# A Unified Approach for Multiobjective Fuzzy Chance Constrained Programming with Joint Normal Distribution

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Abstract- This paper describes some mathematical techniques and modeling aspects for solving fuzzy multiobjective probabilistic decision making problems in which the constraints are jointly distributed and the right sided parameters of the constraints are normally distributed fuzzy random variables. The probabilistic model is first converted into equivalent fuzzy programming model by using incomplete gamma function described in a fuzzy decision making environment. Then independent optimal solution of each objective are determined under the decomposed set of system constraints which are obtained by considering fuzzy nature of parameters involved with them. The tolerance membership function for measuring the degree of satisfaction of the decision maker with the achievement of objective values is defined. The membership functions are then converted into fuzzy goals by assigning unity as aspiration level. Finally a weighted fuzzy goal programming technique is used to achieve the highest degree of each of the defined membership goal to the extent possible by minimizing under deviational variables and thereby obtaining most satisfactory solution in the decision making context which leads to an efficient as well as optimal compromise solution. A numerical example is solved to illustrate the proposed methodology and the solution is compared with some other technique developed earlier.

*Keywords*- Chance Constrained Programming, Incomplete gamma function, Fuzzy random variable, Fuzzy number, Fuzzy goal programming.

### I. INTRODUCTION

Linear programming (LP) is an efficient tool to deal with many real world decision problems which have practical importance. This is especially the case of production problems with linear production functions and linear cost functions, critical path scheduling problems, general network flow problems and others. The formulation of such problems would incorporate technological coefficients on the basis of which a model of the situation could be analyzed.

To apply theories and methods of LP [1] it is required that the coefficients and relationship is completely known. In real life situation, if one wants to become more realistic then this assumption may not be fulfilled. Since most real life decision making problems involve some level of uncertainty about values to be assigned to various parameters or about the occurrence of the components of the problem. When a probabilistic characteristic of a problem is found it is generally treated by using stochastic programming (SP) [2] and also by Chance constrained programming (CCP) technique [3]. Again if some imprecise parameters are involved with the problem, this is handled by using fuzzy programming (FP) method [4, 5].

CCP deals with those kinds of problems where the associated parameters are random variables with some known probability distribution. A class of CCP problems containing multiple and conflicting objectives are known as multiobjective CCP (MOCCP) problems. Among the various types of probability distribution, which are followed by the random variables associated with chance constraints, joint normal distribution sometimes play an important role from the view point of its applicability in different real world planning problems [6].

Further in some practical situations, a decision should have to make on the basis of some data which are not purely probabilistic or only possibilistic but rather a mixture of both kind. In particular, the research works are witnessed by a developing interest in situation where the fuzziness and randomness are considered concurrently in an optimization frame work [7, 8, 9, 10]. This interest has been motivated by the need for basing many human decisions on information which is both fuzzily imprecise and probabilistically uncertain [11]. Leberling [12] proved that solution obtained by fuzzy LP (FLP) is always a compromise solution of the original multiobjective problem. Motivated from the above facts, Sinha et al. [13] applied FP technique to solve multiobjective CCP (MOCCP) problems assuming coefficients of constraints in right side are joint normal random variable.

In recent years, fuzzy goal programming (FGP) approaches to decision making problems having multiplicity of objectives have been extensively investigated [14, 15] and applied to different real life planning problems [16, 17]. An efficient

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methodology for solving CCP problems with single objective by using FGP technique has been studied by Biswas and Modak [18] in the recent past.

In the present study FGP process is adopted to solve fuzzy MOCCP (FMOCCP) problem where the parameters in the right side of the system constraints follow fuzzy joint normal distribution. In model formulation process, the fuzzy probabilistic problem is converted to a FP problem by applying chance constrained methodology with the help of  $\alpha$  - cut of fuzzy numbers and first decomposition theorem on fuzzy sets. Then considering the fuzzy numbers associated with the system constraints, the constraints are decomposed on the basis of tolerance ranges of fuzzy numbers. After that the individual solution of each objective is found to construct the membership goal of the objectives. Finally weighted FGP model is formulated to achieve the highest membership value to the extent possible by minimizing group regret consisting of under deviational variables in the decision making context.

# II. BACKGROUND OF THE PROBLEM FORMULATION

# A. Mathematical tools for combining fuzziness and randomness

To deal with the situation in real life problems in which the fuzziness and randomness occurs simultaneously, the concepts of probability and fuzzy set theory such as probability of fuzzy event [19], linguistic probabilities [20], random fuzzy variable [21], fuzzy random variable [22, 23], and uncertain probabilities [24] are considered together. Before entering into the main part the notion of triangular fuzzy numbers,  $\alpha$ -cuts, uncertain probabilities, fuzzy random variables are discussed briefly.

