Controllability of Linear Time-invariant Dynamical Systems with Fuzzy Initial Condition

Bhaskar Dubey, Raju K. George

***Abstract*—In this paper, we investigate controllability property of the linear time-invariant systems of the form $\dot{x} = Ax(t) + Bu(t)$ with fuzzy initial condition $x(t_0)$ in $(\mathbb{E}^1)^n$ and control $u(t) \in (\mathbb{E}^1)^m$, where A, B, are $n \times n$, $n \times m$ real matrices, respectively, $t_0 \geq 0$, and $(\mathbb{E}^1)^n$ denotes the set of all $n-$dimensional vectors of fuzzy numbers on $\mathbb{R}$. We establish sufficient conditions for the controllability of such systems. Examples are given to substantiate the results obtained.**

***Index Terms*—Fuzzy dynamical systems, Controllability, Fuzzy-number, Fuzzy-state**

I. Introduction

IN most of the physical applications we do not have the exact value of the initial condition from which the dynamical system begins to evolve. This could be due to the fact that the precise measurement of the data could be costly or practically impossible. If the error in estimation of the initial states are not too random, they can be defined by fuzzy sets or fuzzy numbers. Similarly, the desired final state can also be modelled as a fuzzy set. Thus, the problem of steering an initial state of a system to a desired final state in $\mathbb{R}^n$, will essentially become a problem of steering a fuzzy-state to another fuzzy-state in $(\mathbb{E}^1)^n$.

Controllability of fuzzy dynamical control systems is a very important concept in the design of fuzzy systems. Broadly fuzzy systems are classified mainly in three categories, namely pure fuzzy systems in which the dynamics of the fuzzy system is governed by a fuzzy differential equation, T-S fuzzy systems and fuzzy logic systems which uses fuzzifiers and defuzzifiers. Controllability of fuzzy systems has been explored by many authors, for example, Cai and Tang [2], Ding and Kandel ([3], [4]), S. S. Farinwata et al. [8], Y. Feng et el. [9], M.M. Gupta et al. [11]. Recently, Biglarbegian et al. [1] have studied the accessability and controllability properties of T-S fuzzy logic control systems by using differential geometric and Lie-algebraic techniques. In [5], the authors introduced the concept of fuzzy-controllability, a concept weaker than controllability, for the systems of type $\dot{x} = Ax(t)+Bu(t), x(0) = X_0 \in (\mathbb{E}^1)^n$ and established sufficient conditions for such systems to be fuzzy-controllable. In [9], the authors studied a concept of quasi-controllability for the fuzzy dynamical systems described by linear fuzzy differential equations.

In this paper, we consider linear time-invariant systems with fuzzy initial condition and establish results on controllability properties of the system. The results in this paper can be regarded as the extension of some of the results in [5] and [9]. Firstly, the concept of controllability developed in this paper is stronger than the concept of fuzzy-controllability established in [5], that is, controllability implies fuzzy-controllability. Secondly, in [9] the authors assumed the initial condition to be in $\mathbb{R}^n$, whereas we establish our results by assuming the initial condition to be in $(\mathbb{E}^1)^n$, a much wider class than $\mathbb{R}^n$. Furthermore, we prove that the controllability of the pair (A^*, B^*) obtained by flip operations (for flip operations, see Remark III.2 and [5],[15]) on the matrix pair (A, B) is equivalent to the controllability of the pair (A, B) and the pair $(|A|, |B|)$, together.

The organization of the paper is as follows: In Section II, we state preliminary definitions and results on the fuzzy system theory. In Section III, we briefly describe the evolution of solutions of linear time-invariant systems with fuzzy initial condition. In Section IV, we establish the controllability results for linear time-invariant systems with fuzzy initial condition. In Section V, some examples are given to illustrate the results obtained. Finally, we conclude the paper in Section VI.

II. Preliminaries

Let $\mathbb{R}^n$, and $\mathbb{R}^n_+$ denote the set of all n-dimensional real vectors and n-dimensional non-negative real vectors, respectively. Given a real matrix A, $|A|$ denotes the matrix of the size as that of A and whose entries are the absolute values of the corresponding entries in A. $\mathbb{E}^1$ denotes the set of all fuzzy numbers on $\mathbb{R}$.

Definition II.1. *By a fuzzy number on $\mathbb{R}$, we mean a mapping $\mu : \mathbb{R} \to [0, 1]$ with the following properties:*

(i) *μ is upper semi continuous.*
(ii) *μ is fuzzy convex, that is, $\mu(\alpha x + (1 - \alpha)y) \geq min(\mu(x), \mu(y))$ for all $x, y \in \mathbb{R}$.*
(iii) *μ is normal, that is, there exists $x_0 \in \mathbb{R}$ such that $\mu(x_0) = 1$.*
(iv) *Closure of the support of μ is compact, that is, $cl(x \in \mathbb{R} : \mu(x) > 0)$ is compact in $\mathbb{R}$.*

For every $\mu \in \mathbb{E}^1$, an α-level set or α-cut of μ is denoted by μ^α or $[\mu]_\alpha$. For $\alpha \in (0, 1]$, it is defined as follows:

$$\mu^\alpha = \{x : \mu(x) \geq \alpha\}.$$

For $\alpha = 0$, the 0-cut of μ is defined as the closure of union of all non zero α-cuts of μ. That is:

$$\mu^0 = \overline{\bigcup_{\alpha\in(0,1]} \mu^\alpha}.$$

It can be easily shown that for every $\mu \in \mathbb{E}^1$, μ^α is a closed and bounded interval $[\underline{\mu^\alpha}, \overline{\mu^\alpha}]$, where $\underline{\mu^\alpha}$, $\overline{\mu^\alpha}$ are called the

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lower and upper $\alpha-$cut of μ, respectively.

