Proceedings of the World Congress on Engineering and Computer Science 2012 Vol II
WCECS 2012, October 24-26, 2012, San Francisco, USA

Optimal Control Approaches for Some Geometric Optimization Problems

Dan Tiba

Abstract—**This work is a survey on optimal control methods applied to shape optimization problems. The unknown character of the domain where the state system is defined creates major difficulties both for the theoretical analysis and for the implementation of the numerical approximation methods.**

The optimal control and the controllability methods for elliptic equations allow the development of approximation procedures defined in a given domain that contains all the admissible subdomains for the original optimal design problem. This class of methods enters the so-called fixed domain methods that have numerous advantages in the treatment of unknown/variable domains problems. Such methods are very much studied in the scientific literature, both in the case of geometric optimization problems and in the case of free boundary problems.

Index Terms—**exact controllability, approximate controllability, distributed and boundary controls, elliptic equations, optimal design.**

I. Introduction

A typical geometric optimization problem to be discussed in this work is the following:

$$\operatorname*{Min}_{\Omega} \int_E j(x, y_\Omega(x))dx, \tag{1}$$

subject to

$$-\Delta y_\Omega = f \quad \text{in } \Omega, \tag{2}$$

$$y_\Omega = 0 \quad \text{on } \partial\Omega, \tag{3}$$

where $E \subset D \subset R^d$ are some given bounded domains, $j : E \times R \to R$ satisfies integrability assumptions, $f \in L^2(D)$ and Ω, with $E \subset \Omega \subset D$, is unknown and gives the minimization parameter in the problem (1) - (3). Other boundary conditions, more general elliptic operators or cost functionals may be considered as well. The problem (1) - (3) is an example of an optimal design or shape optimization problem. See the monographs of Pironneau [15], Sokolowski and Zolesio [16], Allaire [1], Neittaanmäki, Sprekels and Tiba [12], Pierre and Henrot [8] for basic references in this respect. The main difficulty, both from the theoretical and the computational points of view, is the unknown character of the domain Ω where the partial differential equation governing the problem, is defined. Similar difficulties arise in the study of free boundary problems. Classical techniques to "remove" the geometrical unknown (the free boundary) are the Baiocchi transform for the dam problem [2], the enthalpy method in Stefan problems, Kamenomostskaja [9], regularization in general variational inequalities, Barbu [3]. Then nonlinear partial differential equations have to be solved in a given fixed domain and the free boundary can be recovered, for instance, as a level set.

Manuscript received June 7, 2012, revised July 6, 2012.
This work was supported by Grant 145/2011 CNCS, Romania.
D. Tiba is with the Institute of Mathematics (Romanian Academy) and the Academy of Romanian Scientists, Bucharest, e-mail: dan.tiba@imar.ro.

In order to achieve a similar goal, that is to transform the geometrical optimization problem (1) - (3) into a minimization problems, optimal control and controllabilitty methods may be used. We give here just a simple example from Neittaanmäki and Tiba [13], adapted to the problem (1) - (3). Namely, we associate to it the boundary control problem:

$$\operatorname*{Min}_{u} \int_E j(x, y_u(x))dx, \tag{4}$$

subject to

$$-\Delta y_u = f \quad \text{in } D, \tag{5}$$

$$y_u = u \quad \text{on } \partial D, \tag{6}$$

$$u \le 0 \quad \text{on } \partial D, \tag{7}$$

$$y_u \ge 0 \quad \text{on } E. \tag{8}$$

One may ask supplementary assumptions on f, u such that the unique solution y of (5), (6) is continuous in D and the constraints (7), (8) make sense.

Proposition 1.1 The problem (4) - (8) is a subproblem of (1) - (3).

