Cournot Dynamic Duopoly Model for Homogeneous and Heterogeneous Products

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Abstract: Cournot duopoly is one of the most frequently discussed models in the literature of mathematical economics. Many researchers have used this pioneering work to examine and investigate the problems of oligopoly, which represent the real world rivalries involving product differentiation and multiproduct. This paper uses Stacklberg Model of Duopoly to explain how firms achieve equilibrium under dynamic condition, with both homogeneous and heterogeneous products. Dynamic rivalry combines aspects of supergame rivalry with commitment aspects of two-period games.

The basic super game theory assumes simultaneous price and quantity decisions. In this paper, the dynamic mathematical model for homogeneous and heterogeneous games has been developed, with assumptions of isoelastic demand and constant unit production cost for the output of two firms. The system can result in periodic or dynamic behavior. Stability of Nash equilibrium has been explained and analysed with the help of Jacobian matrix.

Index Terms: Homogenous, Heterogeneous Duopoly, Equilibrium points

I. Linear Model

We have an economy with a monopolistic sector with two firms, each one producing a differentiated good, and a competitive *numeraire* sector. There is a continuum of consumers of the same type with a utility function separable and linear in the *numeraire* good. Therefore, there are no income effects on the monopolistic sector, and we can perform partial equilibrium analysis. The representative consumer maximizes

$U(q_1,q_2)=\sum_{i=1}^{2} p_i\, q_i$ *where* q_i is the amount of goods for i and p_i, its price. *U* is assumed to be quadratic and strictly concave

$$U(q_1,q_2)=\alpha_1 q_1+\alpha_2 q_2-\frac{\left(\beta_1 q_1^2+2\gamma q_1 q_2+\beta_1 q_2^2\right)}{2}$$

where α_i *and* β_i are positive,

i = 1, 2, $\beta_1\beta_2-\gamma^2>0,$

and $\alpha_i\beta_j-\alpha_j\gamma>0$ for $i\neq j,\ i=1,2.$

This utility function gives rise to a linear demand structure. Inverse demands are given by

$$p_1=\alpha_1-\beta_1 q_1-\gamma q_2$$
$$p_2=\alpha_2-\gamma q_1-\beta_2 q_2$$

in the region of quantity space where prices are positive.

Let $\mathrm{a_i}=(\alpha_i\beta_j-\alpha_j\gamma)/\delta$

$$\mathrm{b_i}=\frac{\beta_j}{\delta}\ \text{for}\ \mathrm{j}=1,2\ \text{and}\ \mathrm{c}=\frac{\gamma}{\delta}$$

Where $\delta=\beta_1\beta_2-\gamma^2,$

(note that a_i and b_j are positive because of our assumptions), we can write direct demands as

$$\mathrm{q_1=a_1-b_1p_1+cp_2}\quad \text{and}\quad \mathrm{q_2=a_2+c\,p_1-b_2p_2}$$

provided that these quantities are positive. The goods are substitutes, independent, or complements according to whether $\gamma\ \substack{\le\\ \ge}\ 0$. Demand for good *i* is always downward sloping in its own price and increases (decreases) with increases in the price of the competitor if the goods are substitutes (complements).

When $\alpha_1=\alpha_2$ *and* $\beta_1=\beta_2=\gamma,$ the goods are perfect substitutes.

When $\alpha_1=\alpha_2$ *and* $\gamma^2/\beta_1\beta_2$, expresses the degree of product differentiation, ranging from zero (when the goods are independent) to one (when the goods are perfect substitutes). When γ is positive and $\gamma^2/\beta_1\beta_2$ approaches one, we are close to a homogeneous market.

II. Dynamic Model for Homogeneous and Heterogeneous Duopoly

The study of duopoly model with heterogeneous firms depends upon rational expectations because perfect knowledge of the market may not be available in real

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economics. Since firms have incomplete knowledge of the market, they have to use partial information based on the local market. Each firm increases (decreases) its production q_i at each period (t + 1) if marginal profit is positive (negative).

