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Construction of the Transcomplex Numbers From the Complex Numbers

Tiago S. dos Reis and James A.D.W. Anderson

***Abstract*—A geometrical construction of the transcomplex numbers was given elsewhere. Here we simplify the transcomplex plane and construct the set of transcomplex numbers from the set of complex numbers. Thus transcomplex numbers and their arithmetic arise as consequences of their construction, not by an axiomatic development. This simplifies transcomplex arithmetic, compared to the previous treatment, but retains totality so that every arithmetical operation can be applied to any transcomplex number(s) such that the result is a transcomplex number. Our proof establishes the consistency of transcomplex and transreal arithmetic and establishes the expected containment relationships amongst transcomplex, complex, transreal and real numbers. We discuss some of the advantages the transarithmetics have over their partial counterparts.**

***Index Terms*—transcomplex numbers, transcomplex arithmetic, transreal numbers, transreal arithmetic**

I. Introduction

TRANSARITHMETICS are total over the basic operations of arithmetic: all of addition, subtraction, multiplication and division can be applied to any numbers with the result being a number. Consequently any syntactically correct sentence is semantically correct in the sense that its execution does not cause an exception in an appropriate computer architecture. Transreal arithmetic and transreal numbers are explained in other papers in this proceedings: the removal of exceptions from floating-point arithmetic is discussed in [6], transreal limits are discussed in [7], transreal arithmetic as a basis for paraconsistent logics and their computer implementations is discussed in [9]. We refer the reader to those papers to obtain an insight into transreal arithmetic and to obtain references to tutorial material. However, we do draw the reader's attention to the fact that transreal arithmetic obeys quadrachotomy, not the weaker trichotomy of real arithmetic, and gives a unique and non-trivial meaning to every combination of the relational operators: less-than ($<$), equal-to ($=$), greater-than($>$). See [6] for details.

The reader might hope that the transcomplex numbers can be obtained from the transreal numbers by a simple application of the Cayley-Dickson construction. See [1] for a discussion of the construction. That is a forlorn hope given the current state of knowledge. Firstly non-finite angles cannot be expressed uniquely by Cartesian components so the Cayley-Dickson construction falls at the first hurdle. And, we should add, the complex argument function, $\text{Arg}(z)$, must be generalised for the same reason. Secondly the transreal numbers are necessarily non-distributive at infinity so, again, the Cayley-Dickson construction fails. Thirdly the transcomplex numbers are necessarily non-associative at infinity so, for a third time, the Cayley-Dickson construction fails. Non-associativity follows from the property of transreal arithmetic that $\infty + \infty = \infty$. To see this consider sums of transcomplex numbers of the form (r, θ) where r is the radius and θ is the angle. These terms and arithmetic are defined in the present paper, whence: $[(\infty, 0)+(\infty, 0)]+(\infty, 1) = (\infty, 0)+(\infty, 1) = (\infty, 1/2)$ but $(\infty, 0)+[(\infty, 0)+(\infty, 1)] = (\infty, 0)+(\infty, 1/2) = (\infty, 1/4)$ which is non-associative. We return to the issue of non-associativity in Section V, Discussion.

Given this presently complicated state of affairs, no simple, algebraic method seems powerful enough to generalise the transreal numbers to the transcomplex numbers *ab initio*. Instead we follow the original, geometrical construction of the transcomplexes, simplify the construction and apply algebraic methods to the simplification. Our objective is expressly not to present transcomplex arithmetic as a finished system but to show that the transcomplex numbers, as currently conceived, are consistent with the complex, transreal, and real numbers, and that these sets of numbers have the expected superset/subset relationships. These issues are taken up again in the Discussion.

Transreal arithmetic was generalised to transcomplex arithmetic by giving a geometrical construction for the basic arithmetical operations in a space containing an extended cylinder (or cone) and its axis as a real line, augmented with two non-finite points [5]. This transcomplex space describes both an infinite set of oriented infinities, which may occur at any real angle, and an unoriented infinity. Both kinds of infinity are used in various areas of mathematics [18][14][13][15]. The transcomplex space also contains a single, isolated, non-finite point, nullity, $\Phi = 0/0$, which is essential to totalising both real and complex arithmetic. Despite the intricate structure of this space, complex arithmetical operations are carried out in three simple steps. Firstly a dilatation and translation prepares the data. Secondly complex arithmetic is carried out in the usual geometrical way. Finally the result is made available, following a dilatation and translation. This is satisfactory from a computational point of view but the different treatment of the unoriented transcomplex and the unsigned transreal infinities is inelegant.

