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Transreal Limits Expose Category Errors in IEEE 754 Floating-Point Arithmetic And in Mathematics

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***Abstract*—The IEEE 754 standard for floating-point arithmetic is widely used in computing. It is based on real arithmetic and is made total by adding both a positive and a negative infinity, a negative zero, and many Not-a-Number (NaN) states. The IEEE infinities are said to have the behaviour of limits. Transreal arithmetic is total. It also has a positive and a negative infinity but no negative zero, and it has a single, unordered number, nullity.**

We elucidate the transreal tangent and extend real limits to transreal limits. Arguing from this firm foundation, we maintain that there are three category errors in the IEEE 754 standard. Firstly the claim that IEEE infinities are limits of real arithmetic confuses limiting processes with arithmetic. Secondly a defence of IEEE negative zero confuses the limit of a function with the value of a function. Thirdly the definition of IEEE NaNs confuses *undefined* with *unordered.* Furthermore we prove that the tangent function, with the infinities given by geometrical construction, has a period of an entire rotation, not half a rotation as is commonly understood. This illustrates a category error, confusing the limit with the value of a function, in an important area of applied mathematics – trigonometry. We briefly consider the wider implications of this category error.

Another paper proposes transreal arithmetic as a basis for floating-point arithmetic; here we take the profound step of proposing transreal arithmetic as a replacement for real arithmetic to remove the possibility of certain category errors in mathematics. Thus we propose both theoretical and practical advantages of transmathematics. In particular we argue that implementing transreal analysis in trans-floating-point arithmetic would extend the coverage, accuracy and reliability of almost all computer programs that exploit real analysis – essentially all programs in science and engineering and many in finance, medicine and other socially beneficial applications.

***Index Terms*—transreal arithmetic, transreal analysis, transreal tangent, negative zero, NaN.**

I. Introduction

GILBERT Ryle introduced the concept of a *category mistake* [6], now more popularly called a *category error*. A category error is the ascription to a category of a property it cannot have. In another paper, in this proceedings, [1], we review the IEEE 754 standard for floating-point arithmetic and propose a superior, floating-point arithmetic, based on transreal arithmetic. That paper cites the relevant literature so we do not rehearse it here. We note only that the set of transreal numbers is $\mathbb{R}^T = \mathbb{R} \cup \{-\infty, \infty, \Phi\}$ where $\infty - \infty = \Phi$. We now move directly to showing that the IEEE 754 standard has three category errors. The first is an erroneous definition which has little consequence – claiming that real arithmetic contains limiting processes. We spend little time on this error. The second is a fundamental mathematical error: mistaking the limit of a function for the value of a function. The third is an error only if the reader has shifted to the transmathematical paradigm, where certain non-finite, mathematical results are *unordered* not *undefined.*

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Kahan [4] defends IEEE 754's negative zero in terms of the limits of functions, making an appeal to the real tangent. This trigonometric function is geometrically defined everywhere but it is arithmetically undefined at infinity. In the next section we describe the transreal tangent, which is defined everywhere. We then develop transreal limits as a generalisation of real limits. The main results are: wherever infinities occur as symbols in extended real limits, they occur identically in transreal limits but as definite numbers; wherever the transreal number nullity occurs in transreal limits, the corresponding real limit is undefined. In a carefully nuanced criticism, we show that the geometrical definition of the tangent, as the value of a ratio, leads to different results from the definition of the tangent as the limit of a power series. This example is central to our criticism of negative zero in the IEEE 754 standard. We conclude with a statement of the main original contributions of the paper.

II. Transreal Tangent

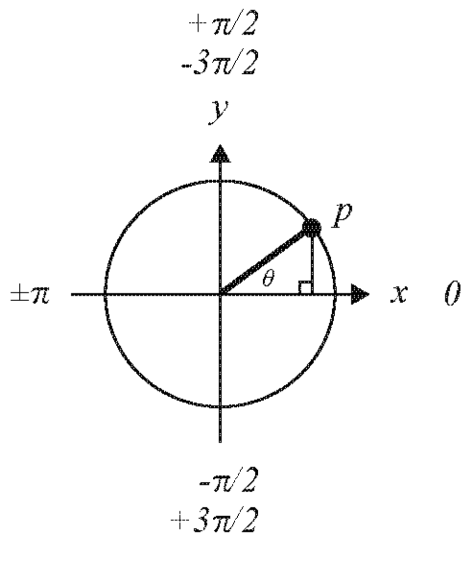

Fig. 1. Geometrical Construction of the Tangent

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Figure 1 shows the well known geometrical construction of the tangent, in which a point, p, lies on a circle, with a unit radius forming the hypotenuse of a right triangle, whose internal angle is θ. When the sides of the triangle are measured in Cartesian co-ordinates, the tangent is defined as $\tan\theta = y/x$. Part of the graph of this function, for real θ, is shown in Figure 2, where the discs, $\bullet$, show where the tangent arrives exactly at a signed infinity and the annuli, $\circ$, show where the tangent asymptotes to a signed infinity, that is where it approaches the infinity but does not arrive at it.