1) Triangular fuzzy number: A fuzzy number is a normal and convex fuzzy set defined on  $\Re$  and always represents a vague datum [19]. Triangular fuzzy number is a kind of fuzzy numbers having triangular shape. For instance, A vague datum "close to a" can be represented by a triangular fuzzy number, which can be denoted by a triple of three real numbers as  $\tilde{a} = (a^L, a, a^R)$ . The membership function of the triangular fuzzy number is of the form

$$\mu_{\tilde{a}}(x) = \begin{cases} 0 \quad if \quad x < a^{L} or x > a^{R} \\ (x - a^{L})/(a - a^{L}) \quad if \quad a^{L} \le x \le a \\ (a^{R} - x)/(a^{R} - a) \quad if \quad a \le x \le a^{R} \end{cases}$$

where  $a^L$  and  $a^R$  represent, respectively, the left and right tolerance values of the fuzzy number  $\tilde{a}$ .

2)  $\alpha$ -cut: Given a fuzzy set  $\tilde{A}$ , its  $\alpha$ -cut  $A[\alpha]$  defined as  $A[\alpha] = \left\{ x \middle| \mu_{\tilde{A}}(x) \ge \alpha \text{ and } x \in U \right\}$  where  $\alpha$  is the confidence level and U is the universe of discourse of the fuzzy set  $\tilde{A}$ . By definition of  $\alpha$ -cut,  $A[\alpha]$  of a fuzzy set  $\tilde{A}$ , is a closed interval on the field of real numbers i.e.,

$$A[\alpha] = \left\{ x \in R \, \Big| \, \mu_{\widetilde{A}}(x) \ge \alpha \right\} = \left[ A^{\alpha L}, A^{\alpha R} \right], \ \alpha \in (0,1]$$

where  $A^{\alpha L}$  and  $A^{\alpha R}$  are the left and right extreme points of the closed interval corresponding to the membership value  $\alpha$ .

3) First Decomposition Theorem on Fuzzy Sets: Every fuzzy set  $\tilde{A}$  defined on *Y*, the universal set of discourse, can be represented in the form  $\tilde{A} = \bigcup_{\alpha \in [0,1]} \alpha \cdot A[\alpha]$ , where the symbol

 $\bigcup$  is considered as standard fuzzy union.

4) Uncertain probability [9]: In probability distribution, if one or more parameters are not known with precision and are modeled by using fuzzy numbers are known as uncertain probability. Using fuzzy arithmetic, basic laws of uncertain probabilities can be developed [24].

Let *X* be a continuous random variable with probability density function f(x, v), where v is a parameter describing the density function. If v is considered as a fuzzy number  $\tilde{v}$ , then *X* becomes a fuzzily described random variable with density  $f(x, \tilde{v})$ , and the event  $P(c \le X < d)$  become a fuzzy set whose  $\alpha$ -cut is defined as

$$\left[\widetilde{P}(c \le X < d)\right][\alpha] = \left\{ \int_{c}^{d} f(x, v) dx \middle| v \in v[\alpha]; \int_{-\infty}^{\infty} f(x, v) dx = 1 \right\}, \text{ for all}$$

 $\alpha \in (0,1]$ .

The first two moments are also defined by their  $\alpha$ -cuts as for all  $\alpha \in (0,1]$ 

$$E(X)[\alpha] = \left\{ \int_{-\infty}^{\infty} xf(x,v) dx \middle| v \in v[\alpha] \text{ and } \int_{-\infty}^{\infty} f(x,v) dx = 1 \right\},$$
  

$$E[X - E(X)]^{2}[\alpha] = \left\{ \int_{-\infty}^{\infty} (x - E(X))^{2} f(x,v) dx \middle| v \in v[\alpha]; E(X) \in E(X)[\alpha]; \int_{-\infty}^{\infty} f(x,v) dx = 1 \right\}$$
  
Also the fuzzy random variables  $\widetilde{X}_{i}$ ,  $(i = 1, 2, ..., n)$  having joint density function  $f(x, x, ..., x \in \widetilde{\Omega})$  and marginal density

density function  $f_i(x_1, x_2, ..., x_n; \tilde{\upsilon})$  and marginal density function  $f_i(x_i; \tilde{\upsilon})$  are said to be independent if

$$f_i(x_1, x_2, ..., x_n; \tilde{\upsilon}) = \prod_{i=1}^n f_i(x_i; \upsilon) \text{ for } \alpha \in (0, 1] \text{ and for all } \upsilon \in \upsilon[\alpha].$$

5) *Fuzzy random variable:* A fuzzy random variable on a probability space  $(\Omega, \Phi, P)$  is a fuzzy valued function