We can easily see that a fuzzy number is characterized by the endpoints of the intervals μ^α. Thus a fuzzy number μ can be identified by a parameterized triple

$$\{(\underline{\mu^\alpha}, \overline{\mu^\alpha}, \alpha)|, \alpha \in [0,1]\}.$$

The following Lemma due to Goetschel and Voxman [10] provides a characterization of fuzzy numbers.

Lemma II.2. *(Goetschel and Voxman [10]) Assume that* $I = [0,1]$*, and* $a : I \to \mathbb{R}$ *and* $b : I \to \mathbb{R}$ *satisfy the conditions:*

(a) $a : I \to \mathbb{R}$ *is a bounded increasing function.*
(b) $b : I \to \mathbb{R}$ *is a bounded decreasing function.*
(c) $a(1) \leq b(1)$
(d) *For* $0 < k \leq 1$, $\lim_{\alpha\to k^-} a(\alpha) = a(k)$ *and* $\lim_{\alpha\to k^-} b(\alpha) = b(k)$
(e) $\lim_{\alpha\to 0^+} a(\alpha) = a(0)$ *and* $\lim_{\alpha\to 0^+} b(\alpha) = b(0)$

Then $\mu : \mathbb{R} \to I$ *defined by*

$$\mu(x) = \sup\{\alpha | a(\alpha) \leq x \leq b(\alpha)\}$$

is a fuzzy number with parametrization given by $\{(a(\alpha), b(\alpha), \alpha) | 0 \leq \alpha \leq 1\}$*. Moreover, if* $\mu : \mathbb{R} \to I$ *is a fuzzy number with parametrization given by* $\{(a(\alpha), b(\alpha), \alpha) | 0 \leq \alpha \leq 1\}$*, then functions* $a(\alpha)$ *and* $b(\alpha)$ *satisfy conditions* $(a) - (e)$*.*

Given two vectors of fuzzy numbers $X_0 = [X_{01}, X_{02}, \ldots, X_{0n}]^T$, $X_1 = [X_{11}, X_{12}, \ldots, X_{1n}]^T$ in $(\mathbb{E}^1)^n$, we say $X_0 \leq X_1$ if $\mu_{X_{0i}}(\cdot) \leq \mu_{X_{1i}}(\cdot)$, $1 \leq i \leq n$ and $X_0 = X_1$ if $\mu_{X_{0i}}(\cdot) = \mu_{X_{1i}}(\cdot)$, $1 \leq i \leq n$, where $\mu_{X_{0i}}(\cdot)$, $\mu_{X_{1i}}(\cdot)$ are the membership functions of X_{0i}, X_{1i}, respectively. Arithmetic fuzzy addition and scalar multiplication in $\mathbb{E}^1$ are defined by using the extension principle [7]. Let $u, v \in \mathbb{E}^1$ and $\beta \in \mathbb{R}$.

$$(u+v)(x) = \sup_{x=y+z} \min(u(y), v(z)), x \in \mathbb{R}$$

$$(\beta u)(x) = \begin{cases} u(\frac{x}{\beta})\} & \text{if } \beta \neq 0 \\ \tilde{0} & \text{if } \beta = 0, \end{cases}$$

where $\tilde{0} \in \mathbb{E}^1$ is defined as follows

$$\tilde{0}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$$

Let the symbol $\mathbb{P}_k(\mathbb{R})$ denote the family of all non empty convex, compact subsets of $\mathbb{R}$.

Definition II.3. *The Hausdorff metric on* $\mathbb{P}_k(\mathbb{R})$ *is defined as*

$$d(A,B) = inf\{\epsilon | A \subset N(B,\epsilon) \, and \, B \subset N(A,\epsilon)\}, \quad (1)$$

where A, $B \in \mathbb{P}_k(\mathbb{R})$ *and* $N(A,\epsilon) = \{x \in \mathbb{R}^n | \|x - y\| < \epsilon$ *for some* $y \in A\}$, $N(B,\epsilon)$ *is similarly defined.*

We can define a metric on E^1 by using Hausdorff metric. Define $D : \mathbb{E}^1 \times \mathbb{E}^1 \to \mathbb{R}^+ \cup \{0\}$ by

$$D(u,v) = \sup_{0\leq\alpha\leq 1} d(u^\alpha, v^\alpha),$$

where d is the Hausdorff metric defined on $\mathbb{P}_k(\mathbb{R})$.

Definition II.4. *A mapping* $F : T = [a,b] \to \mathbb{E}^1$ *is differentiable at* $t_0 \in T$ *if there exists a* $\dot{F}(t_0) \in \mathbb{E}^1$ *such that the limits*

$$\lim_{h\to 0^+} \frac{F(t_0+h) - F(t_0)}{h}, \lim_{h\to 0^+} \frac{F(t_0) - F(t_0-h)}{h},$$

exist and equal to $\dot{F}(t_0)$*. Here the limits are taken in the metric* $(\mathbb{E}^1, D)$*.*

Suppose the parametric form of $F(t)$ is represented by

$$F(t) = \{(F_1(t,\alpha), F_2(t,\alpha), \alpha) : \alpha \in [0,1], t \in T\}.$$

The Seikkala [13] derivative $\dot{F}(t)$ of a fuzzy function $F(t)$ is defined by

$$\dot{F}(t) = \{(\dot{F_1}(t,\alpha), \dot{F_2}(t,\alpha), \alpha) : \alpha \in [0,1], t \in T\} \quad (2)$$

provided that the above equation defines a fuzzy number.