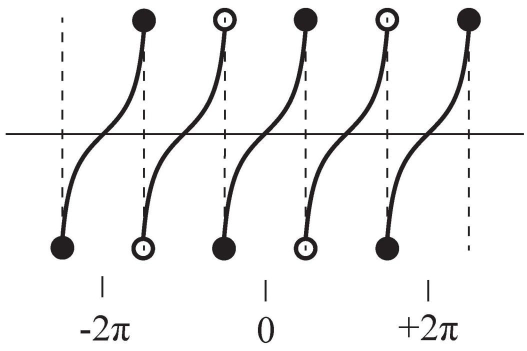

Fig. 2. Graph of the Transreal Tangent

The reader should examine Figure 2. The abscissa shows the angle in radians; the ordinate shows the value of the tangent function. At zero radians the value of the tangent is $\tan = y/x = 0/1 = 0$. As the angle increases: the value increases until it passes exactly through positive infinity at $\tan(\pi/2) = 1/0 = \infty$; the value then jumps discontinuously so that it passes through all negative, real numbers, each of which is finite, until it arrives at $\tan\pi = 0/1 = 0$; the value continues to increase, asymptoting to positive infinity at $3\pi/(2-\epsilon)$ for small, positive ϵ, then jumps discontinuously to negative infinity at $\tan(3\pi/2) = -1/0 = -\infty$; the value of the tangent then increases to zero at $\tan 2\pi = 0/1 = 0$. Notice that the graph has a least, that is principal, period of 2π, not π as is commonly understood. The results for negative angles are similar. For integral k the value of the tangent is positive infinity at $\theta = 2k\pi + \pi/2$ and negative infinity at $\theta = 2k\pi - \pi/2$. The usual graph for the tangent, computed as the limit of a power series, is similar to Figure 2 but with the difference that the tangent is undefined at $2k\pi \pm \pi/2$ for all integral k. Thus the finite values of the geometrical tangent have period π but the extended-real values have period 2π.

As we lack a geometrical construction for the non-finite, transreal angles, we define that the value of the transreal tangent, at non-finite angles, is the limit of the usual power series, evaluated in transreal arithmetic, so that $\tan(-\infty) = \tan\infty = \tan\Phi = \Phi$. This is justified by Observation 15 in Section III-B *Transreal Sequences* below.

We then take the arctangent as usual, for finite values of the tangent, and augment this with $\arctan(-\infty) = -\pi/2$, $\arctan\infty = \pi/2$, $\arctan\Phi = \Phi$.

III. Transreal Analysis

In this section we augment the topology of transreal space, derived transarithmetically from ϵ-neighbourhoods [2], with the usual topology of measure theory and integration theory (anticipating the development of transdifferential and transintegral calculus, which could be presented in a longer paper). Amongst other results we show that transreal space is a compact, separable, Hausdorff space. We then develop transreal sequences and establish the transreal infimum and supremum. Finally we present fundamental results on the limits and continuity of transreal functions. Taken together this implies that transreal analysis contains real analysis.

A. *Transreal Topology*

Transreal arithmetic implies a topology [2], Figure 3, that gives a definite, numerical value to the result of dividing any real number by zero. Infinity, ∞, is the unique number that results when a positive number is divided by zero; negative infinity, $-\infty$, is the unique number that results when a negative number is divided by zero; nullity, Φ, is the unique number that results when zero is divided by zero. Nullity is not ordered, all other transreal numbers are ordered. Infinity is the largest number and negative infinity is the smallest number. Any particular real number is finite; ∞ and $-\infty$ are infinite; Φ is non-finite. The infinite numbers are also non-finite. The real numbers, $\mathbb{R}$, together with the infinite numbers, $-\infty$ and ∞, make up the extended-real numbers, $\mathbb{R}^E$; the real numbers, together with the non-finite numbers, $-\infty, \infty$ and Φ, make up the transreal numbers, $\mathbb{R}^T$.

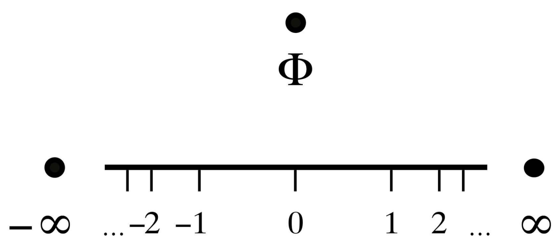

Fig. 3. Transreal Number-Line.

We now define a topology for the whole of $\mathbb{R}^T = \mathbb{R} \cup \{-\infty, \infty, \Phi\}$ which contains the usual topology on $\mathbb{R}^E = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ that is used in measure theory and integration theory. Specifically we note that $\{-\infty\}$, $\{\infty\}$ and $\{\Phi\}$ are singleton sets that are not path connected to any other numbers. This retains compatibility with an older view of the topology of the transreal numbers, based on computing ϵ-neighbourhoods using transreal arithmetic [2]. In our new topology we have that $\{-\infty\}$ and $\{\infty\}$ are closed and not open, while $\{\Phi\}$ is both closed and open. We impose neighbourhoods on $\{-\infty\}$ and $\{\infty\}$ so that the usual topology of measure theory and integration holds, with the possibility that real and extended-real functions have limits of $-\infty$ and ∞. The number Φ is then left as the unique, isolated point, reflecting its status as the unique, unordered number, Φ, in transreal arithmetic. We then rehearse various theorems of sequences, limits and continuity, all of which show that wherever $-\infty$ and ∞ occur as limits in transreal analysis they occur identically in (extended) real-analysis, with the difference that $-\infty$

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and ∞ are abstract symbols in (extended) real analysis and are numbers in transreal arithmetic and transreal analysis. Furthermore $0/0$ is undefined in (extended) real analysis but in transreal arithmetic $\Phi = 0/0$ is a definite number and, in transreal analysis, it is the limit, for example, of constant, transreal functions of the form $f(x) = \Phi$. Thus real analysis is extended to transreal analysis and is extended further in unpublished work that could be presented in a longer version of this paper.

Definition 1: Let $A \subset \mathbb{R}^T$. We say that $x \in \mathbb{R}^T$ is a *transinterior* point related to A if and only if one of the following conditions holds:

1) $x \in \mathbb{R}$ and there is a positive $\varepsilon \in \mathbb{R}$ such that $(x-\varepsilon, x+\varepsilon) \subset A$,
2) $x = -\infty$ and there is $b \in \mathbb{R}$ such that $[-\infty, b) \subset A$,
3) $x = \infty$ and there is $a \in \mathbb{R}$ such that $(a, \infty] \subset A$ or
4) $x = \Phi$ and $\{\Phi\} \subset A$.

