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On the Linear Combination of the Gaussian and Student-t Random Fields and the Geometry of its Excursion Sets

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***Abstract*—In this paper, a random field, denoted GT^{ν}_{β}, is defined from the linear combination of two independent random fields, one is a zero mean Gaussian random field and the second is a student-t random field with ν degrees of freedom scaled by β. The goal is to give the analytical expression of the expected Minkowski functionals of the excursion sets of GT^{ν}_{β} on a subset S of $\mathbb{R}^2$. The motivation comes from the need to model a 3D rough surface topography, where the height measurements distribution is assumed to be resulted from the convolution of both normal and student-t distributions. The expected and empirical Minkowski functionals are compared in order to test the approximation of the model to the real surface measurements.**

***Index Terms*—Gaussian random field, student-t random field, excursion sets, Minkowski functionals, Euler-Poincaré characteristic.**

I. Introduction

The motivation of defining and studying GT^{ν}_{β} random fields comes from the need of modeling 3D rough and anisotropic engineering surfaces used in biomedical and material science applications.

Studying the spatial evolution of a surface or the deformation of its asperities requires combination between different types of random fields. This combination might increase the flexibility of the model, since it defines further statistical parameters such as the higher order moments (skewness, kurtosis, ...) which could interpret the functionality of such surfaces during certain phenomenon, that cannot be involved by only the Gaussian model.

Gaussian and several non-Gaussian random fields, namely χ^2, F, student-t and Hotelling's T^2 random fields, have been studied in [1], [2], [3], [4], [5] in order to detect the local maxima of the random field inside a searching region which refer to certain activations in the brain or anomalies in medical imaging applications. The integral geometric characteristics of the excursion sets of such random fields have been investigated in [6], [3], [2], where the excursion sets are defined as the upper sets that result from thresholding the random field at a giving crossing level value. For example, the excursion set at a height level h of a 3D surface will result from hitting the surface heights by a cutting plane at h. Thus, all the points at which the surface heights will exceed the level h will define this excursion set.

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In this paper, we are interesting on studying the linear combination between a stationary Gaussian random field, denoted G, and a non-Gaussian random field, namely student-t random field with ν degrees of freedom, denoted T^{ν}, i.e, the sum $G(x) + \beta T^{\nu}(x)$, where x denotes the spatial location that belongs to the subset S of $\mathbb{R}^2$. These random fields will be denoted GT^{ν}_{β}. The goal is to present the stationary GT^{ν}_{β} random field, and to calculate analytically the expected Minkowski functionals of its excursion sets on $\mathbb{R}^2$.

The paper is organized as follows. The GT^{ν}_{β} random fields are defined in section II. In section III, the expected Minkowski functionals of the excursion sets of the GT^{ν}_{β} random field are given on $\mathbb{R}^2$. In section IV, an application to a real rough surface modeled by the GT^{ν}_{β} random field is investigated. The model test is illustrated by comparing the empirical Minkowski functionals to the expected ones. Finally, a conclusion is derived in section V.

II. GT^{ν}_{β} random fields

A. Preliminaries

We will suppose that $Y = Y(x), x \in S$, is a stationary real-valued random field defined on a compact subset S of the Euclidean space $\mathbb{R}^d$, with mean μ_Y and variance σ_Y^2. The $N \times N$ covariance matrix of any finite collection $\{Y_i = Y(x_i), i = 1, ..., N, x_i \in S\}$, will be denoted as Ω_Y, where $\Omega_Y(i,j) = \mathbb{E}[(Y_i - \mu_{Y_i})(Y_j - \mu_{Y_j})], (i,j = 1, ..., N)$. The probability density function of Y is denoted as p_Y and the cumulative distribution function is denoted as P_Y. For simplicity, the term $Y(x)$ will be replaced sometimes by Y.