III. Model

Let the linear demand function be

$$p = f(Q) = a - bQ \qquad (1)$$

where $Q(t) = q_i(t) + q_j(t)$ is total supply and 'a' and 'b' are positive constant of demand function.

Let the cost function be

$$C_i(q_i) = c_i q_i \quad i = 1, 2 \qquad (2)$$

C_i is the marginal cost of i^{th} firm.

Profit function of i^{th} firm is given by $\pi_i = pq_i - c_i q_i$

$$\pi_i = q_i(a - bQ) - c_i q_i \quad i = 1, 2$$

$$\pi_i = q_i(a - b(q_i + q_j)) - c_i q_i \quad (3)$$

Now the marginal profit of i^{th} firm is given by

$$\frac{\partial \pi_i}{\partial q_i} = a - c_i - 2bq_i - bq_j$$

$$i, j = 1, 2 \qquad i \neq j$$

For maximum profit $\frac{\partial \pi_i}{\partial q_i} = 0$

$$\Rightarrow \quad a - c_i - 2bq_i - bq_j = 0$$

$$q_i = \left(\frac{a - c_i - bq_j}{2b} \right) \qquad (4)$$

Now at each time period(t) every player must form an expectation of the rival's output in the next time period(t+1), in order to determine the profit maximizing quantities for $(t+1)$ period.

$$q_i(t+1) = \arg_{q_1}^{\max} \ \pi_1[q_1(t), q_2^e(t+1)]$$

$$q_i(t+1) = \arg_{q_2}^{\max} \ \pi_2[q_1^e(t+1), q_2(t)] \qquad (5)$$

Where $q_1^e(t+1)$ represents the expectation of firm 1 about the production decision of firm 2.

In Cournot's model $q_2^e(t+1) = q_1(t)$

[Naïve expectations]

or $\quad q_1^e(t+1) = q_2(t)$

$\Rightarrow \quad q_1(t+1) = f(q_2(t)$ and $q_2(t+1) = g(q_1(t)]$

$\Rightarrow$ There exists a mapping

$T : R^2 \longrightarrow R^2$ given by

$$T: \begin{cases} q_1' &= f(q_2) \\ q_2' &= f(q_1) \end{cases}$$

Where q_1', q_2' represents the one period advancement

The map (6) represent the Duopoly game in the case of homogeneous expectations.

Cournot-Nash equilibrium can be located by the intersection of

$$q_1 = f(q_2) \text{ and } q_2 = f(q_1).$$

IV. Heterogeneous Expectations

The firms use local information based on the marginal profit $\frac{\partial \pi_i}{\partial q_i}$. At each time period t each firm increases (decreases) its production q_i at the period $(t+1)$ if the marginal profit is positive (negative).

$\therefore$ Dynamical equation of this type game is of the form

$$q_i(t+1) = q_i(t) + \alpha_i \, q_i(t) \frac{\partial \pi_i}{\partial q_i(t)}$$

where $i = 0, 1, 2$

where, if $i = 0$ it becomes Cournot naïve expectations and α_i is a positive parameter which represents the speed of adjustment.

Similarly we can obtained equations for two or three players as follows

$$\begin{cases} q_1(t+1) &= q_1(t) + \alpha_1 q_1(t) \dfrac{\partial \pi_1}{\partial q_1(t)} \\ q_2(t+1) &= g(q_1(t)] \end{cases} \qquad (7)$$

Using (4) we can find the dynamical equation of the firm which is boundedly rational player. i.e.,

$$q_1(t+1) = q_1(t) + \alpha \, q_1(t)[a - c_1 - 2bq_1(t) - bq_2(t)]$$

(8)

Similarly we can find the dynamical equation of the second firm which is naïve

$$q_2(t+1) = \frac{1}{2b}[a - c_2 - bq_1(t)]$$

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The two dimensional map $T(q_1,q_2)\to(q_1',\ q_2')$ is completely represent heterogeneous duopoly.