We denote the set of all transinterior points related to A as transint$A$. We say that a set $A \subset \mathbb{R}^T$ is *transopen* if and only if $A = \text{transint}A$.

Notice that for every $A \subset \mathbb{R}^T$ it is the case that transint$A \subset A$.

Theorem 2: The class of all transopen sets in $\mathbb{R}^T$ is a topology on $\mathbb{R}^T$. That is to say:

1) $\emptyset, \mathbb{R}^T$ are transopen,
2) Any union of transopen sets is a transopen set and
3) A finite intersection of transopen sets is a transopen set.

Proof:

1) Notice that transint$\emptyset = \emptyset$ and $\mathbb{R}^T \subset \text{transint}\mathbb{R}^T$ follow directly from the definition of a transopen set.
2) Let I be any set and $A = \bigcup_{\alpha\in I} A_\alpha$, where A_α is transopen for all $\alpha \in I$. If $x \in A$ then $x \in A_\alpha$ for some $\alpha \in I$, whence $x \in \text{transint}A_\alpha$. We have several cases: $x \in \mathbb{R}$, whence there is a positive $\varepsilon \in \mathbb{R}$ such that $(x-\varepsilon, x+\varepsilon) \subset A_\alpha \subset A$; or $x = -\infty$, whence there is $b \in \mathbb{R}$ such that $[-\infty, b) \subset A_\alpha \subset A$; or $x = \infty$, whence there is $a \in \mathbb{R}$ such that $(a, \infty] \subset A_\alpha \subset A$; or $x = \Phi$, whence $\{\Phi\} \subset A_\alpha \subset A$. In every case, $x \in \text{transint}A$, whence $A \subset \text{transint}A$.
3) Let $A_1, A_2 \subset \mathbb{R}^T$ be transopen sets. If $x \in A_1 \cap A_2$ then $x \in A_1$ and $x \in A_2$, whence $x \in \text{transint}A_1$ and $x \in \text{transint}A_2$. If $x \in \mathbb{R}$ then there are positive $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ such that $(x-\varepsilon_1, x+\varepsilon_1) \subset A_1$ and $(x-\varepsilon_2, x+\varepsilon_2) \subset A_2$. Taking $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, we have $(x-\varepsilon, x+\varepsilon) \subset A_1 \cap A_2$. If $x = -\infty$ then there are $b_1, b_2 \in \mathbb{R}$ such that $[\infty, b_1) \subset A_1$ and $[-\infty, b_1) \subset A_1$. Taking $b = \min\{b_1, b_2\}$, we have $[-\infty, b) \subset A_1 \cap A_2$. If $x = \infty$ then there are $a_1, a_2 \in \mathbb{R}$ such that $(a_1, \infty] \subset A_1$ and $(a_2, \infty] \subset A_1$. Taking $a = \max\{a_1, a_2\}$, we have $(a\infty] \subset A_1 \cap A_2$. Finally if $x = \Phi$ then $\{\Phi\} \subset A_1$ and $\{\Phi\} \subset A_2$, whence $\{\Phi\} \subset A_1 \cap A_2$. In any case $x \in \text{transint}(A_1 \cap A_2)$, whence $A_1 \cap A_2 \subset \text{transint}(A_1 \cap A_2)$.

■

Reverting now to ordinary terminology, we call a transopen set an open set, we call a transinterior point an interior point, and we denote transint$A$ by int-A.

We recall that a subset of topological space is closed if and only if its complement is open.

Example 3: The sets $\{\Phi\}$, $(-\infty, x)$, (x, ∞), $[-\infty, x)$, $(x, \infty]$, $(-\infty, \infty) = \mathbb{R}$, $[-\infty, \infty]$, $[-\infty, \infty)$, $(-\infty, \infty]$ and (x, y) are open on $\mathbb{R}^T$ where $x, y \in \mathbb{R}$ and $x < y$.

Example 4: The sets $\{-\infty\}$, $\{\infty\}$, $\{x\}$, $[-\infty, x]$, $[x, \infty]$, $(-\infty, x]$, $[x, \infty)$, $(x, y]$, $[x, y)$ and $[x, y]$ are not open on $\mathbb{R}^T$ where $x, y \in \mathbb{R}$ and $x < y$.

Example 5: The sets $\{\Phi\}$, $\{-\infty\}$, $\{\infty\}$, $\{x\}$, $[-\infty, \infty]$, $[-\infty, x]$, $[x, \infty]$ and $[x, y]$ are closed on $\mathbb{R}^T$ where $x, y \in \mathbb{R}$ and $x < y$. In fact, $\mathbb{R}^T \setminus \{\Phi\} = [-\infty, \infty]$, $\mathbb{R}^T \setminus \{-\infty\} = \mathbb{R} \cup (1, \infty] \cup \{\Phi\}$, $\mathbb{R}^T \setminus \{\infty\} = \mathbb{R} \cup [-\infty, 1) \cup \{\Phi\}$, $\mathbb{R}^T \setminus \{x\} = [-\infty, x) \cup (x, \infty] \cup \{\Phi\}$, $\mathbb{R}^T \setminus [-\infty, \infty] = \{\Phi\}$, $\mathbb{R}^T \setminus [-\infty, x] = (x, \infty] \cup \{\Phi\}$, $\mathbb{R}^T \setminus [x, \infty] = [-\infty, x) \cup \{\Phi\}$ and $\mathbb{R}^T \setminus [x, y] = [-\infty, x) \cup (y, \infty] \cup \{\Phi\}$ are open.

Example 6: The sets $(-\infty, x)$, (x, ∞), $[-\infty, x)$, $(x, \infty]$, $(-\infty, \infty) = \mathbb{R}$, $[-\infty, \infty)$, $(-\infty, \infty]$, $(-\infty, x]$, $[x, \infty)$, (x, y), $(x, y]$ and $[x, y)$ are not closed on $\mathbb{R}^T$ where $x, y \in \mathbb{R}$ and $x < y$.

