Designing a Linear Controller for Enlarging the Robust Domain of Attraction

S. Haghighatnia, R. K. Moghaddam

***Abstract*— This paper considers problem of designing a linear controller for nonlinear systems and obtains the largest robust domain of attraction. Designing controlling parameters is defined in the form of a novel three-level optimization problem that focuses on extending robust domain of attraction. The efficiency of the proposed method is shown in the simulation part by some examples.**

***Index Terms*— robust domain of attraction, uncertain system, linear controller, enlarging robust domain of attraction.**

I. Introduction

DETERMINING the domain of attraction (*DA*) of a stable equilibrium point is an important problem in nonlinear systems theory. In general, *DA* cannot be exactly calculated. Different methods have been proposed to estimate the *DA*. These methods can be classified in two general groups, Lyapunov based and non-Lyapunov based. The first group contains two main steps [1-7].

- A suitable Lyapunov function (*LF*) is suggested based on the structure of the system.
- *DA* is estimated based on the suggested *LF*.

Real systems are often characterized by the presence of uncertain parameters, which represent characteristics that cannot be measured exactly or that are subject to variations. This means that the *DA* is uncertain as well, since in general it depends on such parameters. In such cases, one needs to consider the robust domain of attraction (*RDA*).

In this paper a new approach to enlarge *RDA* in uncertain systems based on design of linear controller is proposed. The problem of enlarging the robust region of attraction is defined in the form of a novel three-level optimization problem that focuses on extending *RDA*. The optimal controlling parameters are found from this optimization problem such that the eigenvalues of the Jacobean matrix of the dynamic system are forced to belong to the left half of the complex space and the *RDA* is enlarged. This optimization consists of three layers. The second and third layers determine *RDA* and the first layer designs the optimization controlling variables in order to find the largest possible *RDA*.

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This work contains four sections. In Section 2 introduces Basic definitions and theorems used along the paper. Section 3 presents the proposed approach and the main problem. In Section 4 the developed methodology is applied on some illustrative examples. Section 5 concludes the paper.

II. Preliminaries

In this section, needed definitions and theorems are introduced.

Consider the following system:

$$\dot{x}_1 = f(x), \qquad x \in R^n, \quad x(t_0) = x_0 \tag{1}$$

Definition 1(Equilibrium Point [9]). A point $x_e \in R^n$ is called an equilibrium point of system (1) if $f(x_e) = 0$. The equilibrium points of system (1) correspond to the intersection of the nullclines of the system, meaning the curves given by $f(x) = 0$.

In the sequel, without loss of generality, we assume that the equilibrium point under study coincides with the origin of the state space of R^n, $(x_e = 0)$.

Definition 2 (Domain of attraction [9]): The domain of attraction of the origin is given by

$$DA = \{x_0 \in R^n \mid \lim_{t\to\infty} x(t, x_0) = 0\} \tag{2}$$

Definition 3 (Robust domain of attraction): Consider an uncertain nonlinear system, with an isolated equilibrium state, x_e of the following form:

$$\dot{x} = f(x,\theta), \quad x \in R^n, \quad \theta = [\theta_1, \theta_2, \ldots, \theta_m], \quad \theta \in B \subset R^m \tag{3}$$

where θ is uncertain parameter vector, B is a bounded set in R^m and *m* is the number of uncertain parameters.

$$RDA = \{x_0 \in R^n \mid \lim_{t\to\infty} x(t, x_0, \theta) = 0, \quad \forall \theta \in B\} \tag{4}$$

Theorem 1 (Estimation of the Domain of Attraction[9]): Let $V(x)$ be a Lyapunov function for the equilibrium $x = 0$ of system (1).

Consider that $dv(x)/dt$ is negative definite in the region:

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$$S(0) = \{x : V(x) \le C, \quad C > 0\} \tag{5}$$

Then, every trajectory initiated within region $S(0)$ tends to $x = 0$ as time tends to infinity.

Theorem 2: Consider the following representation of system (1):

$$f(x) = Ax + f_f(x) \tag{6}$$

where $f_f(x)$ comprises the nonlinear part of function $f(x)$. It can be shown that if the following condition holds, [12]:

$$\frac{\|f_f(x)\|}{\|x\|} \le \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \ \forall x \in B_r \tag{7}$$

V(x) and its time derivatives are positive and negative definite, respectively, within the ball B_r of radius r. It is clear that the larger the ratio, $\frac{\lambda_{\min}}{2\lambda_{\max}(P)}$ the larger the possible choice of r.

III. MAIN METHOD

Consider system (1) with control input u and uncertain parameters θ as follows:

$$\dot{x} = f(x,\theta) + u, \quad x \in R^n, \quad \theta \in B \subset R^m, \quad x(t_0) = x_0 \tag{8}$$

Wheress $f(x,\theta) = A_f(\theta) + F(x,\theta)$, $u = Kx$, $K \in R^n$, vector K contains controller parameters, n is number of states and B is a bounded set in R^m where m is number of uncertain parameters.

Therefore system (5) can be shown as follows:

$$\dot{x} = A_f(\theta) + F(x,\theta) + Kx = (K + A_f(\theta))x + F(x,\theta)$$

Hence

$$\dot{x} = f(x,\theta) + u = A(x,\theta,K) + F(x,\theta) \tag{9}$$

Where $F(x,\theta)$ comprises the nonlinear part of system (8) and $A(x,\theta,K)$ contains the linear part.