III. Evolution of solutions of time-invariant fuzzy dynamical systems

Since we are interested in the controllability of the following system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(t_0) = X_0, \end{cases} \quad (3)$$

where A and B are the $n \times n$, and $n \times m$ real matrices respectively, $X_0 \in (\mathbb{E}^1)^n$, the input $u(t) \in (\mathbb{E}^1)^m$ for each $t \in [t_0, t_1]$ and $u(\cdot)$ is fuzzy-integrable (see [12],[13]) in $[t_0, t_1]$. Therefore, it is important to understand the structure of the solutions of (3). We will now briefly describe the evolution of the solutions of the system (3). The fuzziness in the control and initial condition makes the system (3) a fuzzy dynamical control system. Thus, it is clear that state of the system at any time $t \in [t_0, t_1]$, starting from the initial state X_0, belongs to $(\mathbb{E}^1)^n$, that is, $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ in $(\mathbb{E}^1)^n$. We now introduce new variables which we shall use throughout this paper.

$$\underline{x^\alpha}(t) := [\underline{x_1^\alpha}(t), \underline{x_2^\alpha}(t), \ldots, \underline{x_n^\alpha}(t)]^T$$

$$\overline{x^\alpha}(t) := [\overline{x_1^\alpha}(t), \overline{x_2^\alpha}(t), \ldots, \overline{x_n^\alpha}(t)]^T,$$

where $[\underline{x_k^\alpha}(t), \overline{x_k^\alpha}(t)]$ is the α-cut of $x_k(t)$ for $1 \leq k \leq n$. $\underline{u^\alpha}(t)$ and $\overline{u^\alpha}(t)$ are similarly defined. We denote $x_*^\alpha(t) := [\underline{x^\alpha}(t), \overline{x^\alpha}(t)]^T := [\underline{x_1^\alpha}(t), \underline{x_2^\alpha}(t), \ldots, \underline{x_n^\alpha}(t), \overline{x_1^\alpha}(t), \overline{x_2^\alpha}(t), \ldots, \overline{x_n^\alpha}(t)]^T$ a column vector of size $2n$. $u_*^\alpha(t)$ and $[u(t)]_\alpha$ are similarly defined.

Using these variables we construct a $2n-$dimensional system following the idea suggested in [13] to study the evolution of system (3).

Lemma III.1. *For* $\alpha \in (0,1]$*, let* $x_k^\alpha(t) = [\underline{x_k^\alpha}(t), \overline{x_k^\alpha}(t)]$ *be the* α*-cut of* $x_k(t)$ *for* $1 \leq k \leq n$ *and* $u_j^\alpha(t) = [\underline{u_j^\alpha}(t), \overline{u_j^\alpha}(t)]$ *be the* $\alpha-$*cut of* $u_j(t)$ *for* $1 \leq j \leq m$ *then the evolution of system* (3) *is described by the following* $2n-$*differential equations:*

$$\begin{cases} \dot{\underline{x_k^\alpha}}(t) = min((Az + Bw)_k : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)], \\ \qquad\qquad w_j \in [\underline{u_j^\alpha}(t), \overline{u_j^\alpha}(t)]) \\ \dot{\overline{x_k^\alpha}}(t) = max((Az + Bw)_k : z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)], \\ \qquad\qquad w_j \in [\underline{u_j^\alpha}(t), \overline{u_j^\alpha}(t)]) \\ \underline{x_k^\alpha}(t_0) = \underline{x_{0k}^\alpha} \\ \overline{x_k^\alpha}(t_0) = \overline{x_{0k}^\alpha}, \end{cases} \quad (4)$$

where $1 \leq k \leq n$, *and* $(Az+Bw)_k = \Sigma_{i=1}^{n} a_{ki} z_i + \Sigma_{j=1}^{m} b_{kj} w_j$ *is the* k^{th} *row of* $Az+Bw$.

Proof: A detailed proof of the above Lemma is given in [6]. However, we sketch the brief outline of the proof. The Seikkala [13] derivative $\dot{x}(t)$ of the fuzzy process $x : \mathbb{R}_+ \to (\mathbb{E}^1)^n$ is given by $[\dot{x}_k(t)]_\alpha = [\dot{\underline{x_k^\alpha}}(t), \dot{\overline{x_k^\alpha}}(t)]$, $\alpha \in (0,1]$ and $1 \leq k \leq n$. On the other hand, by using extension principle it can be shown that the $\alpha-$cut of the k^{th} row from R.H.S. of (3) is given by $[min(Az+Bw)_k, max(Az+Bw)_k]$, where $z_i \in [\underline{x_i^\alpha}(t), \overline{x_i^\alpha}(t)]$ for $1 \leq i \leq n$, and $w_j \in [\underline{u_j^\alpha}(t), \overline{u_j^\alpha}(t)]$ for $1 \leq j \leq m$. Hence the lemma is proved. ■

Remark III.2. *By using the above Lemma, the evolution of system* (3) *can be given by a system in a compact form as described below (see also Xu et el. [15], Dubey and George [5])*:
For $\alpha \in [0,1]$, $\dot{x_*^\alpha}(t) = A^* x_*^\alpha(t) + B^* u_*^\alpha(t)$, $x_*^\alpha(t_0) = X_{0\,*}^\alpha$ *in which* A^* *and* B^* *are defined as follows*:

(i) *If A has all its entries non-negative then* $A^* = M$ *and* $B^* = N$, *where*

$$M = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \qquad N = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$$

i.e, M *is a block diagonal matrix of size* $2n \times 2n$ *and* N *is a block diagonal matrix of size* $2n \times 2m$. *We denote* $M = [m_{ij}], 1 \leq i,j \leq 2n$ *and* $N = [n_{ij}], 1 \leq i \leq 2n, 1 \leq j \leq 2m$. *Furthermore the symbol* "$m_{ij} \longleftrightarrow m_{kl}$" *means that the entry in* i^{th} *row and* j^{th} *column of M is swapped by the entry in* k^{th} *row and* l^{th} *column of* M, *and vice versa.* "$n_{ij} \longleftrightarrow n_{kl}$" *is similarly defined.*

(ii) *If A has some of its entries negative then* A^* *is obtained by the following flip operations on the entries of* M.
$m_{ij} \longleftrightarrow m_{i(j+n)}$ *if* $1 \leqslant j \leqslant n$ *and* $m_{ij} < 0$,
$m_{ij} \longleftrightarrow m_{i(j-n)}$ *if* $n < j \leqslant 2n$ *and* $m_{ij} < 0$.