Transreal arithmetic represents the signed infinities, $+\infty = 1/0$ and $-\infty = -1/0$, but does not have an explicit representation of an unsigned infinity that has infinite magnitude and no sign. This unique, unsigned, infinity is obtained, in those mathematical applications that need it, by operating on the modulus or absolute value of the signed infinities. Here we give a new derivation of the transcomplex numbers which has no unoriented infinity so that, as in

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transreal arithmetic, the unoriented infinity may be computed from the modulus of the oriented infinities. This is an elegant solution which deals, in the same way, with the unoriented infinity of transcomplex arithmetic and the unsigned infinity of transreal arithmetic, making them functionally identical. As a side effect, dispensing with the explicitly unoriented, transcomplex infinity allows us to dispense with the part of the whip that is the non-negative part of the real axis [5] so that all that remains is an extended complex plane and a single point at nullity. This makes the treatment of transreal and transcomplex infinities equivalent and enables a natural computation in the new transcomplex plane without the need for pre and post transformations. Indeed the new operations of transcomplex arithmetic are extremely closely related to the usual complex form.

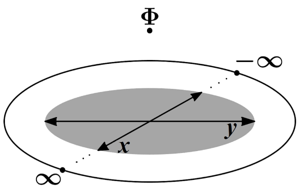

Fig. 1. The transreal numbers, being the extended x-axis and the point at nullity, Φ, as a subset of the transcomplex numbers.

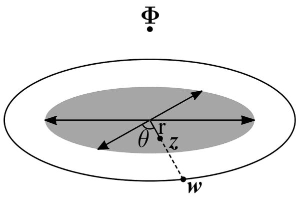

Fig. 2. Entire transcomplex plane described by polar co-ordinates: $z = (r, \theta)$ and $w = (\infty, \theta)$ and $\Phi = (\Phi, \Phi)$ with r, θ finite.

The new transcomplex plane is shown in Figure 1. The usual complex plane is shown as a grey disk. It has no real bound but, after a gap, it is surrounded by a circle at infinity. The point at nullity, Φ, lies off the plane containing the complex plane and the circle at infinity. The transreal number line is shown as the x-axis, together with the point at nullity, Φ. Figure 2 shows that any point in the complex plane and the circle at infinity can be described in polar co-ordinates. The system of polar co-ordinates also describes the point at nullity which lies at nullity distance and nullity angle. Thus every point in the transcomplex plane, including the point at nullity, is described by polar co-ordinates.

The new derivation of transcomplex arithmetic is given in terms of equivalence classes of the form $[c, d]$ where, initially, c is an ordinary complex number and d is unity, $d = 1$, for finite transcomplex numbers and zero, $d = 0$, for non-finite transcomplex numbers. However, the transreal numbers provide explicitly non-finite numbers $-\infty$, ∞, Φ so we may, ultimately, dispense with equivalence classes of the form $[c, d]$ so that the arithmetic is carried out on equivalence class representatives of the form (r, θ) where r and θ are transreal numbers.

The main mathematical work stops with the set of transcomplex numbers, $\mathbb{C}^T$, being represented by polar co-ordinates of a non-negative, transreal radius and a real angle in the principal range $(-\pi, \pi]$. The point at nullity lies at an arbitrary angle which may be taken as zero so that its polar co-ordinates are $(\Phi, 0)$. As usual the point at zero also lies at an arbitrary angle, which is conventionally taken to be zero, so its polar co-ordinates are $(0, 0)$. This set of transcomplex numbers is mathematically elegant and supports a total arithmetic but it is not sufficient for computation where it is required that parameters, here r and θ, are taken from the entire input class of machine, usually binary, representations of, in this case, transreal numbers. A little more work is done to extend the angles from the principal range to the whole set of real numbers so that they wrap to describe a Riemann surface [13]. The real angles are then augmented with the non-finite angles so that, as in previous work [5], all non-finite angles describe the angle nullity. In the new transcomplex plane only the point at nullity lies at angle nullity. This totalises angle over the entire set of transreal numbers. However, we prefer (Φ, Φ) as the representative of the equivalence class of polar co-ordinates that describe the point at nullity because angle Φ makes it immediately obvious that nullity lies outside the extended-complex plane, comprising the complex plane and the circle at infinity: Figures 1 and 2. We then take $(-r, \theta) = (r, \theta + \pi)$ which totalises radius over the transreal numbers. In this form there are no exceptions to the polar co-ordinates.

In future all of these number systems may be referred to as transcomplex numbers, $\mathbb{C}^T$, but it would be helpful for authors to explicitly state the transreal values over which the radius and angle range. Such variety in notation is entirely normal, especially in the early stages of the development of a new mathematical system. In time some one form may be taken as the canonical form of transcomplex numbers.