According to theorem 2, the following three-layer optimization algorithm can be employed to find the best values of controlling parameters which extend *RDA*. According to (5), the larger level set of $v(x,\theta,K)$ leads to better estimated *DA*. In order to find the maximum level set of Lyapunov function which is fully contained in the region of negative definiteness of $\frac{dv}{vt}$, a single point in the state space, which corresponds to a tangential contact of level sets $v(x,\theta,K) = 0$ and $\frac{dv}{dt} = 0$ should be found. If quadratic type Lyapunov functions are adopted (Theorem 2), problem of finding the maximum level set of Lyapunov function which is fully contained in the region of negative definiteness of $\frac{dv}{dt}$ can be reformulated as the third layer in (10). In the third layer the quadratic Lyapunov function $v(x,\theta,K) = x^T P(\theta,K)x$ is considered. The desired solution of third layer in (10) is also a single point in the state space, which corresponds to a contact of the ball B_r of radius r and the surface $\frac{\|F(x,\theta,K)\|}{\|x\|} - \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P(\theta,K))} = 0$.

In the second layer, the intersection of spheres which is obtained from quadratic Lyapunov functions depend on uncertain parameters is considered as *RDA*. Finally, in order to calculate extended *RDA* in the first layer, the optimal controlling parameters are found. So the three-level optimization problem is as follows:

$$R_{\max} = \max_{k,R_k} R_k$$

$$\text{1st layer} \left\{ \begin{array}{l} \mathrm{Re}\{\lambda[A(x_e,\theta,K)]\} < 0, \\ R_k = \min\limits_{\theta,R_{\theta k}} (R_{\theta k}) \\ \text{2nd layer} \left\{ \begin{array}{l} R_{\theta k} = \min\limits_{R_\theta,x,P} R_\theta \\ \text{3rd layer} \left\{ \begin{array}{l} \|x\| - R_\theta = 0 \\ A(x_e,\theta,K)^T P(\theta,K) + P(\theta,K)A(x_e,\theta,K) = -Q \\ \frac{\|F(x,\theta)\|}{\|x\|} - \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P(\theta,K))} = 0 \end{array} \right. \end{array} \right. \end{array} \right. \tag{10}$$

It should be noted that constraints (10) may have many local solutions. In order to avoid dummy solutions, they have to be solved to global optimality therefore in this contribution a standard implementation of a genetic algorithm is employed [13].

IV. EXAMPLES

Consider the following nonlinear system:

$$\frac{dx_1}{dt} = -x_2$$
$$\frac{dx_2}{dt} = x_1 - \theta(1 - x_1{}^2)x_2 + u$$

Where θ is the controlling parameter and $\theta \in [1 \quad 3]$. The analyzed equilibrium is the (0, 0) in θ set. The structure of linear controller is as follows:

Using the optimization algorithm (6), the optimal value of controlling parameters is obtained as $k = [4 \quad -3]$, the related $R_{\max}$ which is the radius of *ERDA,* is 0.6246. As it is shown in figure 1, choosing such controlling parameters leads to a significant increase in radius of the *ERDA*. In absence of controllers, the radius of the estimated *RDA* for nonlinear system is 0.4671.

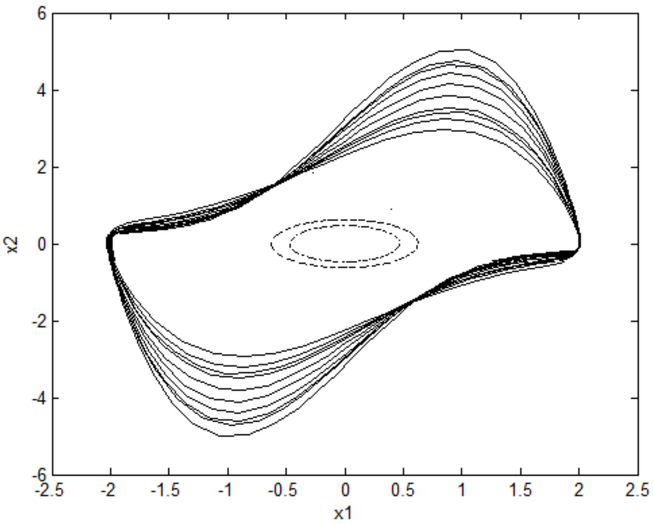

Fig. 1 *RDA* of Van der pol oscillator without controller (-.-.), enlarged *RDA* of Van der pol oscillator (dash line) and *DA*s of Van der pol oscillator for different values of uncertain parameters without controller (solid line)

V. Conclusion

In this work in order to enlarge *RDA* in uncertain systems, a new approach based on design of linear controller is proposed. A three-level optimization problem finds the optimal controlling parameters of this linear controller to extend *RDA*. In the third layer of the optimization problem, the largest estimated *DA* for each uncertain parameters is found then in second layer, intersection of the *DA*s which are depend on uncertain parameters is obtained, finally in the first layer, the optimal controlling parameters which leads to the largest *RDA* are found. The efficiency of proposed methods is shown via simulations. Proposing a nonlinear controller to enlarge *RDA* will be considered in our future work.

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