Classes of Ordinary Differential Equations Obtained for the Probability Functions of Exponential and Truncated Exponential Distributions

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***Abstract*— In this paper, the differential calculus was used to obtain some classes of ordinary differential equations (ODE) for the probability density function, quantile function, survival function, inverse survival function and reversed hazard function of the exponential and truncated exponential distributions. However, the ODE of the hazard function of the exponential distribution cannot be obtained because of the constant failure nature of the distribution. The ODE of the truncated exponential distribution was obtained. The stated necessary conditions required for the existence of the ODEs are consistent with the various parameters that defined the distributions. Solutions of these ODEs by using numerous available methods are new ways of understanding the nature of the probability functions that characterize the distribution.**

***Index Terms*— Exponential distribution, differential calculus, probability density function, survival function, quantile function, reversed hazard function.**

I. Introduction

EXPONENTIAL distribution is used to model the time between events or events between a time intervals. Because of the flexibility of the distribution and constant hazard, the distribution has seen many modifications in form of compounding, exponentiation, transmuted and so on. Some are distributions include: multivariate exponential distribution [1], generalized exponential distribution [2], beta exponential distribution [3], beta generalized exponential distribution [4], quadratic exponential binary distribution [5], generalized exponential Poisson distribution [6], extended Poisson exponential distribution [7], gamma exponentiated exponential distribution [8], convoluted beta-exponential distribution [9], weighted exponential distribution [10], fractional beta exponential distribution [11], exponential Poisson Logarithmic distribution [12], exponentiated Kumaraswamy-exponential distribution [13], Weibull exponential distribution [14], Lindley exponential distribution [15], weighted exponential distribution [16].

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The truncated exponential distribution is a sub model of the parent exponential distribution obtained by the restriction of some selected domain of exponential distribution. The details can be obtained from the notes of [17]. Lominashvili and Patsatsia [18] obtained the estimates of maximum likelihood of the distribution. Korkmaz and Genç [19]observed that their proposed distribution known as the two-sided generalized exponential distribution is the mixture of truncated exponential distribution and truncated generalized exponential distribution earlier proposed by [20].

Exponential has been recently used to model the following: analysis of plates in solid dynamics [21], infection rate [22] and ecology [23].

The aim of this research is to develop ordinary differential equations (ODE) for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF) and reversed hazard function (RHF) of exponential and truncated exponential distributions by the use of differential calculus. Also the ODE of hazard function of the truncated exponential distribution was also obtained. Calculus is a very key tool in the determination of mode of a given probability distribution and in estimation of parameters of probability distributions, amongst other uses. The research is an extension of the ODE to other probability functions other than the PDF. Similar works done where the PDF of probability distributions was expressed as ODE whose solution is the PDF are available. They include: Laplace distribution [24], beta distribution [25], raised cosine distribution [26], Lomax distribution [27], beta prime distribution or inverted beta distribution [28].

I. Exponential Distribution

Probability Density Function

The probability density function of the Cauchy distribution is given by;
The probability density function (PDF) of the exponential distribution is given by;

$$f(x) = \lambda e^{-\lambda x} \tag{1}$$

The first order differential equation for the PDF can be obtained from the differentiation of equation (1);

$$f'(x) = -\lambda^2 e^{-\lambda x} \tag{2}$$

The condition necessary for the existence of the equation is

$\lambda > 0, x \geq 0$

$$f'(x) = -\lambda(\lambda \,\mathrm{e}^{-\lambda x}) = -\lambda f(x) \quad (3)$$

The first order ordinary differential for the probability density function of the exponential distribution is given as;

$$f'(x) + \lambda f(x) = 0 \quad (4)$$

$$f(0) = \lambda \quad (5)$$

To obtain the second order differential equation, differentiate equation (2) to obtain;

$$f''(x) = \lambda^3 \,\mathrm{e}^{-\lambda x} \quad (6)$$

The condition necessary for the existence of the equation is $\lambda > 0, x \geq 0$.