Proposition 7: $\mathbb{R}^T$ is a Hausdorff[1] space.

Proof: Let there be distinct $x, y \in \mathbb{R}^T$. If x or y is Φ, say $x = \Phi$, then it is enough to take $A = \{\Phi\}$, with B a neighbourhood[2] of y, such that $\Phi \notin B$. If one of them is equal to $-\infty$ and the other is equal ∞, say $x = -\infty$ and $y = \infty$, it is enough to take $a, b \in \mathbb{R}$ such that $a < b$, $A = [-\infty, a)$ and $B = (b, \infty]$. If one of them is equal to $-\infty$ and the other is a real number, say $x = -\infty$ and $y \in \mathbb{R}$, it is enough to take a positive $\varepsilon \in \mathbb{R}$, $b \in \mathbb{R}$ such that $b < y - \varepsilon$, $A = [-\infty, b)$ and $B = (y-\varepsilon, y+\varepsilon)$. If one of them is equal to ∞ and the other is a real number, say $x = \infty$ and $y \in \mathbb{R}$, it is enough to take a positive $\varepsilon \in \mathbb{R}$, $a \in \mathbb{R}$ such that $y + \varepsilon < a$, $A = (a, \infty]$ and $B = (y-\varepsilon, y+\varepsilon)$. If $x, y \in \mathbb{R}$, it is enough to take a positive $\varepsilon \in \mathbb{R}$ such that $2\varepsilon < |x-y|$, $A = (x-\varepsilon, x+\varepsilon)$ and $B = (y-\varepsilon, y+\varepsilon)$. In every case, A is a neighbourhood of x, B is a neighbourhood of y and $A \cap B = \emptyset$. ■

Proposition 8: The topology on $\mathbb{R}$, induced by the topology of $\mathbb{R}^T$, is the usual topology of $\mathbb{R}$. That is if $A \subset \mathbb{R}^T$ is open on $\mathbb{R}^T$ then $A \cap \mathbb{R}$ is open (in the usual sense) on $\mathbb{R}$ and if $A \subset \mathbb{R}$ is open (in the usual sense) on $\mathbb{R}$ then A is open on $\mathbb{R}^T$.

Proof: Let $A \subset \mathbb{R}^T$ be an open set on $\mathbb{R}^T$. If $x \in A \cap \mathbb{R}$ then $x \in \text{int}A$ because $x \in A$. This fact, together with $x \in \mathbb{R}$, implies that there is a positive $\varepsilon \in \mathbb{R}$ such that $(x-\varepsilon, x+\varepsilon) \subset A$, whence $(x-\varepsilon, x+\varepsilon) \subset A \cap \mathbb{R}$. Thus $x \in \text{int}(A \cap \mathbb{R})$, where $\text{int}(A \cap \mathbb{R})$ denotes the interior of $A \cap \mathbb{R}$ in the usual topology on $\mathbb{R}$.

Now let $A \subset \mathbb{R}$ be open (in usual sense) on $\mathbb{R}$. If $x \in A$ then there is a positive $\varepsilon \in \mathbb{R}$ such that $(x-\varepsilon, x+\varepsilon) \subset A$. Thus $x \in \text{int}A$. ■

Corollary 9: If $A \subset \mathbb{R}^T$ is closed on $\mathbb{R}^T$ then $A \cap \mathbb{R}$ is closed (in the usual sense) on $\mathbb{R}$.

Proposition 10: $\mathbb{R}^T$ is disconnected[3].

Proof: In fact $\mathbb{R}^T = [-\infty, \infty] \cup \{\Phi\}$ and the sets $[-\infty, \infty]$ and $\{\Phi\}$ are open. ■

[1] A topological space, X, is a Haussdorf space if and only if for any distinct $x, y \in X$, there are open sets $U, V \subset X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. See [5].

[2] A subset U, of a topological space, is a neighbourhood of x if and only if $x \in U$ and U is open.

[3] A topological space, X, is disconnected if and only if there are non-empty, open sets $U, V \subset X$ such that $U \cup V = X$ and $U \cap V = \emptyset$. See [5].

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Notice that Φ is the unique isolated point[4] of $\mathbb{R}^T$.

Proposition 11: $\mathbb{R}^T$ is a separable[5] space.

Proof: $\mathbb{Q} \cup \{\Phi\}$ is dense in $\mathbb{R}^T$. ■

Proposition 12: $\mathbb{R}^T$ is compact[6].

Proof: Let I be any set and $\{A_\alpha;\ \alpha \in I\}$ be an open covering of $\mathbb{R}^T$. We have that $\Phi, -\infty, \infty \in \bigcup_{\alpha\in I} A_\alpha$. Thus there are $\alpha_1, \alpha_2, \alpha_3 \in I$ such that $\Phi \in A_{\alpha_1}$, $-\infty \in A_{\alpha_2}$ and $\infty \in A_{\alpha_3}$. So $\{\Phi\} \subset A_{\alpha_1}$ and there are $a, b \in \mathbb{R}$ with $a < b$ such that $[-\infty, a) \subset A_{\alpha_2}$ and $(b, \infty] \subset A_{\alpha_3}$. Furthermore $[a, b] \subset \bigcup_{\alpha\in I} A_\alpha$, whence $[a, b] \subset \left(\bigcup_{\alpha\in I} A_\alpha\right) \cap \mathbb{R} = \bigcup_{\alpha\in I} (A_\alpha \cap \mathbb{R})$. So $\{A_\alpha \cap \mathbb{R};\ \alpha \in I\}$ is an open covering of $[a, b]$ on $\mathbb{R}$. As $[a, b]$ is compact on $\mathbb{R}$, there are $n \in \mathbb{N}$ and $\alpha_4, \dots, \alpha_n$ such that $[a, b] \subset \bigcup_{i=4}^{n} (A_{\alpha_i} \cap \mathbb{R}) = \left(\bigcup_{i=4}^{n} A_{\alpha_i}\right) \cap \mathbb{R} \subset \bigcup_{i=4}^{n} A_{\alpha_i}$. Thus $\mathbb{R}^T = ([-\infty, a) \cup [a, b] \cup (b, \infty] \cup \{\Phi\}) \subset \bigcup_{i=1}^{n} A_{\alpha_i}$. ■

Corollary 13: Let $A \subset \mathbb{R}^T$. It follows that A is compact if and only if A is closed.