 $X: \Omega \to \Phi_0(\mathfrak{R}), \ \omega \to X_{\omega}$  such that for every Borel set *B* of  $\mathfrak{R}$  and for every  $\alpha \in (0,1], (X[\alpha])^{-1}(B) \in \Phi$ . Here  $\Phi_0(\mathfrak{R})$  and  $X[\alpha]$  denote respectively for the set of fuzzy numbers and the set valued function

 $X[\alpha]: \Omega \to 2^{\Re}, \ \omega \to X_{\omega}[\alpha] = \left\{ x \in \Re | X_{\omega}(x) \ge \alpha \right\}.$ 

By decomposition theorem of fuzzy numbers it is stated that if  $\tilde{X}$  is a fuzzy random variable then it can be represented as  $\tilde{X} = \bigcup_{\alpha \in (0,1]} \alpha X[\alpha]$ . With the consideration of the above

discussions the proposed FMOCCP model is developed in the next section.

# III. FORMULATION OF FMOCCP MODEL

An FMOCCP problem having K number of objectives and the chance constraints, involved with normally distributed fuzzy random variables as right sided parameters, following joint normal distribution is presented as

Min 
$$Z_k = \sum_{j=1}^n c_{kj} x_j, k = 1, 2, ..., K$$

subject to

$$\Pr\left[\sum_{j=1}^{n} \widetilde{a}_{1j} x_j \ge \widetilde{b}_1, \sum_{j=1}^{n} \widetilde{a}_{2j} x_j \ge \widetilde{b}_2, \dots, \sum_{j=1}^{n} \widetilde{a}_{mj} x_j \ge \widetilde{b}_m\right] \ge 1 - p$$

$$x_j \ge 0, \ j = 1, 2, \dots, n \tag{1}$$

where  $\tilde{a}_{ij}$ , i = 1, 2, ..., m; j = 1, 2, ..., n are fuzzy numbers and  $\tilde{b}_i$ , i = 1, 2, ..., m are independent normally distributed fuzzy random variables whose mean and variance are described by fuzzy numbers. The probabilistic constraints in (1) is a joint probabilistic constraint with a specified probability level  $p \in \Re$  with  $0 , and <math>c_{kj} \in \Re$ .

Now the joint probability constraint in (1) is expressed as

$$\prod_{i=1}^{m} \Pr\left[\sum_{j=1}^{n} \widetilde{a}_{ij} x_j \ge \widetilde{b}_i\right] \ge 1 - p \quad , \ i = 1, 2, ..., m$$

$$\tag{2}$$

#### IV. FP MODEL CONSTRUCTION

In this section the FMOCCP model is converted into its equivalent FP model by using  $\alpha$ -cuts and CCP technique for joint probability distribution.

Let the mean,  $\tilde{\delta}_{\tilde{b}_i} = E(\tilde{b}_i)$ , and standard deviation,  $\tilde{\sigma}_{\tilde{b}_i} = \sqrt{\operatorname{var}(\tilde{b}_i)}$ , of the normally distributed fuzzy random variables  $\tilde{b}_i$ , i = 1, 2, ..., m are considered as triangular fuzzy numbers. On the basis of the  $\alpha$ -cuts defined for triangular fuzzy numbers associated with  $\tilde{a}_{ij}$  (i = 1, 2, ..., m; j = 1, 2, ..., n),  $\tilde{\delta}_{\tilde{b}_i}$  and  $\tilde{\sigma}_{\tilde{b}_i}$ , the joint probabilistic constraints in (2) is expressed as

$$\prod_{i=1}^{m} \Pr\left[\left\{ (\widetilde{b}_{i} - \delta_{i}) / \sigma_{i} \right\} \leq \left\{ \left( \sum_{j=1}^{n} u_{ij} x_{j} - \delta_{i} \right) / \sigma_{i} \right\} \right] \geq 1 - p$$
(3)

where  $\delta_i \in \delta_{\tilde{b}_i}[\alpha]$ ,  $\sigma_i \in \sigma_{\tilde{b}_i}[\alpha]$  and  $u_{ij} \in a_{ij}[\alpha]$  for all values of  $\alpha \in (0, 1]$ . Since for all values of  $\alpha \in (0, 1]$ ,  $\delta_i \in \delta_{\tilde{b}_i}[\alpha]$  and  $\sigma_i \in \sigma_{\tilde{b}_i}[\alpha]$ , then  $(\tilde{b}_i - \delta_i)/\sigma_i$  represents a fuzzy standard normal variate.

Therefore, (3) takes the form

$$\prod_{i=1}^{m} \Phi\left[\left(\sum_{j=1}^{n} u_{ij} x_{j} - \delta_{i}\right) \middle| \sigma_{i}\right] \ge 1 - p \tag{4}$$

where  $\Phi(.)$  represents the cumulative distribution function of the fuzzy standard normal variate for all values of  $\alpha \in (0, 1]$ .