$$f''(x) = -\lambda(-\lambda^2 \,\mathrm{e}^{-\lambda x}) = -\lambda f'(x) \quad (7)$$

The second order ordinary differential for the probability density function of the exponential distribution is given as;

$$f''(x) + \lambda f'(x) = 0 \quad (8)$$

$$f'(0) = -\lambda^2 \quad (9)$$

To obtain the third order differential equation, differentiate equation (6) to obtain;

$$f'''(x) = -\lambda^4 \,\mathrm{e}^{-\lambda x} \quad (10)$$

The condition necessary for the existence of the equation is $\lambda > 0, x \geq 0$.

$$f'''(x) = -\lambda(\lambda^3 \,\mathrm{e}^{-\lambda x}) = -\lambda f''(x) \quad (11)$$

The third order ordinary differential for the probability density function of the exponential distribution is given as;

$$f'''(x) + \lambda f''(x) = 0 \quad (12)$$

$$f''(0) = \lambda^3 \quad (13)$$

Similarly, higher order ordinary differential equations can be obtained such as;

$$f^{iv}(x) + \lambda f'''(x) = 0 \quad (14)$$

$$f^{v}(x) + \lambda f^{iv}(x) = 0 \quad (15)$$

QUANTILE FUNCTION

The Quantile function (QF) of the exponential distribution is given by;

$$Q(p) = -\frac{\ln(1-p)}{\lambda} \quad (16)$$

The first order differential equation for the QF can be obtained from the differentiation of equation (16);

$$Q'(p) = \frac{1}{\lambda}(1-p)^{-1} \quad (17)$$

Equation (17) can also be written as;

$$Q'(p) = \frac{1}{\lambda(1-p)} \quad (18)$$

The condition necessary for the existence of the equation is $\lambda > 0, 0 < p < 1$.

Simplify equation (18) to obtain;

$$\lambda(1-p)Q'(p) = 1 \quad (19)$$

The first order ordinary differential for the Quantile function of the exponential distribution is given as;

$$\lambda(1-p)Q'(p) - 1 = 0 \quad (20)$$

$$Q(0.1) = \frac{0.1054}{\lambda} \quad (21)$$

To obtain the second order differential equation, differentiate equation (17) to obtain;

$$Q''(p) = \frac{1}{\lambda}(1-p)^{-2} \quad (22)$$

Equation (22) can also be written as;

$$Q''(p) = \frac{1}{\lambda(1-p)^2} \quad (23)$$

The condition necessary for the existence of the equation is $\lambda > 0, 0 < p < 1$.

Simplify equation (23) to obtain;

$$(1-p)\left[\frac{1}{\lambda(1-p)^2}\right] = \frac{1}{\lambda(1-p)} \quad (24)$$

$$(1-p)Q''(p) = Q'(p) \quad (25)$$

The second order ordinary differential for the Quantile function of the exponential distribution is given as;

$$(1-p)Q''(p) - Q'(p) = 0 \quad (26)$$

$$Q'(0.1) = \frac{10}{9\lambda} \quad (27)$$

To obtain the third order differential equation, differentiate equation (22) to obtain;

$$Q'''(p) = \frac{2}{\lambda}(1-p)^{-3} \quad (28)$$

Equation (28) can also be written as;

$$Q'''(p) = \frac{2}{\lambda(1-p)^3} \quad (29)$$

The condition necessary for the existence of the equation is $\lambda > 0, 0 < p < 1$.