The absence of all exceptions is extremely powerful in computation. It means that it is possible to construct computational systems where all syntactically correct expressions are semantically correct. For example it is possible to guarantee that any program which compiles has no logical run time errors (though it may have physical run time errors due to electrical faults). This is valuable in safety critical systems and in data-flow machines where the absence of exceptions means it can be guaranteed that the flow of data will not be interrupted. The existence of even one total system of Turing complete computation is enough to obtain these advantages so all Turing computations could be described in transreal arithmetic, say via a Gödelisation, but where engineering and scientific computations are wanted in complex arithmetic, it is more efficient to provide a direct totalisation of the complex numbers, as we do here.

In the next section we construct the new transcomplex numbers from the complex numbers and derive the new transcomplex arithmetic. This is followed by a tutorial on transcomplex arithmetic. We then discuss the role of the new transcomplex arithmetic in mathematics, physics and computation.

II. Novelty

The problem of defining division by zero has remained open for a long time. Martinez discusses various approaches that have been taken over the last, approximately, one thousand years [17] ch. 6. In the last one hundred years, the consensus view, among mathematicians, is that the result of dividing by zero is undefined. Some areas of mathematics allude to division by zero as an asymptotic process but do not define exact division by zero. For example the theory of limits allows the calculation of $\lim_{x \to 0^-} \frac{k}{x}$ and $\lim_{x \to 0^+} \frac{k}{x}$, where $k, x \in \mathbb{R}$; similarly the theory of hyperreal numbers allows division by infinitesimal numbers that are infinitesimally close to, but not exactly, zero [19]. Elsewhere, in this proceedings, we take the novel approach of extending real limits and both the real differential and real integral calculus to operate exactly on division by zero [7][11].

The IEEE floating-point standard uses symbols to express a fraction with zero denominator, such as ∞ and NaNs. However the NaNs are not arithmetically well defined numbers, as shown in novel results presented in this proceedings [6].

Exact division by zero is allowed in the theory of Wheels [10]. This theory, like the theory of transreal arithmetic [2][3], is motivated by the syntactic application of the rules for operations on ordinary fractions to fractions with a denominator of zero. There are, however, significant differences between the two theories. The elements of a Wheel are unordered so they do not immediately generalise real numbers, whereas the transreals explicitly do generalise the real numbers. The two theories have different numbers that arise on division by zero. Briefly, a Wheel is a ring adjoined with two new elements, $\infty = 1/0$ and $\perp = 0/0$, whereas the set of transreals is the set of reals adjoined with three new elements: $-\infty = -1/0$, $\infty = 1/0$ and $\Phi = 0/0$. Moreover a Wheel is restricted to an algebraic structure but the transreals occur in a transmetric space [4] with topological properties that, as we have said, extend the concepts of limit, continuity, differentiation and integration [7][11].

Transreal numbers have been enunciated via axioms and have received a machine proof of consistency [8]. The transcomplexes have been enunciated geometrically [5]. A construction of transreal numbers has been proposed in a paper in review for publication. Now we present the first construction of the set of transcomplex numbers from the set of complex numbers. We simplify the set of transcomplexes proposed in [5] and give the first human proof of the consistency of the transcomplexes. A further novelty of the present paper, compared to Wheels, is that we adjoin oriented infinities to the complex numbers.

III. The Set of Transcomplex Numbers

Our aim is to extend the set of complex numbers and their arithmetic to a set where the arithmetic is total. That is, where all results of any arithmetical operation, applied to any elements of the set, belongs to the set. We know that division by zero is not allowed in ordinary complex numbers, $\mathbb{C}$, so we extend the concept of division and, for that, we also need to extend the concept of number.

Any complex number, $z \in \mathbb{C}$, can be written in the form $z = a + bi$, where $a, b \in \mathbb{R}$ and i is the imaginary unit, that is, $i = \sqrt{-1}$. As usual we write the modulus of z as $|z|$, that is, $|z| = \sqrt{a^2 + b^2}$, and we write the principal argument of z, when $z \neq 0$, as $\text{Arg}(z)$, that is, $\theta = \text{Arg}(z)$ if and only if $\cos(\theta) = \dfrac{a}{|z|}$, $\sin(\theta) = \dfrac{b}{|z|}$ and $\theta \in (-\pi, \pi]$.

Definition 1: Let $T := \big\{(x, y);\ x \in \mathbb{C},\ y \in \{0, 1\}\big\}$. Given $(x, y), (w, z) \in T$, we say that $(x, y) \sim (w, z)$, that is, (x, y) is equivalent (w, z) with respect to $\sim$, if and only if there is a positive $\alpha \in \mathbb{R}$ such that $x = \alpha w$ and $y = \alpha z$.