To each admissible control u, we associate the open set $\widetilde{\Omega}_u = \text{int}\{x \in D; y_u(x) \ge 0\}$ and the subdomain Ω_u given by the connected component of $\widetilde{\Omega}_u$ that contains E, $\Omega_u \supset E$. It is clear that the cost (4), corresponding to u, is equal with the cost (1) corresponding to Ω_u. The mapping $u \to \Omega_u$ is one-to-one due to the analyticity of harmonic functions in D. The set of admissible controls for the problem (4) - (8) is nonvoid iff $u \equiv 0$ is admissible (due to the comparison property for (5), (6)).

Remark The domain variation or boundary variation techniques, employed in the literature for the solution of (1) - (3), require in each iteration of the algorithm the definition of a new finite element mesh, the computation of the new mass matrix and are very time consuming. The advantage of the control problem (4) - (8) is that it is defined in the fixed domain D (which avoids remeshing) and it is convex. In [13], it is shown that under a certain controllability hypothesis one

ISBN: 978-988-19252-4-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

can prove a partial converse statement for P 1.1. In this sense, we may say that the solution u^* of (4) - (8) generates a *suboptimal* domain Ω_{u^*} for the shape optimization problem (1) - (3).

In the next section, we discuss various controllability properties for elliptic systems that are useful for shape optimization problems, including applications in free boundary problems. Section 3 is devoted to further applications of optimal control methods.

We end the paper with some short conclusions and hints on future work.

II. Controllability

We indicate several boundary controllability and distributed controllability properties for elliptic equations, with applications to problems involving unknown domains. We recall first the following approximate controllability result from the classical monograph of Lions [10]:

Theorem 2.1 Let $\Omega \in R^d$ be a bounded domain with smooth boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, $\bar\Gamma_1 \cap \bar\Gamma_2 = \emptyset$. For every $u \in L^2(\Gamma_1)$, let $y_u \in L^2(\Omega)$ be the unique solution (in the transposition sense) of the elliptic problem

$$\Delta y_u = 0 \quad in\ \Omega, \tag{9}$$

$$y_u = u\ on\ \Gamma_1;\ y_u = 0\ on\ \Gamma_2. \tag{10}$$

Then $\left\{ \dfrac{\partial y_u}{\partial n}; u \in L^2(\Gamma_1) \right\}$ forms a dense subspace in $H^{-1}(\Gamma_2)$.

Remark A constructive proof of this density result may be found in [12], §5.2.3. The interpretation as on approximate controllability property is a consequence of the fact that the range of the mapping $u \to \dfrac{\partial y_u}{\partial n}$ is dense in $H^{-1}(\Gamma_2)$. An important application is in the identification of the region occupied by the plasma in the void chamber of a tokamak, from outside measurements [5], [12], via an optimization method.

We consider now the case of distributed controls both for the Dirichlet problem (2), (3) and for the case of Neumann boundary conditions. In order to avoid working in unknown domains Ω, the idea is to extend (at least in an approximating sense) the boundary value problem (2), (3) to the given domain D. We denote by χ the characteristic function of Ω in D and we introduce the control system:

$$-\Delta y = f + (1-\chi)u \quad \text{in } D, \tag{11}$$

$$y = 0 \quad \text{on } \partial D, \tag{12}$$

with the cost functional

$$\frac{1}{2}\int_{D\setminus\Omega} y^2 dx. \tag{13}$$

Since the optimal control problem (11) - (13) is ill-posed in general (it may have no solution), a Tikhonov regularization of the cost is considered:

$$\frac{1}{2}\int_{D\setminus\Omega} y^2 dx + \frac{\varepsilon}{2}|u|^2_{L^2(D)},\ \varepsilon > 0. \tag{14}$$

We denote by $[y_\varepsilon, u_\varepsilon] \in [H^2(D) \cap H^1_0(D)] \times L^2(D)$ the unique optimal pair of (11), (12), (14) and by $p_\varepsilon \in H^2(D) \cap H^1_0(D)$ the corresponding adjoint state:

$$-\Delta p_\varepsilon = (1-\chi)y_\varepsilon \text{ in } D,\ p_\varepsilon = 0 \text{ on } \partial D. \tag{15}$$