Under the above context a fuzzy variable  $\hat{\beta}_i$  is introduced whose  $\alpha$ -cut is given by the following expression

$$\beta_i[\alpha] = \left[ \left( \sum_{j=1}^n a_{ij}[\alpha] x_j - \delta_{\tilde{b}_i}[\alpha] \right) \middle/ \sigma_{\tilde{b}_i}[\alpha] \right], \quad i = 1, 2, ..., m$$
(5)

Further, another fuzzy variate  $\tilde{y}_i$  is incorporated to represent the fuzzily described cumulative distribution function as  $\Phi(\xi_i) = \eta_i$  for all  $\xi_i \in \beta_i[\alpha]$  and  $\eta_i \in y_i[\alpha]$ , i = 1, 2, ..., m. (6)

Then 
$$\prod_{i=1}^{m} \eta_i \ge 1 - p \text{ for all } \eta_i \in y_i[\alpha], \quad i = 1, 2, ..., m$$
(7)

Since  $\xi_i \in \beta_i[\alpha]$ , for i = 1, 2, ..., m, is a standard normal variate, then its density function is given by

$$\Phi(\xi_i) = \left(1/\sqrt{2\pi}\right) \int_{-\infty}^{\xi_i} \exp\left(-\left(z^2/2\right)\right) dz \text{ where } \xi_i \in \beta_i[\alpha]$$
(8)

Now assuming  $(z^2/2) = t$ , the above expression transformed into the following forms as

$$\begin{split} \Phi(\xi_i) &= -\left(1/\sqrt{2\pi}\right) \int_{\xi_i^{2/2}}^{\xi_i} \exp(-t) t^{\left(\frac{1}{2}-1\right)} dt \text{, where } \xi_i \in \beta_i[\alpha] \\ \Phi(\xi_i) &= -\left(1/\sqrt{2\pi}\right) \int_{0}^{\xi_i} \exp(-t) t^{\left(\frac{1}{2}-1\right)} dt - \int_{0}^{\xi_i^{2/2}} \exp(-t) t^{\left(\frac{1}{2}-1\right)} dt \\ &= \frac{1}{2} \left[ \frac{1}{\Gamma\left(\frac{1}{2}\right)} \gamma\left(\frac{1}{2}, \frac{\xi_i^2}{2}\right) - 1 \right] = \frac{1}{2} \left[ P\left(\frac{1}{2}, \frac{\xi_i^2}{2}\right) - 1 \right], \text{ where } \xi_i \in \beta_i[\alpha]. \end{split}$$

Here  $P(a, \theta) = \gamma(a, \theta) / \Gamma(a) = (1/\Gamma(a)) \int_{0}^{\theta} \exp(-t)t^{(a-1)} dt$ , a > 0 is incomplete gammas function and satisfies the following conditions: P(a, 0) = 0 and  $P(a, \infty) = 1$  with

 $\gamma(a,\theta) = \exp(-\theta)\theta^a \sum_{r=0}^{\infty} (\Gamma(a)/\Gamma(a+1+r))\theta^r.$ 

Then the equation (6) is converted into the following form as

$$\exp\left(-\xi_i^2/2\right)\sum_{r=0}^{\infty} \left\lfloor \xi_i^{(2r+1)} \middle/ 2^r \Gamma\left(\frac{3}{2}+r\right) \right\rfloor = (2\eta_i+1)\sqrt{2} ,$$

where  $\xi_i \in \beta_i[\alpha]$  and  $\eta_i \in y_i[\alpha]$ . Now the above equation is to be simplified as

$$\sum_{r=0}^{\infty} \left( \xi_i^{(2r+1)} / \prod_{n=0}^r (2n+1) \right) = \sqrt{\pi/2} (2\eta_i + 1) \exp\left( \xi_i^2 / 2 \right)$$
(9)  
where  $\xi_i \in \beta_i[\alpha], \eta_i \in y_i[\alpha]$ .