Notice that the relation, $\sim$, is an equivalence relation[1] on T. Indeed the reflexive property of $\sim$ is immediate. Now let $(x, y), (w, z), (u, v) \in T$ such that $(x, y) \sim (w, z)$ and $(w, z) \sim (u, v)$. Then there are positive $\alpha, \beta \in \mathbb{R}$ such that $x = \alpha w$, $y = \alpha z$, $w = \beta u$ and $z = \beta v$. Since $w = \dfrac{1}{\alpha}x$ and $z = \dfrac{1}{\alpha}y$, so $(w, z) \sim (x, y)$ whence follows the symmetric property and since $x = \alpha\beta u$ and $y = \alpha\beta v$, so $(x, y) \sim (u, v)$ whence follows the transitive property.

For each $(x, y) \in T$, let us write the equivalence class of (x, y) as $[x, y]$. That is, $[x, y] := \big\{(w, z) \in T;\ (w, z) \sim (x, y)\big\}$. Let us call each element of $T/\!\sim$, the quotient set of T with respect to $\sim$, the *transcomplex number* and let us write this set as $\mathbb{C}^T$.

Definition 2: Given $[x, y], [w, z] \in \mathbb{C}^T$ let us define:

a) (addition) $[x, y] + [w, z] :=$

$$\begin{cases} \left[\dfrac{x}{|x|} + \dfrac{w}{|w|}, 0\right], & \text{if } [x, y], [w, z] \in \big\{[u, 0]; u \in \mathbb{C} \setminus \{0\}\big\} \\ [xz + wy, yz], & \text{otherwise} \end{cases}$$

b) (multiplication) $[x, y] \times [w, z] := [xw, yz]$

c) (opposite) $-[x, y] := [-x, y]$

d) (reciprocal) $[x, y]^{-1} := \begin{cases} \left[\dfrac{y}{x}, 1\right], & \text{if } x \neq 0 \\ [y, x], & \text{if } x = 0 \end{cases}$

e) (subtraction) $[x, y] - [w, z] := [x, y] + \big(-[w, z]\big)$

f) (division) $[x, y] \div [w, z] := [x, y] \times [w, z]^{-1}$.

We are conscious that we abuse notation when we reuse the symbols for arithmetical operations on complex numbers to define the arithmetical operations on $\mathbb{C}^T$. However, we emphasise that this is not a problem because the context distinguishes the set to which the symbols refer. For example when we say that $[x, y] + [w, z] = [wy + xz, yz]$ it is clear that the sign "+" on the left hand side of the equality refers to addition in $\mathbb{C}^T$ while the sign "+" on the right hand side of the equality refers to addition in $\mathbb{C}$. Moreover, as will be seen in Theorem 4 and Observation 5, in a suitable sense, $\mathbb{C}$ is a subset of $\mathbb{C}^T$ and when the operations in $\mathbb{C}^T$ are restricted to $\mathbb{C}$ they coincide with the usual operations of $\mathbb{C}$.

Proposition 3: The operations $+$, $\times$, $-$, $^{-1}$ and $\div$ are well defined. That is, $[x, y] + [w, z]$, $[x, y] \times [w, z]$, $-[x, y]$, $[x, y] - [w, z]$, $[x, y]^{-1}$ and $[x, y] \div [w, z]$ are independent of the choice of the representatives of the classes $[x, y]$ and $[w, z]$.

Proof: Let $[x, y], [w, z] \in \mathbb{C}^T$, $(x', y') \in [x, y]$ and $(w', z') \in [w, z]$. We have that there are positives $\alpha, \beta \in \mathbb{R}$ such that $x = \alpha x'$, $y = \alpha y'$, $w = \beta w'$ and $z = \beta z'$.

a) First let us analyse the operation $+$. If $[x, y], [w, z] \in \big\{[u, 0];\ u \in \mathbb{C} \setminus \{0\}\big\}$, then $x \neq 0$, $w \neq 0$, $y = 0$ and $z = 0$ whence $x' \neq 0$, $w' \neq 0$, $y' = 0$, $z' = 0$, $|x| = \alpha|x'|$ and $|w| = \beta|w'|$. Thus $\dfrac{x}{|x|} = \dfrac{\alpha x'}{\alpha|x'|} = \dfrac{x'}{|x'|}$ and $\dfrac{w}{|w|} = \dfrac{\beta w'}{\beta|w'|} = \dfrac{w'}{|w'|}$, whence $\left(\dfrac{x}{|x|} + \dfrac{w}{|w|}, 0\right) = \left(\dfrac{x'}{|x'|} + \dfrac{w'}{|w'|}, 0\right)$. Otherwise $xz + wy = \alpha x'\beta z' + \beta w'\alpha y' = \alpha\beta(x'z' + w'y')$ and $yz = \alpha y'\beta z' = \alpha\beta(y'z')$, whence $(xz + wy, yz) \sim (x'z' + w'y', y'z')$. Hence, in both cases, $[x, y] + [w, z] = [x', y'] + [w', z']$.