B. GT^{ν}_{β} distribution function

Definition 2.1 (GT^{ν}_{β} random variable): Let G be a random variable of standard normal distribution and T^{ν} be a zero mean student-t random variable with ν degrees of freedom independent of G. A random variable Y is said to have GT^{ν}_{β} distribution if it is given from the sum $Y = G + \beta T^{\nu}, \beta \in \mathbb{R}^*$. It will be denoted by $Y \sim GT^{\nu}_{\beta}$. The probability density function of Y, p_Y, results from the convolution between the probability density functions of G and T^{ν} as follows:

$$p_Y(h) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\beta\pi\Gamma\left(\frac{\nu}{2}\right)\sqrt{2\nu}} \times \int_{-\infty}^{\infty} \left(1 + \frac{(h-u)^2}{\beta^2\nu}\right)^{-\frac{\nu+1}{2}} e^{-\frac{u^2}{2}} du \quad (1)$$

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where,

$$\phi(h) = \frac{1}{\sqrt{2\pi}} e^{-h^2/2} \quad (2)$$

is the normal probability density function, and

$$t_\nu(h) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\beta\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}} \left(1 + \frac{h^2}{\nu\beta^2}\right)^{-(\nu+1)/2} \quad (3)$$

is the probability density function of the student-t distribution with ν degrees of freedom and scaled by β.

In the general case, Y could be expressed such that

$$Y = \mu + \sigma G + \beta T^\nu \quad (4)$$

where σ is the scale parameter of the Gaussian distribution, and μ is a location parameter.
This yields applying the transformation $h \to \frac{h-\mu}{\sigma}$ in order to obtain the $GT^\nu_{\frac{\beta}{\sigma}}$ probability density function.

C. Multivariate GT^ν_β distribution function

Let $\boldsymbol{Y}$ be a random vector of N random variables, $\boldsymbol{Y} = (Y_1, ..., Y_N)^t, (N > 1)$. Let Y_i be a $GT^\nu_{\beta_i}$ random variable such that $Y_i = \mu_i + \sigma_i \left(G_i + \frac{\beta_i}{\sigma_i} T^\nu_i\right)$, $i = 1, ..., N$, then $\boldsymbol{Y}$ is said to has N-dimensional $GT^\nu_{\tilde{\boldsymbol{\beta}}}(\Omega)$ distribution with $N \times N$ covariance matrix Ω_Y, and it is expressed as follows:

$$p_{\boldsymbol{Y}}(\mathbf{h}) = \phi(\boldsymbol{h}; \Sigma_1) * t_\nu(\boldsymbol{h}; \tilde{\boldsymbol{\beta}}; \Sigma_2) \quad (5)$$

where $\phi(\boldsymbol{h}; \Sigma_1)$ is the multivariate Gaussian distribution function with the covariance matrix Σ_1, and $t_\nu(\boldsymbol{h}; \tilde{\boldsymbol{\beta}}; \Sigma_2)$ is the multivariate student-t distribution function with ν degrees of freedom, and covariance matrix Σ_2 scaled by the vector $\tilde{\boldsymbol{\beta}}$. The covariance matrix Ω_Y is a function of Σ_1, Σ_2 and $\tilde{\boldsymbol{\beta}}$.

D. GT^ν_β random field

On a subset S in $\mathbb{R}^2$, if any arbitrary N random variables, $Y(x_1), ..., Y(x_N$ has a multivariate GT^ν_β distribution, then for any $x \in S$, $Y(x)$ will define GT^ν_β random field, which yields to the following definition:

Definition 2.2 (GT^ν_β random field): Let G be a stationary, not necessarily isotropic, centered Gaussian random field, defined on a compact subset $S \subset \mathbb{R}^2$, with variance $\sigma^2 = 1$. Let T^ν be a homogeneous student-t random field with ν degrees of freedom, independent of G. Then, the sum given by:

$$Y(x) = G(x) + \beta T^\nu(x), \quad \beta \in \mathbb{R}^* \quad (6)$$

defines a stationary GT^ν_β real-valued random field with ν degrees of freedom.

III. The Expected Minkowski Functionals of the GT^ν_β Excursion Sets

The expected Minkowski functionals of both Gaussian and student-t random excursion sets have been investigated in [7], [5], [8] on $\mathbb{R}^d$, for $d = 1, 2, 3$. This paper focus in giving the analytical expressions of Minkowski functionals for the random field GT^ν_β in $\mathbb{R}^2$.