Proof: Let $A \subset \mathbb{R}^T$. If A is compact, since $\mathbb{R}^T$ is Hausdorff space, A is closed. See [5], Theorem 26.3. If A is closed, since $\mathbb{R}^T$ is compact, A is compact. See [5], Theorem 26.2. ■

B. Transreal Sequences

We use the usual definition for the convergence of a sequence in a topological space. That is a sequence, $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^T$, converges to $x \in \mathbb{R}^T$ if and only if for each neighbourhood, $V \subset \mathbb{R}^T$ of x, there is $n_V \in \mathbb{N}$ such that $x_n \in V$ for all $n \geq n_V$.

Notice that since $\mathbb{R}^T$ is a Hausdorff space, the limit of a sequence, when it exists, is unique.

Observation 14: Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ and let $L \in \mathbb{R}$. Notice that $\lim_{n\to\infty} x_n = L$ in $\mathbb{R}^T$ if and only if $\lim_{n\to\infty} x_n = L$ in the usual sense in $\mathbb{R}$. Furthermore, $(x_n)_{n\in\mathbb{N}}$ diverges, in the usual sense, to negative infinity if and only if $\lim_{n\to\infty} x_n = -\infty$ in $\mathbb{R}^T$. Similarly $(x_n)_{n\in\mathbb{N}}$ diverges, in the usual sense, to infinity if and only if $\lim_{n\to\infty} x_n = \infty$ in $\mathbb{R}^T$.

Observation 15: Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^T$. Notice that $\lim_{n\to\infty} x_n = \Phi$ if and only if there is $k \in \mathbb{N}$ such that $x_n = \Phi$ for all $n \geq k$.

Proposition 16: Every monotone sequence of transreal numbers is convergent.

[4]An element, x, of a topological space, X, is said to be an isolated point if and only if there is a neighbourhood $U \subset X$ of x such that $U \cap V = \emptyset$ for all open $V \subset X$ with $V \neq U$.

[5]A topological space, X, is said to be separable if and only if it has a dense, countable subset. A subset D, of a topological space, X, is dense in X if and only if all element of X are elements or limit points of D. See [5].

[6]A topological space, X, is said to be compact if and only if, for all classes of open subsets of X, $\{U_\alpha;\ \alpha \in I\}$ (where I is an arbitrary set) such that $X \subset \bigcup_{\alpha\in I} U_\alpha$, there is a finite subset $\{U_{\alpha_k};\ 1 \leq k \leq n\}$ (for some $n \in \mathbb{N}$) of $\{U_\alpha,\ \alpha \in I\}$ such that $X \subset \bigcup_{k=1}^{n} U_{\alpha_k}$. See [5].

Proof: Suppose $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^T$ is increasing. The case for decreasing, transreal $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^T$ is similar. If $x_k = \Phi$, for some $k \in \mathbb{N}$, then $x_n = \Phi$, for all $n \in \mathbb{N}$, because $x_i \leq \Phi \leq x_j$ for all $i \leq k$ and $j \geq k$, whence $\lim_{n\to\infty} x_n = \Phi$. If $x_n = -\infty$, for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n = -\infty$. If $x_n \neq \Phi$, for all $n \in \mathbb{N}$, and $x_k \neq -\infty$, for some $k \in \mathbb{N}$, then $x_n > -\infty$ for all $n \geq k$, whence there is $s = \sup\{x_n;\ n \in \mathbb{N}\}$ and $s \in \mathbb{R} \cup \{\infty\}$. If $s = \infty$ then, for each $a \in \mathbb{R}$, there is $n_a \in \mathbb{N}$ such that $x_{n_a} > a$. Since $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, $x_n \in (a, \infty]$ for all $n \geq n_a$, whence $\lim_{n\to\infty} x_n = \infty$. If $s \in \mathbb{R}$ then $(x_{k+n})_{n\in\mathbb{N}}$ is a monotone, bounded sequence of real numbers, thus it is convergent. Hence $(x_n)_{n\in\mathbb{N}}$ is convergent. ■

Theorem 17: Every sequence of transreal numbers has a convergent subsequence.

Proof: Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^T$. If $\{n;\ x_n \neq \Phi\}$ is a finite set then clearly $\lim_{n\to\infty} x_n = \Phi$. If $\{n;\ x_n \neq \Phi\}$ is an infinite set then denote, by $(y_k)_{k\in\mathbb{N}}$, the subsequence of $(x_n)_{n\in\mathbb{N}}$ of all elements of $(x_n)_{n\in\mathbb{N}}$ that are distinct from Φ. Let $J = \{k;\ y_k > y_m \text{ for all } m > k\}$. If J is a infinite set, we write $J = \{k_1, k_2, \dots\}$, with $k_1 < k_2 < \cdots$. Since for each $i \in \mathbb{N}$, $k_i \in J$, we have that $y_{k_i} > y_{k_j}$ for all $i < j$. Thus $(y_{k_i})_{i\in\mathbb{N}}$ is a decreasing subsequence of $(x_n)_{n\in\mathbb{N}}$. If J is finite, let k_1 be greater than all of the elements of J. Since $k_1 \notin J$, there is $k_2 > k_1$ such that $y_{k_2} \geq y_{k_1}$. Since $k_2 > k_1$, it follows that $k_2 \notin J$. So there is $k_3 > k_2$ such that $y_{k_3} \geq y_{k_2}$. By induction we build an increasing subsequence $(y_{k_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$. In both cases, in agreement with Proposition 16, $(y_{k_i})_{i\in\mathbb{N}}$ is convergent. ■