Now, since the series  $\sum_{r=0}^{\infty} \left( \xi_i^{(2r+1)} / \prod_{n=0}^r (2n+1) \right)$  is convergent for any value of  $\xi_i$  where  $\xi_i \in \beta_i[\alpha]$ , it is expanded as:

$$\sum_{r=0}^{\infty} \left( \xi_i^{(2r+1)} / \prod_{n=0}^r (2n+1) \right) = \xi_i \left[ 1 + \frac{1}{3} \xi_i^2 + \frac{1}{15} \xi_i^4 + \frac{1}{105} \xi_i^6 + \dots \infty \right]$$
$$\leq \xi_i \left[ 1 + \frac{1}{3} \xi_i^2 + \frac{1}{3^2} \xi_i^4 + \frac{1}{3^3} \xi_i^6 + \dots \infty \right] = \xi_i \left[ 3 / \left( 3 - \xi_i^2 \right) \right]$$

where  $\xi_i^2 < 3$  and  $\xi_i \in \beta_i[\alpha]$ .

$$\left|3\xi_{i}/(3-\xi_{i}^{2})\right| \geq \sqrt{\pi/2}\left(2\eta_{i}+1\right)\exp\left(\xi_{i}^{2}/2\right) \tag{10}$$

i.e., 
$$\left[3\xi_i / (3 - \xi_i^2)\right] \exp\left(-\xi_i^2 / 2\right) \ge \sqrt{\pi/2} (2\eta_i + 1)$$
 (11)  
where  $\xi \in \mathcal{C}$  [ $\alpha$ ]  $n \in \mathcal{N}$  [ $\alpha$ ]

where  $\xi_i \in \beta_i[\alpha], \eta_i \in y_i[\alpha]$ .

With the help of (5), (7), and (11) the FP model is presented as

$$\begin{aligned} \text{Min } & Z_k = \sum_{j=1}^n c_{kj} x_j, k = 1, 2, ..., K \\ \text{subject to } & \left[ 3\xi_i / \left( 3 - \xi_i^2 \right) \right] \exp\left( - \xi_i^2 / 2 \right) \ge \sqrt{\pi/2} \left( 2\eta_i + 1 \right) \\ & \prod_{i=1}^m \eta_i \ge 1 - p \;, \; \sum_{j=1}^n u_{ij} x_j - \xi_i \sigma_i = \delta_i \;, \; x_j \ge 0 \\ \text{where } & \xi_i \in \beta_i[\alpha], \eta_i \in y_i[\alpha] \;, \; u_{ij} \in a_{ij}[\alpha] \;, \; \delta_i \in \delta_{\tilde{b}_i}[\alpha] \;, \; \sigma_i \in \sigma_{\tilde{b}_i}[\alpha] \\ & i = 1, 2, ..., m; \; j = 1, 2, ..., n \;. \end{aligned}$$

Now applying first decomposition theorem the problem in (12) is converted into the following form as

$$\begin{aligned} \text{Min } & Z_k = \sum_{j=1}^{n} c_{kj} x_j, k = 1, 2, ..., K \\ \text{subject to } & \left[ 3 \widetilde{\beta}_i / \left( 3 - \widetilde{\beta}_i^2 \right) \right] \exp \left( - \widetilde{\beta}_i^2 / 2 \right) \ge \sqrt{\pi / 2} (2 \widetilde{y}_i + 1) \\ & \prod_{i=1}^{m} \widetilde{y}_i \ge 1 - p \;, \; \sum_{j=1}^{n} \widetilde{a}_{ij} x_j - \widetilde{\beta}_i \widetilde{\sigma}_{\widetilde{b}_i} = \widetilde{\delta}_{\widetilde{b}_i} \\ & x_j \ge 0 \;, \; i = 1, 2, ..., m; \; j = 1, 2, ..., n \end{aligned}$$
(13)

Now in the current decision making situation, the parameters  $\tilde{a}_{ij}$ ; mean  $\tilde{\delta}_{\tilde{b}_i}$  and standard deviation  $\tilde{\sigma}_{\tilde{b}_i}$  of  $\tilde{b}_i$ , associated with the system constraints of the above problem (13) are considered as triangular fuzzy numbers with the respective form as

$$\widetilde{a}_{ij} = (a_{ij}^L, a_{ij}, a_{ij}^R), \ \widetilde{\delta}_{\widetilde{b}_i} = \left(\delta_{\widetilde{b}_i}^L, \delta_{\widetilde{b}_i}, \delta_{\widetilde{b}_i}^R\right), \ \widetilde{\sigma}_{\widetilde{b}_i} = \left(\sigma_{\widetilde{b}_i}^L, \sigma_{\widetilde{b}_i}, \sigma_{\widetilde{b}_i}^R\right)$$

On the basis of tolerance ranges of fuzzy numbers, the system constraints in (13) is decomposed as