The first order optimality conditions are given by (11), (12), (15) and

$$\varepsilon u_\varepsilon + (1-\chi)p_\varepsilon = 0 \text{ in } D. \tag{16}$$

One can eliminate u_ε from the above relations and obtain the system:

$$-\Delta y_\varepsilon = f - \frac{1}{\varepsilon}(1-\chi)^2 p_\varepsilon \text{ in } D, \tag{17}$$

$$-\Delta p_\varepsilon = (1-\chi)y_\varepsilon \text{ in } D, \tag{18}$$

$$y_\varepsilon = p_\varepsilon = 0 \text{ on } \partial D \tag{19}$$

which is equivalent with the optimal control problem (11), (12), (14).

Theorem 2.2 We have:

$$p_\varepsilon \to 0\ strongly\ in\ H^1(D\setminus\Omega), \tag{20}$$

$$y_\varepsilon \to 0\ weakly\ in\ L^2(D\setminus\Omega). \tag{21}$$

Remark Since χ is a characteristic function, then $(1-\chi)^2 = (1-\chi)$ and the system (17) - (19) can be written in a more symmetric form. The result of *Thm. 2.2* is an approximate extension result for the equation (2), (3). By the Mazur theorem, strong convergence may be obtained for sequences of convex combinations of the optimal controls. Notice that the extension system has a different form than the initial equation. This may be interpreted as a generalization of usual controllability properties.

In the case of Neumann boundary value problems, the situation is more difficult:

$$\int_\Omega \left[\sum_{i,j=1}^d a_{ij}\frac{\partial y}{\partial x_i}\frac{\partial v}{\partial x_j} + a_0 yv\right] dx = \int_\Omega fv dx, \tag{22}$$

$$\forall\, v \in H^1(\Omega),$$

ISBN: 978-988-19252-4-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

where $a_0, a_{ij} \in L^\infty(D)$ and $(a_{ij})_{i,j=\overline{1,d}}$ is an elliptic matrix. Denote $\Omega_0 = D \setminus \bar{\Omega}$ and assume that $\partial D \subset \bar{\Omega}_0$, that is Ω_0 is a relative neighbourhood of ∂D. Let $\Gamma = \partial\Omega_0 \setminus \partial D$.

We introduce again an optimal control problem:

$$\underset{u\in L^2(\Omega_0)}{\text{Min}} \left\{ \frac{1}{2}|y-w|^2_{H^{1/2}(\Gamma)} + \frac{\varepsilon}{2}|u|^2_{L^2(\Omega_0)} \right\},\ \varepsilon > 0, \quad (23)$$

$$\int_{\Omega_0} \left[\sum_{i,j=1}^{d} a_{ij} \frac{\partial y}{\partial x_i}\frac{\partial z}{\partial x_j} + a_0 y z \right] dx = \int_{\Omega_0} uz dx, \quad (24)$$
$$\forall\ z \in H^1(\Omega_0),$$

where $w \in H^{1/2}(\Gamma)$ is some given element. We denote by $[y_\varepsilon, u_\varepsilon]$ the unique optimal pair of (23), (24). Notice that (24) is the weak formulation of the homogeneous Neumann problem in Ω_0.

Theorem 2.3 We have $y_\varepsilon|_\Gamma \to w$ for $\varepsilon \to 0$, strongly in $H^1(\Omega_0)$.

Remark This approximate controllability result is a distributed variant of Theorem 2.1 since y_ε satisfies in a generalized sense the condition $\frac{\partial y_\varepsilon}{\partial n_A} = 0$ in Γ (the null conormal derivative) as well.

Under smoothness assumptions on Ω_0, w, one can apply directly the trace theorem in Sobolev spaces. The basic idea of *Thm. 2.3* is that y_ε can provide an approximate extension of the solution of (22) from Ω to D, if w is choosen appropriately, that is the trace on Γ of the solution of (22).