[1]Remember [16] that $\sim$ is an equivalence relation on a set A, if and only if, for all $a, b, c \in A$, the three following properties hold:
(reflexivity) $a \sim a$,
(symmetry) if $a \sim b$ then $b \sim a$ and
(transitivity) if $a \sim b$ and $b \sim c$ then $a \sim c$.

b) Next let us analyse the operation $\times$. Notice that $xw = \alpha x'\beta w' = \alpha\beta(x'w')$ and $yz = \alpha y'\beta z' = \alpha\beta(y'z')$, whence $(xw, yz) \sim (x'w', y'z')$. Hence $[x,y] \times [w,z] = [x',y'] \times [w',z']$.

c) Now let us analyse the operation $-$. Note that $-x = -(\alpha x') = \alpha(-x')$ and $y = \alpha y'$. Thus $(-x, y) \sim (-x', y')$. Hence $-[x,y] = -[x',y']$.

d) Finally let us analyse $^{-1}$. If $x \neq 0$ then $\frac{y}{x} = \frac{\alpha y'}{\alpha x'} = \frac{y'}{x'}$, whence $\left(\frac{y}{x}, 1\right) = \left(\frac{y'}{x'}, 1\right)$. If $x = 0$ then $x' = 0$, $y = \alpha y'$ and $x = 0 = \alpha 0 = \alpha x'$, whence $(y,x) \sim (y',x')$. Hence $[x,y]^{-1} = [x',y']^{-1}$.

Notice that $[x,y]-[w,z]$ and $[x,y]\div[w,z]$ are well defined by consequence of the four previous items. ■

It is important to note that $\mathbb{C}^T = \{[x,1]; x \in \mathbb{C}\} \cup \{[w,0]; w \in \mathbb{C}, |w| = 1\} \cup \{[0,0]\}$. Indeed, if $[x,y] \in \mathbb{C}^T$ then $y = 1$ or $y = 0$. If $y = 1$ then $[x,y] \in \{[x,1];\ x \in \mathbb{C}\}$. On the other hand, if $y = 0$ then either $x \neq 0$ implying $x = |x|\frac{x}{|x|}$ and $\left|\frac{x}{|x|}\right| = 1$ whence $[x,y] = \left[\frac{x}{|x|}, 0\right] \in \{[w,0]; w \in \mathbb{C}, |w| = 1\}$ or $x = 0$, whence $[x,y] = [0,0]$. Note also that $\{[x,1];\ x \in \mathbb{C}\}$, $\{[w,0]; w \in \mathbb{C}, |w| = 1\}$ and $\{[0,0]\}$ are pairwise disjoint. Furthermore, if $x \neq w$ then $[x,1] \neq [w,1]$ and if $|x| = 1$, $|w| = 1$ and $x \neq w$ then $[x,0] \neq [w,0]$.

Theorem 4: The set $C := \{[x,1];\ x \in \mathbb{C}\}$ is a field[2].

Proof: The result follows from the fact that $\pi : \mathbb{C} \longrightarrow C$, $\pi(x) = [x,1]$, is a bijective function, from the fact that

(i) $[x,1] + [y,1] = [x+y,1]$ and
(ii) $[x,1] \times [y,1] = [xy,1]$

for any $[x,1],[y,1] \in C$ and from the fact that $\mathbb{C}$ is a field. ■

Notice that, for each $x \in \mathbb{C}$, $-[x,1] = [-x,1]$ and if $x \neq 0$ then $[x,1]^{-1} = [x^{-1},1]$.

Observation 5: Notice that since π is an isomorphism of fields between C and $\mathbb{C}$, we may say that C is a "copy" of $\mathbb{C}$ in $\mathbb{C}^T$. Therefore we may allow an abuse of language and notation: each $[x,1] \in C$ will be written, merely, as x and C will be called set of the complex numbers. In this sense we may say that $\mathbb{C} \subset \mathbb{C}^T$.

Transcomplex arithmetic is total. In particular division by zero is allowed. Thus all transcomplex number can be represented by fractions $\frac{x}{y}$, with $x \in \mathbb{C}$ and $y \in \{0,1\}$ (remember that $\mathbb{C} \subset \mathbb{C}^T$). As usual $\frac{x}{y}$ denotes the result of $x \div y$ in $\mathbb{C}^T$. Indeed, if $z \in \mathbb{C}^T$ then $z = [x,y]$ for some $x \in \mathbb{C}$ and some $y \in \{0,1\}$. So

$$\begin{aligned}[x,y] &= [x \times 1, 1 \times y] &&= [x,1] \times [1,y] \\ &= [x,1] \times [y,1]^{-1} &&= [x,1] \div [y,1] \\ &= x \div y &&= \frac{x}{y}.\end{aligned}$$