Min 
$$Z_k = \sum_{j=1}^n c_{kj} x_j, k = 1, 2, ..., K$$

subject to

$$\begin{bmatrix} 3\beta_{i}^{L} / (3 - \beta_{i}^{L^{2}}) \end{bmatrix} \exp\left(-\beta_{i}^{L^{2}} / 2\right) \ge \sqrt{\pi/2} (2y_{i}^{L} + 1),$$

$$\begin{bmatrix} 3\beta_{i}^{R} / (3 - \beta_{i}^{R^{2}}) \end{bmatrix} \exp\left(-\beta_{i}^{R^{2}} / 2\right) \ge \sqrt{\pi/2} (2y_{i}^{R} + 1)$$

$$\sum_{j=1}^{n} a_{ij}^{L} x_{j} - \beta_{i}^{L} \sigma_{\tilde{b}_{i}}^{L} = \delta_{\tilde{b}_{i}}^{L}, \sum_{j=1}^{n} a_{ij}^{R} x_{j} - \beta_{i}^{R} \sigma_{\tilde{b}_{i}}^{R} = \delta_{\tilde{b}_{i}}^{R},$$

$$\prod_{i=1}^{m} y_{i}^{L} \ge 1 - p, \prod_{i=1}^{m} y_{i}^{R} \ge 1 - p$$

$$x_{j} \ge 0, i = 1, 2, ..., m; j = 1, 2, ..., n \qquad (14)$$

Here  $\beta_i^L$ ,  $y_i^L$  and  $\beta_i^R$ ,  $y_i^R$  represent the left and right tolerance limit of the fuzzy variable  $\tilde{\beta}_i$ ,  $\tilde{y}_i$  respectively, such that

$$\begin{split} \beta_i^L = & \left( \sum_{j=1}^n a_{ij}^L x_j - \delta_{\tilde{b}_i}^L \right) \middle/ \sigma_{\tilde{b}_i}^L , \, \beta_i^R = \left( \sum_{j=1}^n a_{ij}^R x_j - \delta_{\tilde{b}_i}^R \right) \middle/ \sigma_{\tilde{b}_i}^R , \\ y_i^L = \Phi(\beta_i^L) \text{ and } y_i^R = \Phi(\beta_i^R) . \end{split}$$

Now, since  $y_i^L$ ,  $y_i^R$  represent the value of cumulative distribution function, so  $0 \le y_i^L \le 1, 0 \le y_i^R \le 1$ , for i = 1, 2, ..., m and for any values of  $\beta_i^L, \beta_i^R$ .

Each objective is then solved in isolation under the decomposed set of system constraints defined in (14), to define the aspiration level of each of the fuzzy objective goals in the decision making situation.

Let  $(x_1^{kB}, x_2^{kB}, ..., x_n^{kB}, Z_k^B)$  and  $(x_1^{kW}, x_2^{kW}, ..., x_n^{kW}, Z_k^W)$  for k = 1, 2, ..., K be the best and worst value of the *k*-th objective of decision maker. Then the fuzzy goals of the problem is appeared as:  $Z_k \leq Z_k^B$  for k = 1, 2, ..., K.

# A) Construction of Membership Function

In a fuzzy decision making situation, the fuzzy goals are characterized by their associated membership function with the defined tolerance limits for achievement of their aspired goal levels. The membership function for the defined fuzzy goals can be constructed as

$$\mu_{k}(x) = \begin{cases} 1 & \text{if } Z_{k} \leq Z_{k}^{B} \\ \left( Z_{k}^{W} - Z_{k} \right) / \left( Z_{k}^{W} - Z_{k}^{B} \right) & \text{if } Z_{k}^{B} \leq Z_{K} \leq Z_{k}^{W} \\ 0 & \text{if } Z_{k} \geq Z_{k}^{W} \\ & \text{for } k = 1, 2, \dots, K . \end{cases}$$
(15)

# V. WEIGHTED FGP MODEL

The aim of a decision maker is to achieve the highest membership value of each of the associated fuzzy goal of the objectives. But in realistic situation, it is generally not possible to achieve all the membership values to its highest aspiration level simultaneously due to limitation of resources. In such a situation, the FGP technique [16] as an extension of conventional goal programming [25] for multiobjective decision making is used for achievement of the highest membership value of each fuzzy goal of the objectives to the extent possible in a decision making context.

The weighted FGP model of the problem (14) is presented as Find  $X(x_1, x_2, ..., x_n)$ 

so as to Min D = 
$$\sum_{k=1}^{K} M_k W_k d_k^-$$
  
and satisfy  $(Z_1^W - Z_1)/(Z_1^W - Z_1^B) + d_1^- - d_1^+ = 1$ ,  
 $(Z_2^W - Z_2)/(Z_2^W - Z_2^B) + d_2^- - d_2^+ = 1$ ,  
:  
 $(Z_k^W - Z_k)/(Z_k^W - Z_k^B) + d_k^- - d_k^+ = 1$ 

subject to the system constraints in (14) (16)

where  $d_k^-, d_k^+ \ge 0$  (k = 1, 2, ..., K) represent the under- and overdeviational variables, respectively, from the aspired level of the respective fuzzy goals.