As in *Thm. 2.2*, we shall find a system of equations, defined in D, that approximately extends (22) to D. We consider the constrained control problem:

$$\underset{u\in L^2(D)}{\text{Min}} \left\{ \frac{1}{2}\int_\Omega u^2 dx \right\}, \quad (25)$$

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial y_u}{\partial x_i}\right) + a_0 y = f + (1-\chi)u \text{ in } D, \quad (26)$$

$$\frac{\partial y_u}{\partial n_A} = 0 \text{ on } D \quad (27)$$

with the state constraint

$$F(y_u) = \sum_{i,j=1}^{d} a_{ij}\frac{\partial y_u}{\partial x_j}(\nabla g \cdot e_i) = 0 \text{ on } \partial\Omega. \quad (28)$$

Above, χ is the characteristic function of Ω in D and it is assumed that

$$\Omega = \{x \in D; g(x) > 0\} \quad (29)$$

where $g \in C^1(D)$. Then, restriction (28) signifies that $\frac{\partial y_u}{\partial n_A} = 0$ on $\partial\Omega$. In the case of the Laplace operator, (28) has the simple form $\nabla g \cdot \nabla y = 0$ on $\partial\Omega$.

We penalize the state constraint (28) in the cost functional and we define the approximating unconstrained control problem

$$\underset{u\in L^2(D)}{\text{Min}} \left\{ \frac{1}{2}\int_D u^2 dx + \frac{1}{2\varepsilon}\int_{[g\equiv 0]} F(y_u)^2 d\sigma \right\},\ \varepsilon > 0 \quad (30)$$

subject to (26), (27). Here $[g \equiv 0]$ is a notation for $\partial\Omega$ defined in (29). This underlines the fact that the mapping g may be used as the main unknown instead of the unknown domain Ω.

The existence and uniqueness of the optimal pair $[y_\varepsilon, u_\varepsilon] \in H^2(D) \times L^2(D)$, if ∂D and a_{ij} are smooth enough, is standard. One can prove that $u_\varepsilon \to \hat{u}$ strongly in $L^2(D)$, $y_\varepsilon \to \hat{y}$ strongly in $H^2(D)$, where $[\hat{y}, \hat{u}]$ is the unique optimal pair for the control problem (25) - (28).

The adjoint equation and its solution $p_\varepsilon \in L^2(D)$ may be defined by the transposition method, Lions [10]:

$$\int_D p_\varepsilon \left[-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i}\right) + a_0 z \right] dx = \quad (31)$$
$$= \frac{1}{\varepsilon}\int_{[g\equiv 0]} F(y_\varepsilon)F(z) d\sigma,\ \forall\ z \in H^2(D),\ \frac{\partial z}{\partial n_A} = 0 \text{ on } \partial D.$$

The maximum principle has again the form (16) and one can eliminate the control u_ε from the state equation (26):

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial y_\varepsilon}{\partial x_i}\right) + a_0 y_\varepsilon = f - \frac{1}{\varepsilon}(1-\chi)p_\varepsilon \text{ in } D. \quad (32)$$

Remark Relations (31), (32) and the boundary condition (27) give the approximate extension of the Neumann problem (22) from Ω, given by (29), to D. It is possible, via a similar procedure, to obtain an approximating extension of (22) involving Dirichlet conditions on ∂D.

Remark Coming back to the shape optimization problem (1) - (3) (or with (3) replaced by the Neumann condition), we consider that the admissible class of domains Ω is defined by (29) with $g \in X \subset C^1(\bar{D})$ some given subset. The state system in Ω, given by (2) or, alternative, by (22) can be approximated by (17) - (19), respectively (31), (32). We note that $\chi = H(g)$, where H is the Heaviside function. It is possible to regularize $H(\cdot)$ by the Yosida approximation [3] of its maximal monotone extension in $R \times R$.

In this way the initial geometric optimization problem is transformed into an optimal control problem (by the coefficients) with respect to $g \in X$.