Notice that the transcomplex arithmetic, developed here, using numbers in the form $\frac{x}{y}$, is analogous to the arithmetic of fractions of complex numbers. In fact if $\frac{x}{y}, \frac{w}{z} \in \mathbb{C}^T$, where $y,z \in \{0,1\}$, then

(addition) if $y = z = 0$, $x \neq 0$ and $w \neq 0$, so

$$\frac{x}{y} + \frac{w}{z} = \frac{x}{0} + \frac{w}{0} = \frac{\frac{x}{|x|}}{\frac{0}{|x|}} + \frac{\frac{w}{|w|}}{\frac{0}{|w|}} = \frac{\frac{x}{|x|}}{0} + \frac{\frac{w}{|w|}}{0} = \frac{\frac{x}{|x|} + \frac{w}{|w|}}{0},$$

otherwise, $\frac{x}{y} + \frac{w}{z} = \frac{xz + wy}{yz}$.

(multiplication) $\frac{x}{y} \times \frac{w}{z} = \frac{xw}{yz}$.

More than that, the operations of transcomplex arithmetic can be understood geometrically, as set out in the Tutorial, Section IV.

Now let us define *infinity* and *nullity*, respectively, by $\infty := [1,0]$ and $\Phi := [0,0]$. Any complex number can be represented, in polar form, by an ordered pair (r,θ), where $r \in [0,\infty)$ and $\theta \in (-\pi,\pi]$. Note that zero does not have a unique description because $(0,\theta)$ describes zero for all $\theta \in (-\pi,\pi]$. Now we describe Φ by the ordered pair (Φ,θ), where θ is arbitrary in $(-\pi,\pi]$. We represent all transcomplex numbers in the form $[u,0]$ where $u \neq 0$, by the ordered pair (∞,θ), where $\theta = \mathrm{Arg}(u)$. In this way all transcomplex numbers can be represented by an ordered pair, in the form (r,θ), where $r \in [0,\infty] \cup \{\Phi\}$ and $\theta \in (-\pi,\pi]$, observing that $(0,\theta)$ represents zero for all $\theta \in (-\pi,\pi]$ and (Φ,θ) represents Φ for all $\theta \in (-\pi,\pi]$. We can ultimately write

$$\mathbb{C}^T = \mathbb{C} \cup \{(\infty,\theta);\ \theta \in (-\pi,\pi]\} \cup \{\Phi\}.$$

Now the reader can better appreciate figures 1 and 2. Let us refer to the elements of $\mathbb{C}$ as *finite transcomplex numbers*, to the elements of $\{(\infty,\theta);\ \theta \in (-\pi,\pi]\} \bigcup \{\Phi\}$ as *non-finite transcomplex numbers* and, particularly, to the elements of $\{(\infty,\theta);\ \theta \in (-\pi,\pi]\}$ as *infinite transcomplex numbers* then the elements of $\{(\infty,\theta);\ \theta \in (-\pi,\pi]\} \bigcup \{\Phi\}$ are *strictly transcomplex numbers*.

Observation 6: Note that $\mathbb{C}^T$ is a superset of $\mathbb{R}^T$ defined in [8].

Observation 7: As a matter of convenience for computing, we would like to consider every ordered pair, in polar form (r,θ), as transcomplex numbers, where r and θ range over all transreals. This can be done by keeping the equivalence established for (r,θ) when $\theta \in \mathbb{R}$ and establishing that (r,θ) is equivalent to $(\Phi,0)$ for all $\theta \in \{-\infty,\infty,\Phi\}$. That is, $(r,-\infty) \sim (r,\infty) \sim (r,\Phi) \sim (\Phi,0)$ for all $r \in [0,\infty] \cup \{\Phi\}$. Furthermore we observe that, as usual in polar form, $(-r,\theta) \sim (r,\theta+\pi)$ so that r ranges over all transreal numbers. Thus both the radius and angle may be taken from the entire set of transreal numbers.

Notice that (r,θ) is equivalent to $(\Phi,0)$ or, as may be preferred, to (Φ,Φ), for all $\theta \in \{-\infty,\infty,\Phi\}$.

[2]Remember [16] that a set F is a field if and only if F is provided with two binary operations $+$ and $\times$ which, for all $a,b,c \in F$, satisfy the following properties:
(closure under addition and multiplication) $a+b, a \times b \in F$,
(additive and multiplicative identity) there are $0,1 \in F$ such that $a+0 = a$ and $a \times 1 = a$,
(additive and multiplicative inverses) there is $-a \in F$ such that $a + (-a) = 0$ and, if $a \neq 0$ there is $a^{-1} \in F$ such that $a \times a^{-1} = 1$,
(commutativity of addition and multiplication) $a+b = b+a$ and $a \times b = b \times a$,
(associativity of addition and multiplication) $a + (b+c) = (a+b)+c$ and $a \times (b \times c) = (a \times b) \times c$ and
(distributivity of multiplication over addition) $a \times (b+c) = (a \times b) + (a \times c)$.