To assess the relative importance of the fuzzy goals some numerical weights,  $M_k$ , (k = 1, 2, ..., K), are assigned together with the fuzzy weights,  $W_k$ , (k = 1, 2, ..., K), given by [18]  $W_k = 1/(Z_k^w - Z_k^B)(k = 1, 2, ..., K)$ 

The solution process of (16) is straight forward and is illustrated via the following example.

## VI. AN ILLUSTRATIVE EXAMPLE

Considering joint normal distribution associated with system constraints a modified version of the problem discussed by Sinha et al. [13] is considered to expound application potentiality of the proposed methodology. The FMOCCP problem is considered as

Find  $X(x_1, x_2)$ so as to  $Min = x_1 + 2x_2$   $Min = 3x_1 + 2x_2$ subject to  $Pr[\tilde{2}x_1 + \tilde{1}x_2 \ge \tilde{b}_1, \tilde{1}x_1 + \tilde{3}x_2 \ge \tilde{b}_2] \ge 0.85$  $x_1, x_2, x_3 \ge 0.$  (17)

Here  $\tilde{b}_1$  and  $\tilde{b}_2$  are normally distributed fuzzy random variables with mean  $E(\tilde{b}_1) = (5.5, 6, 6.5), E(\tilde{b}_2) = (6.5, 7, 7.5)$  and standard deviation  $\sigma(\tilde{b}_1) = (2.9, 3, 3.1), \sigma(\tilde{b}_2) = (3.9, 4, 4.1)$  and the parameters associated with the left side of the system constraints are described as

 $\tilde{1} = (0.95, 1, 1.05), \tilde{2} = (1.95, 2, 2.05), \tilde{3} = (2.95, 3, 3.05)$  fuzzy

umbers. Considering the proposed methodology, the above problem takes the following FP form as

Find  $X(x_1, x_2)$ 

so as to Min= $x_1 + 2x_2$ 

Ν

$$4 \sin = 3x_1 + 2x_2$$

subject to

 $\begin{aligned} (1.95, 2, 2.05)x_1 + (0.95, 1, 1.05)x_2 - (2.9, 3, 3.1)\beta_1 &= (5.5, 6, 6.5) \\ (1.95, 2, 2.05)x_1 + (2.95, 3, 3.05)x_2 - (3.9, 4, 4.1)\beta_2 &= (6.5, 7, 7.5) \\ 1.253(1+2\tilde{y}_1)(3-\tilde{\beta}_1^{\ 2}) &\leq 3\tilde{\beta}_1 \exp\left(-\tilde{\beta}_1^{\ 2}/2\right) \\ 1.253(1+2\tilde{y}_2)(3-\tilde{\beta}_2^{\ 2}) &\leq 3\tilde{\beta}_2 \exp\left(-\tilde{\beta}_2^{\ 2}/2\right) \\ \tilde{y}_1 &\geq 0.85, \ \tilde{y}_2 &\geq 0.85, \ 0 &\leq \ \tilde{y}_1 &\leq 1, \ 0 &\leq \ \tilde{y}_2 &\leq 1 \\ x_1, x_2, x_3 &\geq 0 \ \tilde{\beta}_1, \ \tilde{\beta}_2 \ \text{ are unrestricted in sign.} \end{aligned}$ (18)

On the basis of the tolerance ranges of the fuzzy numbers and fuzzy variables the problem (18) is decomposed as Find  $X(x_1, x_2)$ 

so as to Min= $x_1 + 2x_2$ 

$$Min = 3x_1 + 2x_2$$

subject to

$$1.95x_{1} + 9.5x_{2} - 2.9\beta_{1}^{L} = 5.5; \quad 2.05x_{1} + 1.05x_{2} - 3.1\beta_{1}^{R} = 6.5$$
  

$$0.95x_{1} + 2.95x_{2} - 3.9\beta_{2}^{L} = 6.5; \quad 1.05x_{1} + 3.05x_{2} - 4.1\beta_{2}^{R} = 7.5$$
  

$$1.253(1 + 2y_{1}^{L})(3 - \beta_{1}^{L^{2}}) \leq 3\beta_{1}^{L} \exp\left(-\beta_{1}^{L^{2}}/2\right)$$
  

$$1.253(1 + 2y_{1}^{R})(3 - \beta_{1}^{R^{2}}) \leq 3\beta_{1}^{R} \exp\left(-\beta_{1}^{R^{2}}/2\right)$$
  

$$1.253(1 + 2y_{2}^{L})(3 - \beta_{2}^{L^{2}}) \leq 3\beta_{2}^{L} \exp\left(-\beta_{2}^{L^{2}}/2\right)$$
  

$$1.253(1 + 2y_{2}^{R})(3 - \beta_{2}^{R^{2}}) \leq 3\beta_{2}^{R} \exp\left(-\beta_{2}^{L^{2}}/2\right)$$
  

$$1.253(1 + 2y_{2}^{R})(3 - \beta_{2}^{R^{2}}) \leq 3\beta_{2}^{R} \exp\left(-\beta_{2}^{R^{2}}/2\right)$$