Remark Proofs, examples and more details on this analytic approach in shape optimization problems may be found in [11], [14]. For another fixed domain approach in optimal design, using finite elements, we quote [6], [17].

We close this section with an approximate controllability result for the coincidence set in variational inequalities of obstacle type. The classical formulation of this problem is:

$$\Delta y = 0 \text{ in } \{x \in D; y(x) > \varphi(x)\}, \tag{33}$$

$$\Delta y \leq 0 \text{ in } D, \tag{34}$$

$$y \geq \varphi \text{ in } D, \tag{35}$$

$$y = u \text{ on } \partial D. \tag{36}$$

Here D is a bounded domain in R^d, $d \geq 2$ with ∂D a smooth manifold of class C^m, $u \in H^{m-1/2}(\partial D)$ and $\varphi \in C^m(\bar{D})$ is the "obstacle" (with $u \geq \varphi$ on ∂D assumed in order that (33) - (36) make sense). The natural number m is "big" such that y, φ and u are continuous by the Sobolev lemma. We define the coincidence set associated to (33) - (36) by

$$C_u = \{x \in D; y(x) = \varphi(x)\}.$$

If $\Omega \subset D$ is some given subdomain of class C^m, we study the boundary controllability problem

$$\text{find } u \in H^{m-1/2}(\partial D) \text{ such that } \Omega \subset C_u. \tag{37}$$

We denote by $\Omega_0 = D \setminus \Omega$. An alternative formulation of (37) is:

$$\text{find } u \in H^{m-1/2}(\partial D) \text{ such that } y_u = \varphi \text{ on } \partial\Omega \tag{38}$$
$$\text{and } y_u \geq \varphi \text{ in } \Omega_0,$$

where $y_u \in H^m(\Omega_0)$ is the solution to the elliptic problem

$$\Delta y_u = 0 \text{ in } \Omega_0, \tag{39}$$

$$\frac{\partial y_u}{\partial n} = \frac{\partial \varphi}{\partial n} \text{ on } \partial\Omega,\ y_u = u \text{ on } \partial D. \tag{40}$$

Notice that the problem (38) - (40) is a variant of the one studied in *Thm. 2.1*, with the notable difference that $y^u \geq \varphi$ in Ω_0 (a state constraint is imposed). One can compare the problem (38) - (40) with (5) - (8) as well and we shall give now an approximation result.

The idea of replacing (37) by (38) - (40) is that y_u may be extended in Ω by φ, preserving regularity in $H^2(D)$. Consequently, $\Omega \subset C_u$ and (33) - (36) are satisfied. See [4] for details, where the following approximate geometrical controllability results are established.

Theorem 2.4 If $\Delta\varphi \leq 0$ in Ω and $\dfrac{\partial \varphi}{\partial n} \geq 0$ on $\partial\Omega$, there is a sequence $\{(u_n, y_n)\} \subset H^{m-1/2}(\partial D) \times H^m(\Omega_0)$, satisfying (39), (40) and $y_n \geq \varphi$ in Ω_0 and having the following property:

For every smooth part Γ of ∂D there is no domain $\Pi \subset \Omega_0$ such that $\partial\Pi \cap \partial\Omega = \Gamma$, $\partial\Pi \cap \partial D \neq \emptyset$ is a smooth submanifold of ∂D and

$$y_{n_k}(x) > \varphi(x),\ \forall\, x \in \Pi,$$

for some subsequence $n_k \to \infty$.

Theorem 2.5 For every $\varepsilon > 0$, there is a connected open subset Q_ε of Ω_0 such that $\text{meas}(\Omega_0 \backslash Q_\varepsilon) \leq \varepsilon$ *and a sequence $\{(u_n, y_n)\} \subset H^{m-1/2}(\partial D) \times H^m(\Omega_0)$ satisfying (39), (40) and*

$$y_n \geq \varphi \ a.e. \ in \ Q_\varepsilon,$$
$$y_n \to \varphi \ weakly \ in \ L^2(\partial\Omega).$$

Remark Such geometrical approximate boundary controllability properties may be obtained in the case of Neumann boundary conditions as well.