A. Notifications

1) Let $Y(x), x \in \mathbb{R}^2$, be a zero mean real-valued GT^ν_β random field given by the sum $G(x) + \beta T^\nu(x)$ and defined on $S = [a, b]^2 \subset \mathbb{R}^2$. G is supposed to be a stationary and centered Gaussian random field with unit variance, $\sigma^2 = 1$, and T^ν is a homogeneous student-t random field with $\nu \geq 2$. T^ν is defined by [8] from a homogeneous, independent and identically distributed Gaussian random fields as follows:

$$T^\nu = \frac{\sqrt{\nu} G_0}{\left[\sum_{k=1}^\nu G_k^2\right]^{1/2}} \quad (7)$$

where $G_k, k = 0, ..., \nu$ are $\nu + 1$ independent Gaussian random fields with zero means and unite variances.
Let $E_h(Y, S)$ be the excursion set, [4], [8], of Y inside S, above a threshold h, and it is defined as follows:

$$E_h(Y, S) = \{x \in S : Y(x) \geq h\} \quad (8)$$

2) Let Λ_G be the 2×2 variance-covariance matrix of G (i.e., the covariance of all the first order partial derivatives $\partial G / \partial x_i$ [4]) such that:

$$\lambda_{G_{ij}} = \mathbb{E}\left(\frac{\partial G}{\partial x_i}, \frac{\partial G}{\partial x_j}\right) \quad (9)$$

For the student-t random field, the Gaussian components are identically distributed and they all have the same variance-covariance matrix denoted Λ such that:

$$\lambda_{ij} = \mathbb{E}\left(\frac{\partial G_0}{\partial x_i}, \frac{\partial G_0}{\partial x_j}\right) = \mathbb{E}\left(\frac{\partial G_1}{\partial x_i}, \frac{\partial G_1}{\partial x_j}\right) = ... \quad (10)$$

(i=1,2;j=1,2)
The matrix Λ_G and Λ are called roughness matrix since their parameters determine the wavelength or the distance between the dependent points of the Gaussian random fields in all the spatial directions. In the Gaussian case, they are expressed by $\Lambda_G = \frac{\Sigma_1^{-1}}{2}$, $\Lambda = \frac{\Sigma_2^{-1}}{2}$, and in general case, they are given from the covariance between the first order partial derivatives of the random field.

B. Expectations

The expected Minkowski functionals including Euler-Poincaré characteristic of the excursion sets, $E_h(Y, S)$, for random fields $\mathbb{R}^d$, are given as follows, [7]:

$$\mathbb{E}[\mathcal{L}_i(E_h(Y, S))] = \sum_{j=0}^{d-i} \mathcal{L}_{i+j}(S)\rho_j(h) \quad (11)$$

where $\mathcal{L}_j(S)$ is called is called the 'j-th dimensional size', [9], of S (j-th dimensional Minkowski functionals of S), and $\mathcal{L}_i(E_h(Y, S))$ is the i-th dimensional Minkowski functionals of the excursion sets $E_h(Y, S)$. In this paper, S is a rectangular subset $[a, b]^2 \in \mathbb{R}^2$, then the expected Minkowski functionals of $E_h(Y, S)$ are:

$$\mathbb{E}[\chi(E_h(Y, S))] = \mathcal{L}_2(S)\rho_2(h) + \mathcal{L}_1(S)\rho_1(h) + \mathcal{L}_0(S)\rho_0(h) \quad (12)$$

$$\mathbb{E}[P(E_h(Y, S))] = \mathcal{L}_1(S)\rho_1(h) + \mathcal{L}_0(S)\rho_0(h) \quad (13)$$

$$\mathbb{E}[A(E_h(Y, S))] = \mathcal{L}_0(S)\rho_0(h) \quad (14)$$

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where
$\chi(E_h(Y,S)) = \mathcal{L}_0(E_h(Y,S))$ is Euler-Poincaré characteristic of $E_h(Y,S)$.
$P(E_h(Y,S)) = \mathcal{L}_1(E_h(Y,S))$ is the half boundary length of $E_h(Y,S)$.
$A(E_h(Y,S)) = \mathcal{L}_2(E_h(Y,S))$ is the area of $E_h(Y,S)$.

and $\mathcal{L}_0(S) = 1$ is the Euler-Poincaré characteristic of S, $\mathcal{L}_1 = a+b$ is the half boundary length of S, and $\mathcal{L}_2 = a \times b$ is the two dimensional area of S.
The coefficients, $\rho_j(h)$, $(j = 0,1,2)$, are called the j-th dimensional Minkowski coefficients of the random excursion set, $E_h(Y,S)$, for a given threshold h.

In the following, the expected Minkowski functionals are expressed analytically in the general case when Y is expressed as follows:

$$Y(x) = \mu + \sigma G(x) + \beta T^{\nu}(x) \tag{15}$$

where μ, σ, β are constants for all $x \in S$.