(iii) *If B has some of its entries negative then* B^* *is obtained by the following flip operations on the entries of* N.
$n_{ij} \longleftrightarrow n_{i(j+m)}$ *if* $1 \leqslant j \leqslant m$ *and* $n_{ij} < 0$,
$n_{ij} \longleftrightarrow n_{i(j-m)}$ *if* $m < j \leqslant 2m$ *and* $n_{ij} < 0$.

(iv) *If* $u(t) \in \mathbb{R}^m$ *is a crisp vector instead of being a vector of fuzzy numbers, then* B^* *can be taken as* N *and in this case* $u_*^\alpha(t) = [u(t), u(t)]^T$.

The flip operations in Remark III.2 are illustrated by an example in the Appendix B.

IV. Main Results

In this section, we establish some controllability results for the system (3). Before proving the main results we will briefly state some controllability results for the crisp systems which we shall use in establishing the controllability results for the fuzzy dynamical systems. Consider the linear time-invariant system $\dot{x}(t) = Ax(t) + Bu(t), x(t_0) = x_0 \in \mathbb{R}^n$, where A, B are $n \times n$, $n \times m$ real matrices, respectively. The system is completely controllable during time interval $[t_0, t_1]$ or the pair (A, B) is controllable during $[t_0, t_1]$ if any of the following conditions hold (see [14]):

(i) Controllability Grammian $W(t_0, t_1)$ defined by
$$W(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_0,\tau) B B^T \Phi^T(t_0,\tau) d\tau$$
is non-singular, where $\Phi(t,\tau)$ denotes the transition matrix for the system $\dot{x}(t) = Ax(t)$.

(ii) No eigenvector of A^T lies in the kernel of B^T (PBH Test).

(iii) Rank of controllability matrix $[B|AB|A^2B|\ldots|A^{n-1}B] = n$.

We will now define the controllability for the fuzzy system (3).

Definition IV.1. *(Controllability) The system* (3) *with fuzzy initial condition* $x(t_0) = X_0 \in (\mathbb{E}^1)^n$ *is said to be controllable to a fuzzy-state* $X_1 \in (\mathbb{E}^1)^n$ *at* $t_1 (> t_0)$ *if there exists a fuzzy-integrable control* $u(t) \in (\mathbb{E}^1)^m$ *for* $t \in [t_0, t_1]$ *such that the solution of system* (3) *with this control satisfies* $x(t_1) = X_1$.

Remark IV.2. *A concept of fuzzy-controllability, weaker than the controllability defined in the Definition IV.1, was introduced in [5]. In fuzzy-controllability, we do not require* $x(t_1) = X_1$ *instead one looks for a control* $u(t) \in (\mathbb{E}^1)^m$ *with which the solution of system* (3) *satisfies* $x(t_1) \leq X_1$.

We will now give sufficient conditions for the controllability of fuzzy dynamical system (3). If the pair (A^*, B^*) is controllable, where A^* and B^* are obtained by the process defined in Remark III.2 of Section 3, then a control $u(\cdot)$ which steers a state x_0 in $\mathbb{R}^{2n}$ to a desired state x_1 in $\mathbb{R}^{2n}$ during time interval $[t_0, t_1]$ is given by

$$\begin{aligned} u(t) &\triangleq \eta(t, t_0, t_1, x_0, x_1) \\ &:= B^{*T} \Phi^{*T}(t_0, t) W^{*-1}(t_0, t_1)[\Phi^*(t_0, t_1)x_1 - x_0], \end{aligned}$$

where $\Phi^*(t,\tau)$ denotes the transition matrix for the system $\dot{x}(t) = A^* x(t)$ and $W^*(t_0, t_1)$ is the controllability Grammian for the system $\dot{x}(t) = A^* x(t) + B^* u(t)$.

Theorem IV.3. *The system* (3) *with fuzzy initial condition* $X_0 \in (\mathbb{E}^1)^n$ *is controllable to* $X_1 \in (\mathbb{E}^1)^n$ *during time interval* $[t_0, t_1]$ *if*

(i) *The Pair* (A^*, B^*) *is controllable.*

(ii) *The function* $u(\cdot)$, *characterized by* $[u(t)]_\alpha = [\underline{u^\alpha}(t), \overline{u^\alpha}(t)]$, *where* $\underline{u^\alpha}(t), \overline{u^\alpha}(t)$ *are defined by* $[\underline{u^\alpha}(t), \overline{u^\alpha}(t)]^T := \eta(t, t_0, t_1, X_{0\,*}^\alpha, X_{1\,*}^\alpha)$, *belongs to* $\mathbb{E}^m$.