III. Optimal control approach

We fix now the general optimal design problem

$$\underset{\Omega}{\text{Min}} \int_E j(x, y_\Omega(x))dx, \tag{41}$$

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial y_\Omega}{\partial x_i}\right) + a_0 y_\Omega = f \text{ in } \Omega, \tag{42}$$

$$B y_\Omega = 0 \text{ on } \partial\Omega. \tag{43}$$

All the notations are as before and B is a general boundary operator: identity for Dirichlet conditions, $By = \dfrac{\partial y}{\partial n_A}$ for the Neumann problem, $By = \dfrac{\partial y}{\partial n_A} + \alpha y$ ($\alpha \in R$) for the Robin condition, etc. The admissible domains are Lipschitzian and contain E (given).

To the problem (41) - (43), we associate the optimal control problem

$$\underset{u \in L^2(D)}{\text{Min}} \left\{ \int_E j(x, y_u(x))dx + \eta \int_{E y_u} (u - f)^2 dx \right\} \tag{44}$$

subject to

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial y_u}{\partial x_i}\right) + a_0 y_u = u \text{ in } D, \tag{45}$$

$$B y_u = 0 \text{ on } \partial D, \tag{46}$$

where $\eta > 0$ is some "big" constant and $E_{y_u} \subset D$ is the smallest (or some) Lipschitzian subdomain such that $E \subset E_{y_u} \subset D$ and $By_u = 0$ on ∂E_{y_u}. One may allow E_{y_u} to be an open set, not necessarily connected (for instance, if E is not connected).

For any Ω Lipschitzian, admissible for the problem (41) - (43), by the trace theorem, we can find $\widetilde{y} \in H^2(D)$ with $B\widetilde{y} = 0$ on ∂D and $\widetilde{y}|_\Omega = y_\Omega$.
In particular, $B\widetilde{y} = 0$ on $\partial\Omega$ too.

Define $v \in L^2(D \setminus \Omega)$ by

$$v = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial \widetilde{y}}{\partial x_j}\right) + a_0\widetilde{y}, \tag{47}$$

$$\widetilde{u} = \begin{cases} f & \text{in } \Omega \\ v & \text{in } D \setminus \Omega \end{cases} \tag{48}$$

It is clear that $\widetilde{y}$ is a strong solution of (45), (46) corresponding to $\widetilde{u}$ given by (39), (40). The open set $E_{\widetilde{y}}$ is contained in Ω since $B\widetilde{y} = 0$ on $\partial\Omega$.

Then, for every $\eta > 0$, the second term in the cost (44) is zero since $\widetilde{u} = f$ in $E_{\widetilde{y}} \subset \Omega$.

Proposition 3.1 For any $\eta > 0$, the cost (36) corresponding to $\widetilde{u}$ is equal with the cost (41) corresponding to Ω. Moreover

$$\mathrm{Inf}_2 \leq \mathrm{Inf}_1 \tag{49}$$

where Inf_1, Inf_2 *denote the infimimum of the problem (41) - (43), respectively (44) - (46).*

A partial converse of P 3.1 may be formulated for solutions with cost close to the infimal value.

Theorem 3.2 Let $\eta \in R_+$. If $[y_\eta, u_\eta]$ is a δ_η-optimal pair for (44) - (46), then E_{y_η} is an ε_η-optimal domain for (41) - (43), where $\varepsilon_\eta > 0$ depends on $\delta_\eta > 0$ and decreases with it.

We recall that δ_η-optimal pair means that the cost (44) associated to $[y_\eta, u_\eta]$ is less than $\mathrm{Inf}_2 + \delta_\eta$. Similar definition for ε_η-optimal domain. Related results and proofs, numerical examples may be found in [12].

Remark In the case of the Dirichlet boundary conditions, E_y is just a level set. For the Neumann boundary condition $\frac{\partial y}{\partial n} = 0$ (i.e. corresponding to the Laplace operator), the boundary of E_y is orthogonal to the level surfaces.