Proposition 18: Let $x, y \in \mathbb{R}^T$ and let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subset \mathbb{R}^T$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. It follows that:

1) If $x, y \in \{-\infty, \infty\}$ and $x + y = \Phi$ do not occur simultaneously then $\lim_{n\to\infty} (x_n + y_n) = x + y$;
2) If $x, y \in \{0, \infty, -\infty\}$ and $xy = \Phi$ do not occur simultaneously then $\lim_{n\to\infty} (x_n y_n) = xy$;
3) If $y \neq 0$ then $\lim_{n\to\infty} (y_n^{-1}) = y^{-1}$ and
4) If $y = 0$ and there is $k \in \mathbb{N}$ such that $y_n < 0$ for all $n \geq k$ then $\lim_{n\to\infty} (y_n^{-1}) = -(y^{-1})$. If $y = 0$ and there is $k \in \mathbb{N}$, such that $y_n > 0$ for all $n \geq k$, then $\lim_{n\to\infty} (y_n^{-1}) = y^{-1}$.

Theorem 19 (Sandwiches): Let $L \in \mathbb{R}^T$ and let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \subset \mathbb{R}^T$ such that $\lim_{n\to\infty} x_n = L$ and $\lim_{n\to\infty} z_n = L$. If there is $N \in \mathbb{N}$, such that $x_n \leq y_n \leq z_n$ for all $n \geq N$, then $\lim_{n\to\infty} y_n = L$.

Proof: Let $L \in \mathbb{R}^T$, let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \subset \mathbb{R}^T$ and let $N \in \mathbb{N}$ such that $\lim_{n\to\infty} x_n = L$, $\lim_{n\to\infty} z_n = L$ and $x_n \leq y_n \leq z_n$ for all $n \geq N$.

If $L = \Phi$, the result follows immediately from Observation 15.

If $L \in \mathbb{R}$, let there be an arbitrary, positive $\varepsilon \in \mathbb{R}$. Since $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L$, there are $N_1, N_2 \in \mathbb{N}$ such that $L - \varepsilon < x_n$ for all $n \geq N_1$ and $z_n < L + \varepsilon$ for all $n \geq N_2$. Taking $N_3 = \max\{N, N_1, N_2\}$, we have that $L - \varepsilon < x_n \leq y_n \leq z_n < L + \varepsilon$ for all $n \geq N_3$.

If $L = -\infty$, let there be an arbitrary $b \in \mathbb{R}$. Since $\lim_{n\to\infty} z_n = L$, there is $N_1 \in \mathbb{N}$ such that $z_n \in [-\infty, b)$

for all $n \geq N_1$. Taking $N_2 = \max\{N, N_1\}$, we have that $y_n \leq z_n < b$ for all $n \geq N_2$, whence $y_n \in [-\infty, b)$ for all $n \geq N_2$.

If $L = \infty$, the result follows similarly to the previous case. ■

Definition 20: Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^T \setminus \{\Phi\} = \mathbb{R}^E$. Let $v_n = \inf\{x_k,\ k \geq n\}$ and let $u_n = \sup\{x_k,\ k \geq n\}$. We define and denote the *lower limit* and the *upper limit* of $(x_n)_{n\in\mathbb{N}}$, respectively, by

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} v_n \text{ and } \limsup_{n\to\infty} x_n := \lim_{n\to\infty} u_n.$$

Notice that $(v_n)_{n\in\mathbb{N}}$ is increasing and $(u_n)_{n\in\mathbb{N}}$ is decreasing, whence $\lim_{n\to\infty} v_n = \sup_{n\in\mathbb{N}}(v_n)$ and $\lim_{n\to\infty} u_n = \inf_{n\in\mathbb{N}}(u_n)$. Therefore the notations $\sup_{n\in\mathbb{N}} \inf_{k\geq n}(x_k)$ and $\inf_{n\in\mathbb{N}} \sup_{k\geq n}(x_k)$ denote, respectively, the lower limit and the upper limit.

Proposition 21: Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^E$. It follows that there is a limit $\lim_{n\to\infty} x_n$ if and only if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$. In this case, $\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$.

C. Limit and Continuity of Transreal Functions

We remember that if X is a topological space then $x_0 \in A \subset X$ is a limit point of A if and only if for every neighbourhood V of x_0 it follows that $V \cap (A \setminus \{x_0\}) = \emptyset$. The set of all limit points of A is denoted as A'.

We use the usual definition of the limit of functions in a topological space. That is, if A is a subset of $\mathbb{R}^T$, $f : A \to \mathbb{R}^T$ is a function, x_0 is a limit point of A and L is a transreal number, we say that $\lim_{x\to x_0} f(x) = L$ if and only if, for each neighbourhood V of L, there is a neighbourhood U of x_0 such that $f(A \cap U \setminus \{x_0\}) \subset V$.

Observation 22: Notice that given $x_0, L \in \mathbb{R}$, the transreal limit $\lim_{x\to x_0} f(x) = L$ in $\mathbb{R}^T$ exists if and only if the real limit $\lim_{x\to x_0} f(x) = L$ exists in the usual sense in $\mathbb{R}$. The same can be said about $\lim_{x\to x_0} f(x) = -\infty$, $\lim_{x\to x_0} f(x) = \infty$, $\lim_{x\to -\infty} f(x) = L$, $\lim_{x\to -\infty} f(x) = -\infty$, $\lim_{x\to -\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = L$, $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$.