Proof: Let X_0 be the initial fuzzy-state at time t_0 and X_1 be the prescribed fuzzy-state at time t_1. The dynamics of the system (3), under the assumptions when $x(t) \in (\mathbb{E}^1)^n$ and $u(t) \in (\mathbb{E}^1)^m$, is given by the following levelwise set of equations:

$$\dot{x_*^\alpha}(t) = A^* x_*^\alpha(t) + B^* u_*^\alpha(t), \alpha \in (0,1] \tag{5}$$

Using condition (i) it follows that for each $\alpha \in (0,1]$, there exists a control $\tilde{u}_*^\alpha(t) := \eta(t, t_0, t_1, X_{0\,*}^\alpha, X_{1\,*}^\alpha)$ with which the solution of (5) with initial crisp state $x_*^\alpha(t_0) = X_{0\,*}^\alpha$ satisfies $x_*^\alpha(t_1) = X_{1\,*}^\alpha$. Condition (ii) now implies that there exists a function $\tilde{u}(\cdot)$ such that $\tilde{u}(t) \in (\mathbb{E}^1)^m$ for each $t \in [t_0, t_1]$ and $[\underline{\tilde{u}^\alpha}(t), \overline{\tilde{u}^\alpha}(t)]^T = \eta(t, t_0, t_1, X_{0\,*}^\alpha, X_{1\,*}^\alpha)$. Since $\tilde{u}_*^\alpha(t)$ is integrable in $[t_0, t_1]$, therefore $\int_{t_0}^{t_1} \underline{\tilde{u}^\alpha}(t)$ and

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$\int_{t_0}^{t_1} \overline{u^\alpha}(t)$ are well defined, which implies that $\tilde{u}(t)$ is fuzzy-integrable in $[t_0, t_1]$ (see [13]). Hence $\tilde{u}(t)$ is a fuzzy-controller with which the solution of (3) with fuzzy initial condition $x(t_0) = X_0$ satisfies $x(t_1) = X_1$. Hence system (3) with initial condition X_0 is controllable to X_1 during $[t_0, t_1]$. ■

Remark IV.4. *The condition (ii) of Theorem IV.3 inherently states that controllability of system* (3) *not only depends on matrices A and B but also on initial and final fuzzy-states, whereas crisp-controllability of system* (3) *depends only on matrices A and B. Therefore, given any arbitrary initial state $X_0 \in (\mathbb{E}^1)^n$ it may not be possible to control the system to an arbitrary state $X_1 \in (\mathbb{E}^1)^n$. However, if the initial state is crisp, that is, $X_0 \in \mathbb{R}^n$ then the set of all reachable fuzzy states from X_0 can be characterized more precisely by using a result due to Feng et el. [9][Theorem 3.4]. Thus we have the following theorem.*

Theorem IV.5. *The fuzzy control system $\dot{x} = Ax(t) + Bu(t)$ with the arbitrary initial condition $x_0 \in \mathbb{R}^n$ can be steered to any fuzzy state in the admissible controllable state subset $(\mathbb{E}_0^1)^n$ of $(\mathbb{E}^1)^n$ if and only if the pair (A^*, B^*) is controllable. And the admissible controllable state subset $(\mathbb{E}_0^1)^n$ of $(\mathbb{E}^1)^n$ is given by:*

$$(\mathbb{E}_0^1)^n = \{V \in (\mathbb{E}^1)^n \mid \overline{V}^1 - \underline{V}^1 \in \bigcap_{t_0 \le t \le t_1} (\Psi(t))^{-1}\mathbb{R}_+^m \text{ and}$$

$$\frac{d}{d\alpha}\begin{pmatrix} \underline{V}^\alpha \\ -\overline{V}^\alpha \end{pmatrix} \in \bigcap_{t_0 \le t \le t_1} (\Psi^*(t))^{-1}\mathbb{R}_+^{2m}, \tag{6}$$

$$\alpha \in (0, 1], \} \tag{7}$$

where $\Psi(t)$, $\Psi^*(t)$ are defined as follows:

$$\Psi(t) = |B|^T \Phi_{|A|}^T(t_1, t) W_1^{-1}(t_1, t_0)$$

where $\Phi_{|A|}(t, s)$ is the transition matrix for the system $\dot{x} = |A|x$ and $W_1(t_1, t_0)$ is defined by

$$W_1(t_1, t_0) = \int_{t_0}^{t_1} \Phi_{|A|}(t_1, s)|B||B|^T \Phi_{|A|}^T(t_1, s) ds.$$

$$\Psi^*(t) = |B^*|^T \Phi_{|A^*|}^T(t_1, t) W_2^{-1}(t_1, t_0),$$

where $\Phi_{|A^*|}(t, s)$ is the transition matrix for the system $\dot{x} = |A^*|x$ and $W_2(t_1, t_0)$ is defined by

$$W_2(t_1, t_0) = \int_{t_0}^{t_1} \Phi_{|A^*|}(t_1, s)|B^*||B^*|^T \Phi_{|A^*|}^T(t_1, s) ds.$$

Proof: It can be shown that the controllability of the pair (A^*, B^*) is equivalent to the controllability of pair $(|A^*|, |B^*|)$(see Appendix A). Now the proof follows along the similar lines of the proof of Theorem 3.4 *of Feng et al.[9].* ■

We will now provide a closed form formula for the steering fuzzy control that can be applied to the systems of type (3) with the matrices A, B having non-negative entries. For a matrix A, $A \geq 0$, we mean that all the entries of A are non-negative. When $A \geq 0$ and $B \geq 0$, we have the following result.

Theorem IV.6. *Let $A, B \geq 0$ in system* (3) *and $W(t_0, t_1)$ is non singular then a fuzzy-controller, which steers an initial fuzzy-state $X_0 \in (\mathbb{E}^1)^n$ to a desired fuzzy-state $X_1 \in (\mathbb{E}^1)^n$ during time interval $[t_0, t_1]$, is given by*

$$u(t) = B^T \Phi^T(t_0, t) W^{-1}(t_0, t_1)(\Phi(t_0, t_1)\widetilde{X_1} - X_0) \tag{8}$$

provided $\widetilde{X_1} \in \mathbb{E}^n$ with α-level sets given by $[\widetilde{X_1}]_\alpha = [\underline{X_1^\alpha} + \Phi(t_1, t_0)(\overline{X_0^\alpha} - \underline{X_0^\alpha}), \overline{X_1^\alpha} - \Phi(t_1, t_0)(\overline{X_0^\alpha} - \underline{X_0^\alpha})]$.