Simplify equation (29) to obtain;

$$\frac{(1-p)}{2}\left[\frac{2}{\lambda(1-p)^3}\right] = \frac{1}{\lambda(1-p)^2} \quad (30)$$

$$\frac{(1-p)}{2}Q'''(p) = Q''(p) \quad (31)$$

The third order ordinary differential for the Quantile function of the exponential distribution is given as;

$$(1-p)Q'''(p) - 2Q''(p) = 0 \quad (32)$$

$$Q''(0.1) = \frac{100}{81\lambda} \quad (33)$$

Similarly, higher order ordinary differential equations can be obtained such as;

$$(1-p)Q^{iv}(p) - 3Q'''(p) = 0 \quad (34)$$

$$(1-p)Q^{v}(p) - 4Q^{iv}(p) = 0 \quad (35)$$

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SURVIVAL FUNCTION

The survival function (SF) of the exponential distribution is given by;

$$S(t) = \mathrm{e}^{-\lambda t} \tag{36}$$

The first order differential equation for the SF can be obtained from the differentiation of equation (36);

$$S'(t) = -\lambda\,\mathrm{e}^{-\lambda t} \tag{37}$$

The condition necessary for the existence of the equation is $\lambda > 0, t \geq 0$.

$$S'(t) = -\lambda(\mathrm{e}^{-\lambda t}) = -\lambda S(t) \tag{38}$$

The first order ordinary differential for the survival function of the exponential distribution is given as;

$$S'(t) + \lambda S(t) = 0 \tag{39}$$

$$S(0) = 1 \tag{40}$$

To obtain the second order differential equation, differentiate equation (37) to obtain;

$$S''(t) = \lambda^2\,\mathrm{e}^{-\lambda t} \tag{41}$$

The condition necessary for the existence of the equation is $\lambda > 0, t \geq 0$.

$$S''(t) = -\lambda(-\lambda\,\mathrm{e}^{-\lambda t}) = -\lambda S'(t) \tag{42}$$

The second order ordinary differential for the survival function of the exponential distribution is given as;

$$S''(t) + \lambda S'(t) = 0 \tag{43}$$

$$S'(0) = \lambda \tag{44}$$

Similarly, higher order ordinary differential equations can be obtained such as;

$$S'''(t) + \lambda S''(t) = 0 \tag{45}$$

$$S^{iv}(t) + \lambda S'''(t) = 0 \tag{46}$$

$$S^{v}(t) + \lambda S^{iv}(t) = 0 \tag{47}$$

INVERSE SURVIVAL FUNCTION

The inverse survival function (ISF) of the exponential distribution is given by;

$$Q(p) = -\frac{\ln p}{\lambda} \tag{48}$$

The first order differential equation for the ISF can be obtained from the differentiation of equation (48);

$$Q'(p) = -\frac{1}{\lambda p} \tag{49}$$

The condition necessary for the existence of the equation is $\lambda > 0, 0 < p < 1$.

Simplify equation (49) to obtain;

$$\lambda p Q'(p) = -1 \tag{50}$$

The first order ordinary differential for the inverse survival function of the exponential distribution is given as;

$$\lambda p Q'(p) + 1 = 0 \tag{51}$$

$$Q(0.1) = \frac{2.3026}{\lambda} \tag{52}$$

To obtain the second order differential equation, differentiate equation (49) to obtain;

$$Q''(p) = \frac{1}{\lambda p^2} \tag{53}$$

The condition necessary for the existence of the equation is $\lambda > 0, 0 < p < 1$.

Simplify equation (53) to obtain;

$$p\left[\frac{1}{\lambda p^2}\right] = \frac{1}{\lambda p} \tag{54}$$

$$pQ''(p) = -Q'(p) \tag{55}$$

The second order ordinary differential for the inverse survival function of the exponential distribution is given as;

$$pQ''(p) + Q'(p) = 0 \tag{56}$$

$$Q'(0.1) = -\frac{10}{\lambda} \tag{57}$$

To obtain the third order differential equation, differentiate equation (53) to obtain;

$$Q'''(p) = -\frac{2}{\lambda p^3} \tag{58}$$

The condition necessary for the existence of the equation is $\lambda > 0, 0 < p < 1$.