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IV. Tutorial

The reader is perfectly free to perform transcomplex arithmetic by operating on fractions with a complex numerator and zero or unit denominator, as given in the construction of the transcomplex numbers above. However, it may be helpful to present transcomplex arithmetic in others terms. We assume the reader is fluent in real and complex arithmetic. The reader should then learn transreal arithmetic. The easiest way to learn is by studying the tutorial in [5]. We now present a series of lessons, each of which teaches a different way of doing transcomplex arithmetic. Readers may then use whichever method best suits their temperament or the problem at hand.

A. By Abstract Cases

There are six abstract cases to consider:

(a) Multiplication and division are the usual dilatation and rotation but taken in the whole of the transcomplex plane. (See below.)
(b) When nullity is combined arithmetically with any transcomplex number the result is nullity.
(c) When opposite infinities are added the result is nullity.
(d) When non-opposite infinities are added the result is infinity along the unique bisector of the given infinities.
(e) When infinity is added to a finite number the result is the given infinity.
(f) Complex arithmetic holds in all finite cases.

B. By Geometrical Cases

Transcomplex arithmetic can be understood geometrically.

(a) Multiplication and division are a generalisation of the usual rotation and dilatation where dilatation of a finite radius by ∞ is ∞ and dilatation of any radius by Φ is Φ.
(b) Addition is performed using a generalisation of the usual parallelogram rule where addition of an infinite number and a finite number involves a parallelogram whose one side has infinite length and whose other side has finite length such that the diagonal has infinite length and lies at the same angle as the infinite side.
(c) The sum of two, non-opposite, infinite numbers involves a parallelogram with sides of equal and infinite length such that the sum is the infinitely long diagonal.
(d) The sum of any number with Φ is a diagonal of length Φ.
(e) The sum of finite numbers is given by the ordinary parallelogram rule.

C. By Polar Arithmetic

Consider transcomplex numbers in the polar form (r, θ) with $r, \theta \in \mathbb{R}^T$.

We prefer to reduce all arithmetical results to canonical form. As usual, we accept $(0, 0)$ as the canonical form of $(0, \theta)$ with $\theta \in \mathbb{R}$. For arbitrary $r, \theta \in \mathbb{R}^T$, we may rewrite any of $(\Phi, \Phi) = (\Phi, 0) = (\Phi, \theta) = (r, -\infty) = (r, \infty) = (r, \Phi)$ by whichever of these forms we prefer as the canonical form. Two of these seem natural: (Φ, Φ) and $(\Phi, 0)$. We have a slight preference for (Φ, Φ) because $\theta = \Phi$ makes it immediately apparent that the point at nullity lies off the extended complex plane. See figure 2. Therefore we recommend (Φ, Φ) as the canonical form for the point at nullity.

We re-write any transcomplex number with a negative radius, $(-r, \theta)$, as the corresponding transcomplex number with positive radius $(r, \theta + \pi)$ before applying any arithmetical operator.

The usual formula for polar-complex multiplication applies to the transcomplex numbers without side conditions. Thus: $(r_1, \theta_1) \times (r_2, \theta_2) = (r_1 \times r_2, \theta_1 + \theta_2)$.

The usual formula for polar-complex division applies to the transcomplex numbers without side conditions. Thus: $(r_1, \theta_1) \div (r_2, \theta_2) = (r_1 \div r_2, \theta_1 - \theta_2)$.

A sum over a common radius, r, is written as $(r, \theta_1) + (r, \theta_2) = (rr', \mathrm{Arg}(x, y))$ where $x = \cos\theta_1 + \cos\theta_2$, $y = \sin\theta_1 + \sin\theta_2$ and $r' = \sqrt{x^2 + y^2}$. Notice that applying Arg to the x and y components of a complex number $x + iy$ is an abuse of notation. The computer programmer will be familiar with the use of a function commonly called *arctan2* to obtain the result of $\mathrm{Arg}(x, y)$.

The sum of transcomplex number with distinct radii is computed as follows. Without loss of generality let $r_1 \not< r_2$. Compute $r_2' = r_2 \div r_1$ then $(r_1, \theta_1) + (r_2, \theta_2) = (r' r_1, \mathrm{Arg}(x, y))$ where $x = \cos\theta_1 + r_2' \cos\theta_2$, $y = \sin\theta_1 + r_2' \sin\theta_2$ and $r' = \sqrt{x^2 + y^2}$.