Proof: Under the condition $A, B \geq 0$, the evolution of the system (3) with the control given in (8), is given by the following set of levelwise decomposed linear differential equations.(see Remark III.2)

$$\begin{cases} \dot{\underline{x^\alpha}}(t) = A\underline{x^\alpha}(t) + B\underline{u^\alpha}(t) \\ \dot{\overline{x^\alpha}}(t) = A\overline{x^\alpha}(t) + B\overline{u^\alpha}(t) \\ \underline{x^\alpha}(t_0) = \underline{X_0^\alpha} \\ \overline{x^\alpha}(t_0) = \overline{X_0^\alpha}, \end{cases} \tag{9}$$

where $\alpha \in (0, 1]$. Using (8), $\underline{u^\alpha}(t)$ and $\overline{u^\alpha}(t)$ are obtained as below.

$$\underline{u^\alpha}(t) = B^T \Phi^T(t_0, t) W^{-1}(t_0, t_1)(\Phi(t_0, t_1)\underline{\widetilde{X_1}^\alpha} - \overline{X_0^\alpha}),$$

$$\overline{u^\alpha}(t) = B^T \Phi^T(t_0, t) W^{-1}(t_0, t_1)(\Phi(t_0, t_1)\overline{\widetilde{X_1}^\alpha} - \underline{X_0^\alpha}).$$

The solution of system (9) is given by following two equations:

$$\underline{x^\alpha}(t) = \Phi(t, t_0)\underline{X_0^\alpha} + \int_{t_0}^{t} \Phi(t, \tau)B\underline{u^\alpha}(\tau)d(\tau) \tag{10}$$

$$\overline{x^\alpha}(t) = \Phi(t, t_0)\overline{X_0^\alpha} + \int_{t_0}^{t} \Phi(t, \tau)B\overline{u^\alpha}(\tau)d(\tau) \tag{11}$$

From (10) we have,

$$\begin{aligned} \underline{x^\alpha}(t_1) &= \Phi(t_1, t_0)\underline{X_0^\alpha} + \int_{t_0}^{t_1} \Phi(t_1, \tau)B\underline{u^\alpha}(\tau)d(\tau) \\ &= \Phi(t_1, t_0)\underline{X_0^\alpha} + \int_{t_0}^{t_1} \Phi(t_1, \tau)BB^T\Phi^T(t_0, \tau) \\ &\qquad W^{-1}(t_0, t_1)(\Phi(t_0, t_1)\underline{\widetilde{X_1}^\alpha} - \overline{X_0^\alpha})d(\tau) \\ &= \Phi(t_1, t_0)\underline{X_0^\alpha} + \\ &\qquad \Phi(t_1, t_0)WW^{-1}(\Phi(t_0, t_1)\underline{\widetilde{X_1}^\alpha} - \overline{X_0^\alpha}) \\ &= \underline{\widetilde{X_1}^\alpha} - \Phi(t_1, t_0)(\overline{X_0^\alpha} - \underline{X_0^\alpha}) = \underline{X_1^\alpha} \end{aligned} \tag{12}$$

Similarly from (11) we can show that

$$\overline{x^\alpha}(t_1) = \overline{\widetilde{X_1}^\alpha} + \Phi(t_1, t_0)(\overline{X_0^\alpha} - \underline{X_0^\alpha}) = \overline{X_1^\alpha} \tag{13}$$

Equations (12) and (13) together imply that $x(t_1) = X_1$. Hence system (3) with the control $u(\cdot)$ given in (8) steers X_0 to X_1 during time interval $[t_0, t_1]$. ■

Remark IV.7. *If $A, B \geq 0$ then the controllability of pair (A^*, B^*) is equivalent to the controllability of the pair (A, B). In general, checking the controllability conditions for the pair (A^*, B^*) is computationally inefficient due to the fact that the sizes of A^* and B^* are twice that of the original matrices A and B, respectively. However, alternatively, the controllability of the pair (A^*, B^*) can be checked in an efficient way as expressed by the following result.*

Lemma IV.8. *Pair (A^*, B^*) is controllable if and only if the pair (A, B) and the pair $(|A|, |B|)$ are both controllable. (For proof see Appendix-A).*

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V. Numerical Examples

In this section, we provide examples which demonstrate controllability of time-invariant systems with fuzzy initial condition. Example V.1, V.2 apply to Theorem IV.6 and Theorem IV.3, respectively.

Example V.1. *Let*

$$\dot{x}(t) = x(t) + 2u(t)$$

and $x(0) = X_0$ *and* $x(1) = X_1$*, where* X_0 *and* X_1 *are in* $\mathbb{E}^1$ *and are defined as follows:*

$$X_0(s) = \begin{cases} ee^{-\frac{1}{1-4s^2}} & |s| \le \frac{1}{2} \\ 0 & |s| \ge \frac{1}{2} \end{cases},$$

$$X_1(s) = \begin{cases} ee^{-\frac{4}{4-s^2}} & |s| \le 2 \\ 0 & |s| \ge 2. \end{cases}$$

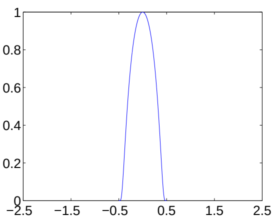
In the setting of above example, we have $\Phi(t,\tau) = e^{t-\tau}$ *and* $W(0,1) = 2(1-e^2)$*. Using equation* (8) *the fuzzy-controller, which steers the initial fuzzy state* X_0 *to target fuzzy state* X_1 *during time-interval* $[0,1]$*, is given by*

$$u(t) = \frac{e^{-t}}{(1-e^2)}[e^{-1}\widetilde{X_1} - X_0],$$

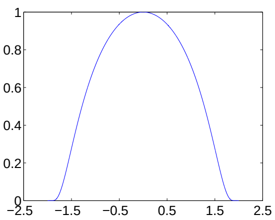
where the fuzzy number $\widetilde{X_1}$ *is defined as follows:*

$$\forall \alpha \in (0,1], [\widetilde{X_1}]_\alpha = [\underline{X_1^\alpha} + e(\overline{X_0^\alpha} - \underline{X_0^\alpha}), \overline{X_1^\alpha} - e(\overline{X_0^\alpha} - \underline{X_0^\alpha}).$$

