Sun, "A high order difference scheme for a nonlocal boundary value problem for the heat equation," Computatinal methods in applied mathematics, vol. 1, pp.398 – 414, 2001.
[9] M. Akram and M. A. Pasha, "A numerical method for the heat equation with a nonlocal boundary condition," Intern. Jour. of Information and systems sciences, vol. 1 n°2, pp.162- 171, 2005.
[10] M.Siddiqe., "Numerical Computation of Two-dimensinal Di¤usion equation with nonlocal boundary condions," IAENG International Journal of Applied Mathematics, vol. 40, N° 1, 2004.
[11] A. B. Gumel, "On the numerical solution of the diffusion equation subject to the specification of mass," J. Austral. Math.Soc. Ser. B 40, pp. 475 – 483, 1999.

ISBN: 978-988-19251-7-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)