In the case of the Robin boundary condition (in R^2) the angle between the boundary of E_y and the level lines of y is given by:

$$\cos\theta(x) = \pm\frac{\alpha y(x)}{|\nabla y(x)|}$$

where the sign depends on the choice of the tangent direction.

IV. Conclusion

In this work, we review various controllability and optimal control techniques for the solution of shape optimization or free boundary problems. The main idea is to replace problems involving geometric unknowns (domains or boundaries) by analytic formulations, approximating the original problems.

Such control approaches will be extended in the future by the use of characteristic functions of the unknown geometry and of shape functions, as briefly mentioned in (29). An important role will be played by global implicit function theorems, applied to certain classes of functions [7], [18]. The obtained control problems are of control by coefficients type.

References

[1] G. Allaire, “Shape optimization by the homogenization method,” Springer-Verlag, New-York (2001).
[2] C. Baiocchi, “Su un problema a frontiera libera conessa a questioni di idraulica,” *Ann. Mat. Pura Appl.*, 92 (1972), pp. 107-127.
[3] V. Barbu, “Optimal control of variational inequalities,” *RNM 100*, Pitman, London (1984).
[4] V. Barbu, D. Tiba, “Boundary controllability of the coincidence set in the obstacle problem,” *SIAM J. Control Optim.*, vol. 29, no. 5 (1991), pp. 1150-1159.
[5] J. Blum, “Numerical simulation and optimal control in plasma physics,” J. Wiley & Sons, Chichester and Gauthier-Villars, Montrouge (1989).
[6] D. Chenais and E. Zuazua, “Finite-element approximation of $2D$ elliptic optimal design,” *J. Math. Pures Appl.*, (9), 85 (2006), pp. 225-249.
[7] M. Cristea, “A Note On Global Implicit Function Theorem,” *JIPAM*, vol. 8, iss. 4, art. 100, 2007.
[8] A. Henrot and M. Pierre, “Variation et optimisation de formes,” Springer, Berlin (2005).
[9] S.L. Kamenomostskaja, “On Stefan’s problem,” *Math. Sb.*, 53 (1961), pp. 489-514.
[10] J.L. Lions, “Contrôle optimale des systèmes gouvernés par des équations aux dérivées partielles,” Paris, Dunod (1968).
[11] P. Neittaanmäki, A. Pennanen and D. Tiba, “Fixed approaches in shape optimization problems with Dirichlet boundary conditions,” *Inverse Problems*, 25 (2009), pp. 1-18.
[12] P. Neittaanmäki, J. Sprekels and D. Tiba, “Optimization of elliptic systems. Theory and applications,” Springer, New-York (2006).
[13] P. Neittaanmäki and D. Tiba, “An embedding of domains approach in free boundary problems and optimal design,” *SIAM J. Control Optim.*, 33(5) (1995), pp. 1587-1602.
[14] P. Neittaanmäki and D. Tiba, “A fixed domain approach in shape optimization problems with Neumann boundary conditions, in *Partial Differential Equations*,” R. Glowinski, P. Neittaanmäki eds., Springer Verlag, Barcelona (2008), pp. 235-245.
[15] O. Pironneau, “Optimal shape design for elliptic systems,” Springer-Verlag, Berlin (1984).
[16] J. Sokolowski and J.-P. Zolesio, “Introduction to shape optimization,” Springer-Verlag, Berlin (1992).
[17] D. Tiba, “Finite element approximation for shape optimization problems with Neumann and mixed boundary conditions,” *SIAM J. Control Optim.*, 49(3) (2011), pp. 1064-1077.
[18] W. Zhang, S. Sam Ge, “A global Implicit Function Theorem without initial point and its applications to control of non-affine systems of high dimensions,” *J. Math. Anal. Appl.*, 313 (2006), pp. 251-261.

ISBN: 978-988-19252-4-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)