Simplify equation (58) to obtain;

$$\frac{p}{2}\left[-\frac{2}{\lambda p^3}\right] = -\frac{1}{\lambda p^2} \tag{59}$$

$$\frac{p}{2}Q'''(p) = -Q''(p) \tag{60}$$

The third order ordinary differential for the inverse survival function of the exponential distribution is given as;

$$pQ'''(p) + 2Q''(p) = 0 \tag{61}$$

$$Q''(0.1) = \frac{100}{\lambda} \tag{62}$$

REVERSED HAZARD FUNCTION

The reversed hazard function (RHF) of the exponential distribution is given by;

$$j(t) = \frac{\lambda\,\mathrm{e}^{-\lambda t}}{1 - \mathrm{e}^{-\lambda t}} \tag{63}$$

The first order differential equation for the RHF can be obtained from the differentiation of equation (63);

$$j'(t) = \left\{-\frac{\lambda\,\mathrm{e}^{-\lambda t}}{\mathrm{e}^{-\lambda t}} - \frac{\lambda\,\mathrm{e}^{-\lambda t}(1 - \mathrm{e}^{-\lambda t})^{-2}}{(1 - \mathrm{e}^{-\lambda t})^{-1}}\right\} j(t) \tag{64}$$

The condition necessary for the existence of the equation is $\lambda > 0, t \geq 0$.

$$j'(t) = -\left(\lambda + \frac{\lambda\,\mathrm{e}^{-\lambda t}}{1 - \mathrm{e}^{-\lambda t}}\right) j(t) \tag{65}$$

$$j'(t) = -(\lambda + j(t))\,j(t) \tag{66}$$

The first order ordinary differential for the reversed hazard function of the exponential distribution is given as;

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$$j'(t)+j^2(t)+\lambda j(t)=0 \quad (67)$$

$$j(1)=\frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}=\frac{\lambda}{e^{\lambda}-1} \quad (68)$$

II. TRUNCATED EXPONENTIAL DISTRIBUTION

PROBABILITY DENSITY FUNCTION

The probability density function (PDF) of the truncated exponential distribution is given by;

$$f(x)=\frac{\frac{1}{\lambda}e^{-\frac{x}{\lambda}}}{1-e^{-\left(\frac{b}{\lambda}\right)}} \quad (69)$$

The first order differential equation for the PDF can be obtained from the differentiation of equation (69);

$$f'(x)=-\frac{\frac{1}{\lambda^2}e^{-\frac{x}{\lambda}}}{1-e^{-\left(\frac{b}{\lambda}\right)}} \quad (70)$$

The condition necessary for the existence of the equation is $\lambda, b>0, x\geq 0$.

Simplify equation (70) using (69) to obtain;

$$f'(x)=-\frac{f(x)}{\lambda} \quad (71)$$

The first order ordinary differential for the probability density function of the truncated exponential distribution is given as;

$$\lambda f'(x)+f(x)=0 \quad (72)$$

$$f(0)=\frac{1}{\lambda(1-e^{-\left(\frac{b}{\lambda}\right)})} \quad (73)$$

To obtain the second order differential equation, differentiate equation (70) to obtain;

$$f''(x)=\frac{\frac{1}{\lambda^3}e^{-\frac{x}{\lambda}}}{1-e^{-\left(\frac{b}{\lambda}\right)}} \quad (74)$$

The condition necessary for the existence of the equation is $\lambda, b>0, x\geq 0$.

Simplify equation (74) to obtain;

$$f''(x)=-\frac{f'(x)}{\lambda} \quad (75)$$

The second order ordinary differential for the probability density function of the truncated exponential distribution is given as;

$$\lambda f''(x)+f'(x)=0 \quad (76)$$

$$f'(0)=-\frac{1}{\lambda^2(1-e^{-\left(\frac{b}{\lambda}\right)})} \quad (77)$$

Similarly, higher order ordinary differential equations can be obtained such as;

$$\lambda f'''(x)+f''(x)=0 \quad (78)$$

$$\lambda f^{iv}(x)+f'''(x)=0 \quad (79)$$

$$\lambda f^{v}(x)+f^{iv}(x)=0 \quad (80)$$

QUANTILE FUNCTION

The Quantile function (QF) of the truncated exponential distribution is given as;

$$Q(p)=-\lambda\ln(1-p+pe^{-\left(\frac{b}{\lambda}\right)}) \quad (81)$$

The first order differential equation for the QF can be obtained from the differentiation of equation (81);

$$Q'(p)=-\frac{\lambda(e^{-\frac{b}{\lambda}}-1)}{1-p+pe^{-\frac{b}{\lambda}}} \quad (82)$$