V. Boundary Equilibrium and Nesh Equilibrium

When we coupling the dynamic equation we get

$$T:\left\{\begin{array}{l} q_1' = q_1 + \alpha q_1(a-c_1-2bq_1-bq_2) \\ q_2' = \dfrac{1}{2b}(a-c_2-bq_1) \end{array}\right\}$$

Using $q_1'=q_1$ and $q_2'=q_2$ we can find non-negative solution of algebraic equation

$$q_1(a-c_1-2bq_1-bq_2)=0$$

$$a-c_2-bq_1-2bq_2=0$$

$$\Rightarrow \quad q_1=0 \text{ or } a-c_1-2bq_1-bq_2=0;$$

$$a-c_2-bq_1-2bq_2=0$$

When $q_1=0$ $\quad q_2=\dfrac{a-c_2}{2b}$

We get fixed point $E_0=\left(0,\ \dfrac{a-c_2}{2b}\right)$ which is called boundary equilibrium

On solving $a-c_1-2bq_1-bq_2=0$

and $a-c_2-bq_1-2bq_2=0$

$$2a-2c_1-4bq_1-2bq_2=0$$

$$\underline{a-c_2-bq_1-2bq_2=0}$$

$$a+c_2-2c_1-3bq_1=0$$

$$q_1^*=\frac{a+c_2-2c_1}{3b}$$

and $\quad q_2^*=\dfrac{a+c_1-2c_2}{3b}$

$$q_1^*=\frac{a-(2c_1-c_2)}{3b}$$

$$q_2^*=\frac{a-(2c_2-c_1)}{3b}$$

Then $E^*(q_1^*,q_2^*)$ represent Nash equilibrium provided $a>2c_1-c_2$ and $a>2c_2-c_1$.

VI. Eigen Values

The study of local stability of equilibrium solution is based on the localization, on the complex plane of the eigen values of the Jacobian matrix of the two dimensional map.

Jacobian matrix

$$J=\left[\frac{q_1',\ q_2'}{q_1,q_2}\right]=\begin{bmatrix} \dfrac{\partial q_1'}{\partial q_1} & \dfrac{\partial q_1'}{\partial q_2} \\ \dfrac{\partial q_2'}{\partial q_1} & \dfrac{\partial q_2'}{\partial q_2} \end{bmatrix}$$

$$=\begin{bmatrix} 1+\alpha(a-4bq_1-bq_2-c_1) & -\alpha bq_1 \\ -\dfrac{1}{2} & 0 \end{bmatrix}$$

$$|J|=-\frac{1}{2}\alpha b q_1$$

This dynamical system for $(\alpha bq_1)<2$ become **dissipative** [In homogeneous].

VII. Nash Equilibrium Based on Partial Information

Next study of Nash Equilibrium based on partial information or local stability.

$$J(E^*)=\begin{bmatrix} 1-2\alpha bq_1^* & -\alpha bq_1^* \\ -\dfrac{1}{2} & 0 \end{bmatrix}$$

Characteristic equation of $J(E^*)$ is

$$P(\lambda)=\lambda^2 - (\text{Trace})\lambda + \text{Det} = 0$$

Where Trace = $1-2\alpha bq_1^*$ and Det = $-\dfrac{1}{2}\alpha bq_1^*$

$$\lambda^2-(1-2\alpha bq_1^*)\lambda-\frac{1}{2}\alpha bq_1^*=0$$

Which is quadratic equation in λ with discriminants.

$$D=b^2-4ac$$

$$=(1-2\alpha bq_1^*)^2-4.1.\left(-\frac{1}{2}\alpha bq_1^*\right)$$

$$=(1-2\alpha bq_1^*)^2+2\alpha bq_1^*$$

$$D>0 \text{ (always)}$$

$\Rightarrow$ Eigen values of Nash equilibrium are real.

VIII. Conclusion

Dynamic Models for homogenous and heterogeneous games are of great concern for today's researchers. Boundary and Nash equilibrium points are discussed with homogenous and heterogeneous products. During research, it was found that the boundary point is unstable and varies according to the change in time and that is why the model needed to be dynamic. Further Nash equilibrium is located on the partial information. Researchers can take the opportunity to develop the model for three to five players, which would be

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really helpful for the industry. Care should be taken as the cost function would then become nonlinear in nature.

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