Observation 23: Let $x_0 \in \mathbb{R}^T$, notice that $\lim_{x\to x_0} f(x) = \Phi$ if and only if there is a neighbourhood U of x_0 such that $f(x) = \Phi$ for all $x \in U \setminus \{x_0\}$.

Proposition 24: Let $A \subset \mathbb{R}^T$, $f : A \to \mathbb{R}^T$, $x_0 \in A'$ and $L \in \mathbb{R}^T$. The following two statements are equivalent:

1) $\lim_{x\to x_0} f(x) = L$,
2) $\lim_{n\to\infty} f(x_n) = L$ for all $(x_n)_{n\in\mathbb{N}} \subset A \setminus \{x_0\}$ such that $\lim_{n\to\infty} x_n = x_0$.

Proof: Let $A \subset \mathbb{R}^T$, $f : A \to \mathbb{R}^T$, $x_0 \in A'$ and $L \in \mathbb{R}^T$. Suppose that $\lim_{x\to x_0} f(x) = L$. Let $(x_n)_{n\in\mathbb{N}} \subset A \setminus \{x_0\}$ such that $\lim_{n\to\infty} x_n = x_0$. Let V be an arbitrary neighbourhood of L. Then there is a neighbourhood, U, of x_0 such that $f(A \cap U \setminus \{x_0\}) \subset V$. Since $\lim_{n\to\infty} x_n = x_0$ there is an n_U such that $x_n \in U$ for all $n \geq n_U$. Thus $f(x_n) \in f(A \cap U \setminus \{x_0\}) \subset V$ for all $n \geq n_U$.

Now suppose $\lim_{x\to x_0} f(x) \neq L$. Then there is a neighbourhood, V, of L such that, for each $n \in \mathbb{N}$, there is $x_n \in A$ such that $0 < |x_n - x_0| < \frac{1}{n}$ (if $x_0 \in \mathbb{R}$) or $x_n \in (-\infty, -n)$ (if $x_0 = -\infty$) or $x_n \in (n, \infty)$ (if $x_0 = \infty$), and $f(x_n) \notin V$. Hence $(x_n)_{n\in\mathbb{N}} \subset A \setminus \{x_0\}$, $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} f(x_n) \neq L$. ■

Proposition 25: Let $L, M \in \mathbb{R}^T$, $A \subset \mathbb{R}^T$, with functions $f, g : A \to \mathbb{R}^T$, and $x_0 \in A'$ such that $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$. It follows that:

1) If $L, M \in \{-\infty, \infty\}$ and $L + M = \Phi$ do not occur simultaneously then $\lim_{x\to x_0} (f+g)(x) = L + M$;
2) If $L, M \in \{0, \infty, -\infty\}$ and $LM = \Phi$ do not occur simultaneously then $\lim_{x\to x_0} (fg)(x) = LM$;
3) If $M \neq 0$ then $\lim_{x\to x_0} \left(\frac{1}{f}\right)(x) = \frac{1}{M}$ and
4) If $M = 0$ and there is a neighbourhood, U, of x_0, such that $g(x) < 0$ for all $x \in U \setminus \{x_0\}$, then $\lim_{x\to x_0} \left(\frac{1}{g}\right)(x) = -(M^{-1})$. If $M = 0$ and there is a neighbourhood, U, of x_0, such that $g(x) > 0$ for all $x \in U \setminus \{x_0\}$, then $\lim_{x\to x_0} \left(\frac{1}{g}\right)(x) = M^{-1}$.

We use the usual definition of continuity in a topological space. That is if $A \subset \mathbb{R}^T$, $f : A \to \mathbb{R}^T$ is a function and $x_0 \in A$, we say that f is continuous in x_0 if and only if, for each neighbourhood V of $f(x_0)$, there is a neighbourhood U of x_0 such that $f(A \cap U) \subset V$.

Observation 26: Notice that given $x_0 \in \mathbb{R}$, f is continuous in x_0 in $\mathbb{R}^T$ if and only if f is continuous in x_0 in the usual sense in $\mathbb{R}$.

Observation 27: Notice that if $\Phi \in \mathrm{Dm}(f)$ ($\mathrm{Dm}(f)$ denote the domain of f) then f is continuous in Φ.

Proposition 28: Let $A \subset \mathbb{R}^T$, $f : A \to \mathbb{R}^T$ and $x_0 \in A$. The following two statements are equivalent:

1) f is continuous in x_0,
2) $\lim_{n\to\infty} f(x_n) = f(x_0)$ for all $(x_n)_{n\in\mathbb{N}} \subset A$ and $\lim_{n\to\infty} x_n = x_0$.

Proposition 29: Let $A \subset \mathbb{R}^T$; $f, g : A \to \mathbb{R}^T$ and $x_0 \in A$ such that f and g are continuous in x_0. It follows:

1) If $f(x_0), g(x_0) \in \{-\infty, \infty\}$ and $(f+g)(x_0) = \Phi$ do not occur simultaneously then $f + g$ is continuous in x_0;
2) If $f(x_0), g(x_0) \in \{0, \infty, -\infty\}$ and $(fg)(x_0) = \Phi$ do not occur simultaneously then fg is continuous in x_0;
3) If $g(x_0) \neq 0$ then $\frac{1}{g}$ is continuous in x_0 and
4) If $g(x_0) = 0$ and there is a neighbourhood, U, of x_0, such that $g(x) \geq 0$ for all $x \in U$, then $\frac{1}{g}$ is continuous in x_0.

Notice that if $g(x_0) = 0$ and there is no neighbourhood, U, of x_0, such that $g(x) \geq 0$ for all $x \in U$, then $\frac{1}{g}$ is not continuous in x_0.