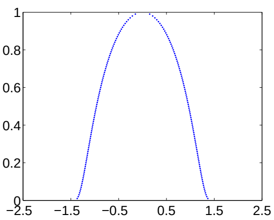
The propagated state at time $t = 1$ *(Fig. 1d) coincides with the desired target state (Fig. 1b). In Fig. 2, lower and upper cuts of the control and system-states are plotted corresponding to* $\alpha = .5$*. It can be seen in the figure (2b) that* $[X_0]_{.5}$ *is steered to* $[X_1]_{.5}$ *during time-interval* $[0,1]$*.*



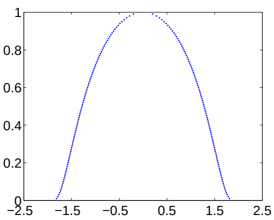
(a) Initial state at t=0



(b) Desired target state at $t = 1$



(c) System-state at $t = 3/4$



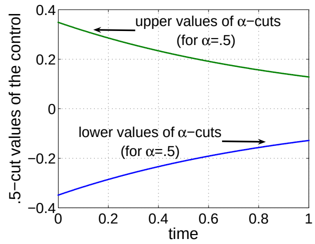
(d) System-state at $t = 1$

Fig. 1: Initial, target and propagated states of the system

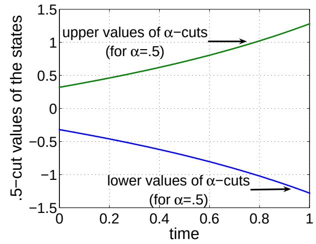
Example V.2. *Let*

$$\dot{x}(t) = -x(t) - 2u(t)$$

and $x(0) = X_0$ *and* $x(1) = X_1$*, where* X_0 *and* X_1 *are in* $\mathbb{E}^1$ *and are defined as follows:*



(a) Control plot for $\alpha = .5$



(b) State plot for $\alpha = .5$

Fig. 2: Control and state plots for $\alpha = .5$ during $[0,1]$

$$X_0(s) = \begin{cases} 2s & 0 \le s \le \frac{1}{2} \\ 2-2s & \frac{1}{2} \le s \le 1 \end{cases},$$

$$X_1(s) = \begin{cases} \frac{s}{4} & 0 \le s \le 4 \\ 2-\frac{s}{4} & 4 \le s \le 8. \end{cases}$$

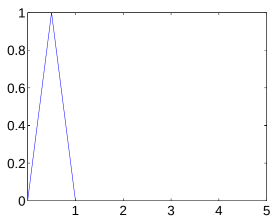
In this case, the evolution of system is given by the following level-wise equations:

$$\begin{pmatrix} \dot{\underline{x^\alpha}}(t) \\ \dot{\overline{x^\alpha}}(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \underline{x^\alpha}(t) \\ \overline{x^\alpha}(t) \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} \underline{u^\alpha}(t) \\ \overline{u^\alpha}(t) \end{pmatrix}.$$

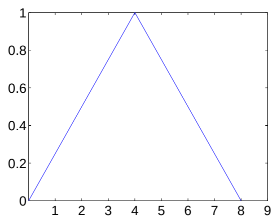
Using Theorem IV.3, the fuzzy-controller $u(\cdot)$ *which steers* X_0 *to* X_1 *during time-interval* $[0,1]$*, is given by the following* α*-cut representation:*

$$\begin{aligned} [u(t)]_\alpha =& [-1.399e^t + e^{-t}(1.1238\alpha - 1.349), \\ & -1.399e^t + e^{-t}(-1.1238\alpha + 1.349)]. \end{aligned} \quad (14)$$

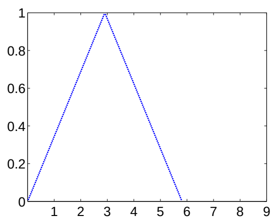
It is clear from Fig. 3 that the initial fuzzy-state X_0 *is steered to the desired target state* X_1 *during time-interval* $[0,1]$*. In Fig. 4, lower and upper* α*-cuts of the control and system-states are plotted corresponding to* $\alpha = .5$*. It can be easily seen in the figure (4b) that* $[X_0]_{.5}$ *is steered to* $[X_1]_{.5}$ *during time-interval* $[0,1]$*.*



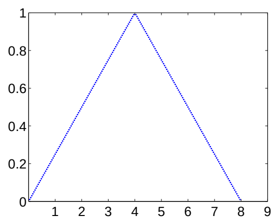
(a) Initial state $t = 0$



(b) Desired target state at $t = 1$



(c) System-state at $t = 3/4$



(d) System-state at $t = 1$

Fig. 3: Initial, target and propagated states of the system

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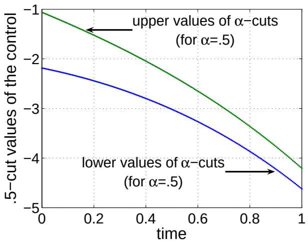
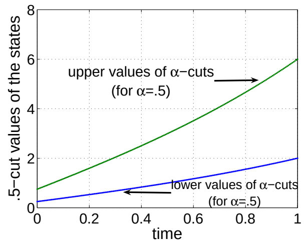


(a) Control plot for $\alpha = .5$



(b) State plot for $\alpha = .5$

Fig. 4: Control and state plots for $\alpha = .5$ during $[0, 1]$

VI. Conclusion

In this article, a new concept of controllability, a concept sharper than the fuzzy-controllability (see [5]), is introduced and sufficient conditions are established for the controllability of linear time-invariant systems with fuzzy initial conditions. The results obtained are seemingly important for the controllability of systems with uncertain parameters like initial condition. Furthermore, we feel that the results can be extended to time-varying systems by using some of the results in [9]. Also, the results can be further generalized to the systems with uncertain plant parameters by considering the matrices A and B to be fuzzy. Obviously, the present investigation enriches our knowledge about controllability of such systems.