D. By Trigonometric Components

The transcomplex numbers were originally given [5] as triples (r, c, s), where r is the radius and $c = \cos\theta$ and $s = \sin\theta$ with $\theta \in \mathbb{R}^T$. The original algorithms are effective with the new definition of the transcomplex numbers when the whip is collapsed onto the point at nullity. The reader who is skilled in computer algorithms will recognise opportunities to simplify the algorithms given in [5].

V. Discussion

We have developed a generalisation of complex arithmetic that provides binary operations of addition, subtraction, multiplication and division. This is sufficient to establish the consistency of transcomplex arithmetic, as currently conceived, but the non-associativity of the addition of transcomplex numbers, with infinite radius, may militate against having binary operators for addition and subtraction. In future we may prefer to have a single addition operator of arbitrary arity or we may allow the numerators of transcomplex numbers to be summed and differenced associatively, with a separate operator that reduces the sums and differences to a canonical transcomplex number. That is a matter for the future which might best be explored by examining generalisations of vector algebra and differential geometry or by examining the interrelationship between the transcomplex exponential and transcomplex logarithm. Thus we identify the partial non-associativity (and partial non-distributivity) of transcomplex arithmetic as subjects for future work.

Leaving aside these concerns, it may help the reader if we discuss how the transcomplex numbers relate to more familiar number systems and how they can be exploited in computer applications and the design of novel computers.

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The first thing to say is that every transcomplex number is exact. It is described by exactly one point in the transcomplex plane which is composed of the complex plane, the circle at infinity and the point at nullity. Zero is an exact real number. It is the only real number which has neither a positive sign nor a negative sign. In order to make sign total, zero is said to have sign zero. In the same way complex zero is said to have angle zero. Similarly transreal nullity has no negative, zero or positive sign. In order to make sign total, nullity is said to have sign nullity. In the same way transcomplex nullity is said to have angle nullity. Whereas each transcomplex number is described by a unique point in the transcomplex plane, it is described by a conventional representative, a least terms form, drawn from its equivalence class. It shares this two-fold property of a unique point and non-unique representative with the real and complex number systems. But the transnumbers have a profound difference from the ordinary treatment of numbers where division by zero introduces an indefinite or undefined result. There are no indefinite or undefined results in transcomplex or transreal arithmetic. All transnumbers are defined and definite.

It may take the reader some time to appreciate that all transnumbers are defined, definite and exact. For example it is never an arithmetical error to divide any transnumber by zero. Dividing a number by zero might or might not be intended by the mathematician or programmer but that is a question of how the numbers are being used, in other words what they are being used to model, rather than being a property of the number system itself. One is free to use the non-finite transreal numbers to model, say, indefinite numbers in calculus but, as we show in a paper in preparation, one can equally read calculus as operating at and on the exact non-finite transnumbers. This involves a paradigm shift in thinking: division by zero produces exact solutions. In another paper, also in preparation, we show that Newton's laws of motion apply on division by zero so that we obtain exact solutions at mathematical singularities. Fundamentally, transnumbers allow us to consider that infinity is a number, not only an asymptotic form, as in calculus, nor only a cardinality, as in Cantor's set theory. This allows us to define non-finite distance in a generalised metric. Metrics are usually defined in terms of real numbers but transreal numbers give a natural description of non-finite distance [12].

Of course the reader would be more comfortable if all of the consequences of division by zero had been worked out but we are at an early stage in the development of the transnumbers. Results will necessarily appear in a more or less haphazard order. Today we present a construction of the transcomplex numbers from the complex numbers. A construction of the transreal numbers from the reals, given by the first named author of the present paper, is under review for publication. No doubt it will appear at some future time, despite having been written earlier and being logically prior to the present paper. Such irregularities are a natural part of the advancement of science in a new area.

A consequence of totality is that no checking for division by zero need be done at a program's run time nor in the hardware that executes it. A suitably designed, total, computer system has the property that any program that compiles for the machine, executes without any run time errors, aside from physical errors and unintended, but programmed, behaviours. This is certainly beneficial in data-flow machines and may be beneficial in safety critical systems. In the longer term, we may find physical systems where the solution at a singularity has some practical benefit. In the mean time, the present paper records the state of the art in the development of the transcomplex numbers.

VI. Conclusion

We derive the set of transcomplex numbers from the complex numbers and describe a transcomplex arithmetic which totalises the operations of complex arithmetic so that any complex number can be divided by zero. This establishes that transcomplex arithmetic is consistent and that we obtain the expected containments of transcomplex, transreal, complex and real arithmetic. Specifically transcomplex arithmetic contains transreal arithmetic, which contains real arithmetic and transcomplex arithmetic contains complex arithmetic which contains real arithmetic.

Transcomplex arithmetic may find application in mathematical physics where solutions of complex systems are wanted at singularities. Also data-flow machines, operating on transcomplex data, can be guaranteed to run without any interruption to the flow of data, as would otherwise occur on arithmetical exceptions.

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