Theorem 3.1: The j-th dimensional Minkowski coefficients, $\rho_j(.)$, $j = 0,1,2$ for a random field Y expressed by the linear combination of isotropic Gaussian random field and a homogeneous student-t random field with ν degrees of freedom, $\nu > 2$, on $\mathbb{R}^2$, are defined at a given h by:

$$\begin{aligned}\rho_0(h) &= \mathbb{P}\left[Y \geq h\right] \\ &= \frac{\sigma\Gamma\left(\frac{\nu+1}{2}\right)}{(2\pi)\beta\sqrt{\nu/2}\Gamma\left(\frac{\nu}{2}\right)} \int_h^{\infty}\int_{-\infty}^{\infty}\left(1+\frac{(u-\mu-\sigma y)^2}{\beta^2\nu}\right)^{-\frac{\nu+1}{2}} \\ &\times e^{-y^2/2} du dy\end{aligned} \tag{16}$$

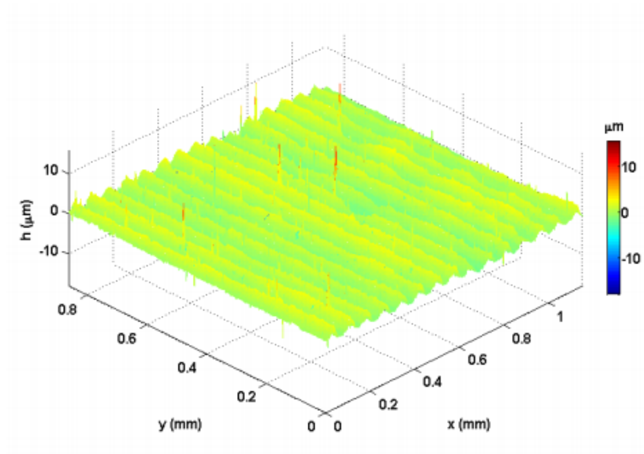
$$\begin{aligned}\rho_1(h) &= \frac{\lambda^{1/2}}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\left(1+\frac{(h-\mu-\sigma y)^2}{\beta^2\nu}\right)^{-\frac{\nu-1}{2}} e^{-y^2/2}dy \\ &+ \frac{\sigma\lambda_G^{1/2}\Gamma\left(\frac{\nu+1}{2}\right)}{(2\pi)^{3/2}\Gamma\left(\frac{\nu}{2}\right)\beta\sqrt{\nu}}\int_{-\infty}^{\infty} e^{-y^2/2} \\ &\times \left(1+\frac{(h-\mu-\sigma y)^2}{\beta^2\nu}\right)^{-\frac{\nu+1}{2}} dy\end{aligned} \tag{17}$$

$$\begin{aligned}\rho_2(h) &= \frac{2^{\frac{1}{2}}\lambda\Gamma\left(\frac{\nu+1}{2}\right)}{(2\pi)^2\Gamma\left(\frac{\nu}{2}\right)}\int_{-\infty}^{\infty}\frac{(h-\mu-\sigma y)}{\beta\sqrt{\nu}}e^{-y^2/2} \\ &\times \left(1+\frac{(h-\mu-\sigma y)^2}{\beta^2\nu}\right)^{-\frac{\nu-1}{2}} dy + \frac{\sigma\lambda_G\Gamma\left(\frac{\nu+1}{2}\right)}{(2\pi)^2\Gamma\left(\frac{\nu}{2}\right)\beta\sqrt{\nu/2}} \\ &\times \int_{-\infty}^{\infty} y\left(1+\frac{(h-\mu-\sigma y)^2}{\beta^2\nu}\right)^{-\frac{\nu+1}{2}} e^{-y^2/2}dy\end{aligned} \tag{18}$$