Proposition 30: Let $A, B \subset \mathbb{R}^T$, $f : A \to \mathbb{R}^T$ and $g : B \to \mathbb{R}^T$ such that $f(A) \subset B$. If f is continuous in x_0 and g is continuous in $f(x_0)$ then $g \circ f$ is continuous in x_0.

Proposition 31: Let A be an open set such that $A \subset \mathbb{R}^T$ and let $f : A \to \mathbb{R}^T$. It follows that f is continuous in A if and only if $f^{-1}(B)$ is open, for all open $B \subset \mathbb{R}^T$.

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IV. Category Errors

The IEEE 754 standard [3] (page 34, section 6.1) says, "The behaviour of infinity in floating-point arithmetic is derived from the limiting case of real arithmetic with operands of arbitrarily large magnitude, when such a limit exists. Infinities shall be interpreted in the affine sense, that is: $-\infty < \{\text{everyfinitenumber}\} < \infty$." This is erroneous. There are no limiting cases in real arithmetic; to think otherwise is a category error which confuses real arithmetic with real analysis. It is also back to front. Transreal arithmetic is total and can be used to derive limits, as above. Crucially these limits are a *superset* of real limits so transreal arithmetic cannot be derived from real limits. Also the property $-\infty < r < \infty$, for all real r, is a theorem of transreal arithmetic not an axiom (See citations in [1]) so it need not be stated. Attempting to derive a total arithmetic from real analysis is unlikely to succeed and in the case of IEEE 754 floating point arithmetic it fails, as shown in [1] and next.

Kahan defends IEEE 754's negative zero in a paper which deals with the solution of complex functions defined as the limits of power series. Within that setting, Kahan's treatment is valid but problems arise when he considers the calculation of (real) functions that have an alternative, geometrical definition as the value, not the limit, of some expression. Commenting on the APL language standard that specifies an unsigned zero, he says [4] (page 186), "... like zero, $1/0$ has no sign and therefore $\arctan(1/0)$ has to be either undefined or else chosen arbitrarily from $\{\pm\pi/2\}$." But, as shown in section II, above, transreal arithmetic has an unsigned zero but $1/0$ is positive (since 0 is not negative it cannot toggle the sign of 1) and $\arctan = \pi/2$ is uniquely determined in the principal range, all other positive infinities of $\tan\theta$ occurring at $\theta = 2k\pi + \pi/2$ for all integers k. Kahan maintains the thesis that negative zero, produced by underflow from a negative number, preserves information about the limit of a function and that this leads to the correct calculation of the function's value. But this is contradicted by the geometrical construction of the tangent. Suppose we calculate $\tan\theta = 1/(-\epsilon)$ for some small, positive ϵ that underflows to IEEE 754's negative zero then $\tan\theta = -1/0 = -\infty$. This is correct at $\theta = 2k\pi - \pi/2$ and is wrong at $\theta = 2k\pi + \pi/2$ for all integers k. The limit at $\theta = 2k\pi \pm \pi/2$ indicates only the infinite magnitude of the tangent at the given angle and gives no information about its sign, despite the fact that both the magnitude and sign of the value of the function are completely determined for all integral k. Believing that the value of a function can always be computed from its limit is a category error. Conversely believing that where the limit is not uniquely defined the value of the function must be undefined or arbitrary is also a category error. The limit and value of a function are in fundamentally different categories of mathematical object – they exist, or else do not exist, entirely independently of each other.

Kahan defines many complex functions in terms of principal functions involving the complex logarithm. We conjecture that real and complex trigonometric identities allow the geometrical specification of functions, such as the tangent, to spread to all trigonometrical functions so that their behaviour is completely determined. This is a matter for future research.

We have already criticised the IEEE 754 NaNs [1] but we take up this issue again. The tangent is defined, geometrically, for all right triangles with a unit hypotenuse. It is mapped to other strictly positive hypotenuses by dilatation but what of zero, infinite and nullity hypotenuses? A triangle with hypotenuse zero or nullity has all sides, respectively, zero or nullity, whence all geometrically defined trigonometric ratios are nullity. When the hypotenuse is infinity, at least one other side is infinity and the remaining side may be finite or infinity. The trigonometric ratios are then one of zero, nullity, positive or negative infinty as the case may be. In every case the geometrically defined trigonometric ratios have a completely determined transreal value. In particular $0/0 = \Phi$ is an *unordered* number not an *undefined* NaN. This matter is taken further in the development of transcomplex numbers in polar form where angle and radius are first-class citizens from which non-bijective Cartesian tangents and other non-bijective functions may arise [1].

We have shown in [1] that trans-floating-point arithmetic is more accurate than IEEE 754 arithmetic. Here we have introduced transreal analysis with a wider coverage than real analysis and have shown how to perform calculations more reliably than with IEEE 754's negative zero. We expect many computer applications to benefit from transmathematics.

V. Conclusion

We add the usual topology of measure theory and integration theory to the space of transreal numbers and prove that this space is a compact, separable, Hausdorff space. Using these results we extend the limit and continuity of real functions to transreal functions. Separately we show that the usual geometrical construction of the tangent is defined for infinite values of the function when it is calculated using transreal arithmetic. We introduce the transreal tangent and transreal arctangent as total functions of transreal numbers. We show that while the finite values of the transreal tangent have a primitive period of π, the function has a primitive period of 2π when the infinite values are considered because the transreal tangent alternately asymptotes to and arrives at an infinity on alternate periods of π. We use this result to show that IEEE 754's negative zero computes the wrong result for alternate periods of the transreal tangent and diagnose this fault to a category error where the limit of a function is confused with the value of a function. We propose that all trigonometric functions, including complex trigonometric functions, could be totalised by forcing their power series definitions to be faithful to the boundary conditions demanded by geometrical constructions, such as the geometrical construction of the tangent. We believe that adopting transreal arithmetic, in place of real arithmetic, would increase the coherence of mathematics and would bring both theoretical and practical advantages to computing.

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