Appendix A
Proof of the Lemma IV.8

Proof: Assume that pair (A^*, B^*) is controllable. We want to show that (A, B) and $(|A|, |B|)$ are also controllable. We will prove it by the method of contradiction. Suppose first that the pair (A, B) is not controllable, then by PBH test of controllability, there exists a non-zero vector $v \in \mathbb{R}^n$ such that

$$A^T v = \lambda v \text{ and } B^T v = 0. \tag{15}$$

Define a vector $w = [v, v]^T$, then from (15) we have

$$A^{*T} w = \lambda w \text{ and } B^{*T} w = 0. \tag{16}$$

By PBH test, the last equation implies that the pair (A^*, B^*) is not controllable contrary to the assumption. Similarly, if the pair $(|A|, |B|)$ is not controllable then there exists a non zero vector $v \in \mathbb{R}^n$ such that

$$|A|^T v = \lambda v \text{ and } |B|^T v = 0. \tag{17}$$

Now, by taking $w = [v, -v]^T$, (16) follows from (17), which is again a contradiction.

Conversely, assume that (A, B) and $(|A|, |B|)$ are controllable, we want to show that pair (A^*, B^*) is controllable. Suppose (A^*, B^*) is not controllable, then there exists a non zero vector $x = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{2n}) \in \mathbb{R}^{2n}$ such that

$$A^{*T} x = \lambda x \text{ and } B^{*T} x = 0. \tag{18}$$

Now define a vector $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ such that $v_i = x_i + x_{n+i}$ for each $i = 1, 2, \ldots, n$. Then, from (18) it follows that

$$A^T v = \lambda v \text{ and } B^T v = 0. \tag{19}$$

The last equation is contrary to the fact that pair (A, B) is controllable. Hence the lemma. ■

Remark A.1. *Following closely the proof given above, it can also be shown that pair $(|A^*|, |B^*|)$ is controllable if and only if the pair (A, B) and the pair $(|A|, |B|)$ are both controllable.*

Appendix B
Illustrations of the Flip Operations

Example B.1. *Let*

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}, M = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 2 & -1 \end{bmatrix}$$

Then A^* is given by

$$A^* = \begin{bmatrix} 0 & 2 & -1 & 0 \\ 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 \\ 0 & -1 & 2 & 0 \end{bmatrix}$$

In A^*, the negative entries m_{11}, m_{22}, m_{33}, m_{44} of the matrix M are flipped by m_{13}, m_{24}, m_{31} and m_{42}, respectively.

References

[1] M. Biglarbegian, A. Sadeghian, W. Melek, "On the accessibility/controllability of fuzzy control systems," *Information Science*, Vol. 202, pp. 58-72, 2012.
[2] Z. Cai and S. Tang, "Controllability and robustness of T-fuzzy control systems under directional disturbance," *Fuzzy Sets and Systems*, Vol. 115, pp. 279-285, 2000.
[3] Z. Ding and A. Kandel, "On the controllability of fuzzy dynamical systems (I)," *Journal of Fuzzy Mathematics*, Vol. 8(1), pp. 203-214, 2000.
[4] Z. Ding and A. Kandel, "On the controllability of fuzzy dynamical systems (II)," *Journal of Fuzzy Mathematics*, Vol. 8(2), pp. 295-306, 2000.
[5] Bhaskar Dubey and Raju K. George, "Estimation of controllable initial fuzzy states of linear time-invariant dynamical systems," *Communications in Computer and Information Science*, Springer, Vol. 283, pp. 316-324, 2012.
[6] Bhaskar Dubey and Raju K. George, "A note on the evolution of solutions of a system of ordinary differential equations with fuzzy initial conditions and fuzzy-inputs," *Journal of Uncertain Systems*, Accepted for publication, July-2013.
[7] D. Dubois, H.Prade, "Towards fuzzy differential calculas. Part 3," *Fuzzy Sets and Systems*, Vol. 8, pp. 225-233, 1982.
[8] S. S. Farinwata and G. Vachtsevanos, "Survey on the controllability of fuzzy logic systems," *Proceedings of* 32^{th} *IEEE Conference on Decision and Control*, pp. 1749-1750, 1993.
[9] Y. Feng, L. Hua, "On the quasi-controllability of continuous-time dynamic fuzzy control systems," *Chaos, Solitons & Fractals*, Vol. 30 (1), pp. 177-188, 2006.
[10] R. Goetschel, W. Voxman, "Elementary calculus," *Fuzzy Sets and Systems*, Vol. 18, pp. 31-43, 1986.
[11] M. M. Gupta et al., "Controllability of fuzzy control systems," *IEEE Transactions on Systems, Man, and Cybernetics*, Vol. 16, pp. 576-582, 1985.
[12] O. Kaleva, "Fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 24, pp. 301-317, 1987.
[13] S. Seikkala, "On the fuzzy initial value problem," *Fuzzy Sets and Systems*, Vol. 24, pp. 319-330, 1987.
[14] F. Szidarovszky, A.Terry Bahil, "Linear system theory," CRC Press, 1998.
[15] Jiuping Xu, Zhigao Liao, Zhineng Hu, "A class of linear differential dynamical systems with fuzzy initial condition," *Fuzzy Sets and Systems*, Vol. 158, pp. 2339-2358, 2007.