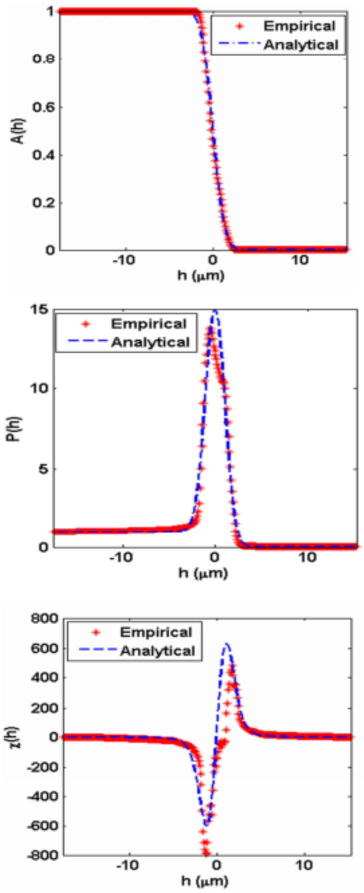
where $\lambda_G^{1/2} I_{2\times 2} = \Lambda_G$, and $\lambda^{1/2} I_{2\times 2} = \Lambda$.

IV. Application

The stochastic model has been tested on a real 3D rough and anisotropic microstructure surface of a UHMWPE component (Ultra High Molecular Weight Polyethylene),[10]. The surface has been measured by a non-contact white light interferometry, (Bruker nanoscope Wyko® NT 9100), on a lattice of 480×640 points with spatial sampling steps equal to $1.8\mu m$ in both X and Y directions, see figure 1(a).



(a)



(b)

Fig. 1. (a) A real 3D surface roughness topography digitized on a lattice of 480 points with a spatial sampling steps equal to $1.8\mu m$ in X and Y directions. (b) Fitting the expected and the empirical Minkowski functionals, $A(h)$, $P(h)$ and $\chi(h)$, respectively, between the model and the real surface.

The surface is composed of a large-scale features, which could be characterized by the covariance function and modeled by a Gaussian random field, and the small-scale features including the noise which are modeled by the student-t

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random field. Since the surface tends to be one directional anisotropy, so Minkowski coefficients $\rho_1(h)$ and $\rho_2(h)$ has been modified to be:

$$\rho_1(h) = \frac{\sigma\lambda^{1/2}}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \left(1 + \frac{(h-\mu-\sigma y)^2}{\beta^2\nu}\right)^{-\frac{\nu-1}{2}} e^{-y^2/2} dy + \frac{\lambda_{G_{11}}^{1/2} + \lambda_{G_{22}}^{1/2}}{2} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{(2\pi)^{3/2}\Gamma\left(\frac{\nu}{2}\right)\beta\sqrt{\nu}} \int_{-\infty}^{\infty} e^{-y^2/2} \times \left(1 + \frac{(h-\mu-\sigma y)^2}{\beta^2\nu}\right)^{-\frac{\nu+1}{2}} dy \tag{19}$$

$$\rho_2(h) = \frac{2^{\frac{1}{2}}\lambda\Gamma\left(\frac{\nu+1}{2}\right)}{(2\pi)^2\Gamma\left(\frac{\nu}{2}\right)} \int_{-\infty}^{\infty} \frac{(h-\mu-\sigma y)}{\beta\sqrt{\nu}} e^{-y^2/2} \times \left(1 + \frac{(h-\mu-\sigma y)^2}{\beta^2\nu}\right)^{-\frac{\nu-1}{2}} dy + \frac{\sigma\lambda_{G_{11}}^{1/2}\lambda_{G_{22}}^{1/2}\Gamma\left(\frac{\nu+1}{2}\right)}{(2\pi)^2\Gamma\left(\frac{\nu}{2}\right)\beta\sqrt{\nu/2}} \times \int_{-\infty}^{\infty} g\left(1 + \frac{(h-\mu-\sigma y)^2}{\beta^2\nu}\right)^{-\frac{\nu+1}{2}} e^{-y^2/2} dy \tag{20}$$

where $\Lambda_G = \begin{pmatrix} \lambda_{G_{11}} & 0 \\ 0 & \lambda_{G_{22}} \end{pmatrix}$

The model parameters have been estimated from minimizing the error between the empirical and the expected Minkowski functionals, which yields to $\lambda_{G_{11}} = 117$, $\lambda_{G_{22}} = 15$, $\nu = 5$, $\beta = 0.2$, $\sigma = 2$ and $\lambda = 190.3$. Figure 1(b) shows the fitting result between the expected and the empirical characteristic functions of the excursion sets of the GT_β^ν random field and the real surface.