 $y_{1}^{L} \cdot y_{2}^{L} \ge 0.85, \quad y_{1}^{R} \cdot y_{2}^{R} \ge 0.85$   $y_{1}^{L}, y_{1}^{R}, y_{2}^{L}, y_{2}^{R} \ge 0; y_{1}^{L}, y_{1}^{R}, y_{2}^{L}, y_{2}^{R} \le 1 \ x_{1}, x_{2}, x_{3} \ge 0 \text{ and}$  $\beta_{1}^{L}, \beta_{1}^{R}, \beta_{2}^{L}, \beta_{2}^{R} \text{ are unrestricted in sign.}$ (19)

Solving each of the objectives independently under the same system constraints, the best and worst solution achieved as  $\begin{pmatrix} x^B & x^B \end{pmatrix} = (3.480.2.74)^2 \times (3.480.2.74)^2$ 

$$(x_1, x_2; z_1) = (3.480, 2.741; 8.962),$$
  
 $(x_1^B, x_2^B; z_2^B) = (3.394, 2.813; 15.809)$   
and  $(x_1^W, x_2^W; z_1^W) = (3.394, 2.813; 9.021);$   
 $(x_1^W, x_2^W; z_2^W) = (3.480, 2.741; 15.922).$   
Hence the fuzzy goal of the objective

Hence the fuzzy goal of the objectives are appeared as  $z_1 \leq 8.962$  and  $z_2 \leq 15.809$  respectively

Using the above numerical values the membership function for the defined fuzzy goals of the objectives are obtained as

$$\mu_{z_1}(x_1, x_2) = \begin{cases} 1 & \text{if} \quad z_1 \le 8.962 \\ 16.949(9.021 - z_1) & \text{if} \quad 8.962 \le z_1 \le 9.021 \\ 0 & \text{if} \quad z_1 \ge 9.021 \end{cases}$$
$$\mu_{z_2}(x_1, x_2) = \begin{cases} 1 & \text{if} \quad z_2 \le 15.809 \\ 8.85(15.922 - z_2) & \text{if} \quad 15.809 \le z_2 \le 15.922 \\ 0 & \text{if} \quad z_2 \ge 15.922 \end{cases}$$

Then the executable weighted FGP model can be written as Find  $X(x_1, x_2)$ 

so as to Min  $D = 90.186d_1^- + 88.496d_2^-$ 

satisfying  $16.949(9.021 - (x_1 + 2x_2)) + d_1^- - d_1^+ = 1$ 8 85(15922 - (3x + 2x )) +  $d_1^- - d_1^+ = 1$ 

$$8.85(15.922 - (3x_1 + 2x_2)) + d_2 - d_2 = 1$$

subject to system constraints in (19) with  $d_1^-, d_1^+, d_2^-, d_2^+ \ge 0$ ;

$$d_1^- \cdot d_1^+ = d_2^- \cdot d_2^+ = 0 \tag{20}$$

The problem (20) is solved by using the software LINGO (6.0). The optimal solution of the problem is obtained as  $x_1 = 3.437$  and  $x_2 = 2.777$  with the achieved objective values  $Z_1 = 8.991_* Z_2 = 15.864$ 

The solution achieved here is most satisfactory in terms of achieving aspired goal levels of the objectives of the decision maker.

The solution obtained by using the methodology developed by Sinha et al. [13] is  $x_1 = 3.887$ ,  $x_2 = 3.221$  and the corresponding objective values are  $Z_1 = 10.33$  and  $Z_2 = 18.106$ . Both the achieved values of the objectives obtained by Sinha et. al. [13] is inferior to the achieved objective values obtained by using the developed methodology.

### VII. CONCLUSIONS

This paper captures the idea of fuzziness and randomness simultaneously which is inherent in model formulation process due to the DM's vague understanding of the nature of parameters associated with the problem. The superiority of the proposed technique has also been reflected by comparing with other existing technique. The model is flexible enough to assign different goal values of the objectives of the DMs. The

proposed methodology can also be extended to solve FMOCCP problems having fuzzy random variables which follow other types of joint probability distributions. Further the developed technique can be applied to solve FMOCCP model involving some fuzzily defined parameters or fuzzy random variables with the objectives. Also the proposed methodology can be used to solve bilevel or multilevel optimization problems in a fuzzy stochastic decision making arena. However it is concluded that the described methodology may add a new dimension into the way of solving FMOCCP in a fuzzily defined probabilistic decision making environment.

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