The results show that the model approximates the real measurements especially at high thresholds when $h \geq 2.5\mu m$. So, one can obtain an approximation of the number of peaks over the significant threshold which will be equal to Euler-Poincaré characteristics, or the number of valleys when getting the complement of the Excursion sets. The aim is to study the changes of the surface roughness during certain mechanical phenomenon such as friction and wear. The model defined in this paper is symmetric, (no skewness). However, most of real surfaces tend to have asymmetric heights, and so the skewness parameter becomes a significant statistical parameter that can not be neglected. Our future work is to adapt the model in order to include the skewness parameter besides the kurtosis.

V. Conclusion

A random field model is defined from the linear combination of a stationary Gaussian random field and a homogeneous student-t random field with ν degrees of freedom. The Gaussian random field is uniquely characterized by its covariance matrix, whereas the student-t is characterized by the covariance matrix and the degree of freedom. The analytical expressions of Minkowski functionals of this random field are given on $\mathbb{R}^2$. The characteristic functionals, on $\mathbb{R}^2$, are the area function, the half boundary length function and Euler-Poincaré characteristic. The stochastic model, in this paper, can be extended to higher dimensions. An application is reported on modeling the topography of a real 3D rough surface of a finished polyethylene component, [10], measured by a 3D non-contact white light interferometry. The expected Minkowski functionals enabled to fit the model with the real measurements with good approximation.

Appendix
Proof of theoreme 3.1

The proof of the theorem is based on a previous proofs and Lemmas given in [3], [8], [2], [1], [11].
The expected j-th dimensional Minkowski coefficient, $\rho_j(h)$, of the excursion set, $E_h(Y,S)$, of an isotropic random field Y, is represented in [3] as follows:

$$\rho_j(h) = (-1)^{j-1}\mathbb{E}\left[(\dot{Y}_j > 0)\det(\ddot{Y}_{|j-1})|\dot{Y}_{|j-1} = 0, Y = h\right] \times p_{\dot{Y}_{|j-1}}(0,h)p_Y(h) \tag{21}$$

where the term $|j-1$ represent the sub-matrix of the first $j-1$ rows and columns of Y, and j refers to the j-th component of the matrix Y. $p_{\dot{Y}_{|j-1}}(0,h)$ is the probability density of $\dot{Y}$ at zero conditional on $Y = h$. $p_Y(h)$ is the probability density of Y.
We will use this representation for calculating $\rho_j(h)$ of $E_h(Y,S)$ where Y is the defined GT_β^ν random field.
Basing on the conditional expectations and the last representation in 21, $\rho_j(h)$ becomes in our case:

$$\rho_j(h) = (-1)^{j-1}\mathbb{E}_G\left\{\mathbb{E}_W\left\{\mathbb{E}\left[(\dot{Y}_j > 0)\det(\ddot{Y}_{|j-1})| \dot{Y}_{|j-1} = 0, Y = h\right] p_{\dot{Y}_{|j-1}}(0,W,G,h)\right\} p_Y(h,G)\right\}\phi(h) \tag{22}$$

where $p_{\dot{Y}_{|j-1}}(0,W,G,h)$ is the joint probability density of $\dot{Y}$ at zero conditional on $Y = h$, W and G. $p_Y(h,G)$ is the probability density of $Y = h$ conditional on G, and ϕ is the probability density function of G.

Let G be isotropic Gaussian random field with zero mean and unit variance $\sigma_G^2 = 1$ defined on $S \subset \mathbb{R}^d$. Let the first and second order spatial derivatives of G be denoted as $\dot{G}, \ddot{G}$ respectively. Then the following two results have been proven [2]:

- the first derivative $\dot{G}$ of the Gaussian random field is Gaussian such that $\dot{G} \sim \text{Normal}_d(0,\Lambda_G)$ which is independent of both G and $\ddot{G}$.
- conditioning on G, the second derivative $\ddot{G}$ of G is a Gaussian random field which satisfies that
$$\ddot{G}|G \sim \text{Normal}_{d\times d}(-G\Lambda_G, M(\Lambda_G))$$
where $M(\Lambda_G)$ is symmetric and it satisfies that
$$M(\Lambda_G) = \mathbb{E}\left[\ddot{G_{ij}}, \ddot{G_{kl}}|G\right] = cov(\ddot{G}_{ij}, \ddot{G}_{kl}|G)$$
$(i,j,k,l = 1,..,d)$. Hence, the second order spatial derivative of G could be expressed conditioning on G such that $\ddot{G}|G = -G\Lambda_G + V$, where $V \sim \text{Normal}_d(0, M(\Lambda_G))$.

The first and second order spatial derivatives of Y, conditional on $Y = h, G, W$ and $\dot{Y} = 0$, can be expressed as

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follows:

$$\begin{aligned}
\dot{Y} &= \dot{G} + \beta\nu^{1/2}\left(1+\frac{(h-G)^2}{\beta^2\nu}\right)W^{-1/2}Z_1 \\
\ddot{Y} &= -G\Lambda_G + V + \beta\nu^{1/2}\left(1+\frac{(Y-G)^2}{\beta^2\nu}\right)W^{-1} \\
&\times\left\{-\frac{(Y-G)}{\beta\sqrt{\nu}}Q + W^{1/2}H\right\}
\end{aligned} \tag{23}$$

where, Z_1, Z_2, H are independent Gaussian random fields such that $Z_1 \sim \text{Normal}_d(0,\Lambda), Z_2 \sim \text{Normal}_d(0,\Lambda)$, $H \sim \text{Normal}_d(0, M(\Lambda))$, W is the Chi-squared random field with $\nu+1$ degrees of freedom, $W \sim \chi^2_{\nu+1}$, and Q is the Wishart random field, [11], $Q \sim Wishart(\Lambda, \nu-1)$.
Then,

$$\begin{aligned}
&\mathbb{E}\left[\det(-\ddot{Y}_{|j-1})|\dot{Y}_{j-1}=0, Y=h, G, W\right] = (2\pi)^{-1/2}\lambda_{G_j}^{1/2} \\
&+ (2\pi)^{-1/2}\lambda^{1/2}\beta\nu^{1/2}\left(1+\frac{(Y-G)^2}{\beta^2\nu}\right)W^{-1/2}K(h)
\end{aligned} \tag{24}$$

where $K(h)$ is a polynomial function of $j-1$ degrees :

$$\begin{aligned}
K(h) &= \det{}_{j-1}(\Lambda)\sum_{i=0}^{\lfloor (j-1)/2\rfloor}\sum_{k=0}^{j-1-2i}\sum_{m=0}^{\lfloor (j-1-2i-k)/2\rfloor}\binom{j-1}{k} \\
&\times\frac{(-1)^i}{2^i i!}c^{2i}r^k\frac{(\nu-1)!}{(\nu-1-k)!}(2i+k)!\det{}_{j-1-2i-k}(\Lambda_G\Lambda^{-1}) \\
&\times\frac{(-1)^{j-1-2i-k-m}(2m)!}{2^m m!}G^{j-1-2i-k-2m}
\end{aligned} \tag{25}$$

The joint probability density function of the $j-1$ first derivatives of $\dot{Y}$, $p_{\dot{Y}_{|j-1}}(0, G, W, h)$, conditioning on $G, W, Y=h$, is a Gaussian probability density function and it is given by:

$$\begin{aligned}
&p_{\dot{Y}_{|j-1}}(0,G,W,h) = (2\pi)^{-\frac{j-1}{2}}\Bigg\{\det{}_{j-1}(\Lambda_G)^{1/2}+ \\
&\det{}_{j-1}(\Lambda)^{1/2}\left[\beta\nu^{1/2}\left(1+\frac{(h-G)^2}{\beta^2\nu}\right)W^{-1/2}\right]^{j-1} + \\
&\sum_{k=1}^{j-2}\binom{j-2}{k}\det{}_{j-2-k}(\Lambda_G)^{1/2}\det{}_k(\Lambda)^{1/2} \\
&\times\left[\beta\nu^{1/2}\left(1+\frac{(h-G)^2}{\beta^2\nu}\right)W^{-1/2}\right]^k\Bigg\}^{-1}
\end{aligned} \tag{26}$$

The joint probability density function of $Y=h$ and G conditioning on G, $p_Y(h,G)$, is the student-t probability density function with ν degrees of freedom. Furthermore, $\mathbb{E}[W^j] = 2^j\frac{\Gamma((\nu+1)/2+j)}{\Gamma((\nu+1)/2)}$. Putting these results together in equation 22, the coefficients $\rho_j(h), (j=1,2)$ of the excursion set $E_h(Y,S)$ could be obtained.
For $j=0$, $\rho_0(h)$ becomes the cumulative distribution function, $\mathbb{P}[Y(x)\geq h]$, of Y at each point x.

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