The condition necessary for the existence of the equation is $\lambda, b>0, 0<p<1$.

Let $A=e^{-\frac{b}{\lambda}}$ (83)

Substitute equation (83) into equation (82) to obtain;

$$Q'(p)=-\frac{\lambda(A-1)}{1-p+pA}$$

(84) The first order ordinary differential for the Quantile function of the truncated exponential distribution is given as; $(1-p+pA)Q'(p)+\lambda(A-1)=0$

(85) $Q(0)=0$

(86)

To obtain the second order differential equation, equation (82) can be written as;

$$Q'(p)=\lambda(1-e^{-\frac{b}{\lambda}})(1-p+pe^{-\frac{b}{\lambda}})^{-1} \quad (87)$$

Differentiate equation (87) to obtain;

$$Q''(p)=\lambda(1-e^{-\frac{b}{\lambda}})^2(1-p+pe^{-\frac{b}{\lambda}})^{-2} \quad (88)$$

The condition necessary for the existence of the equation is $\lambda, b>0, 0<p<1$.

Re-arranging and simplifying equation (88), to obtain

$$Q''(p)=\frac{\lambda(1-e^{-\frac{b}{\lambda}})^2}{(1-p+pe^{-\frac{b}{\lambda}})^2} \quad (89)$$

$$\lambda Q''(p)=\frac{\lambda^2(1-e^{-\frac{b}{\lambda}})^2}{(1-p+pe^{-\frac{b}{\lambda}})^2} \quad (90)$$

$$\lambda Q''(p)=Q'^2(p) \quad (91)$$

The second order ordinary differential for the Quantile function of the truncated exponential distribution is given as; $\lambda Q''(p)-Q'^2(p)=0$

(92) $Q'(0)=\lambda(1-e^{-\frac{b}{\lambda}})$

(93) To obtain the third order differential equation, differentiate equation (88) to obtain;

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$$Q'''(p) = 2\lambda(1-\mathrm{e}^{-\frac{b}{\lambda}})^3(1-p+p\,\mathrm{e}^{-\frac{b}{\lambda}})^{-3} \quad (94)$$

The condition necessary for the existence of the equation is $\lambda, b > 0, 0 < p < 1$.

Re-arranging and simplifying equation (91), to obtain

$$Q'''(p) = \frac{2\lambda(1-\mathrm{e}^{-\frac{b}{\lambda}})^3}{(1-p+p\,\mathrm{e}^{-\frac{b}{\lambda}})^3} \quad (95)$$

$$\lambda^2 Q'''(p) = \frac{2\lambda^3(1-\mathrm{e}^{-\frac{b}{\lambda}})^3}{(1-p+p\,\mathrm{e}^{-\frac{b}{\lambda}})^3} \quad (96)$$

$$\lambda^2 Q'''(p) = Q'^3(p) \quad (97)$$

The third order ordinary differential for the Quantile function of the truncated exponential distribution is given as; $\lambda^2 Q'''(p) - 2Q'^3(p) = 0$

(98) $Q''(0) = \lambda(1-\mathrm{e}^{-\frac{b}{\lambda}})^2$

(99) Similarly, higher order ordinary differential equations can be obtained such as;

$$\lambda^3 Q^{iv}(p) - 3Q'^4(p) = 0 \quad (100)$$

$$\lambda^4 Q^{v}(p) - 4Q'^5(p) = 0 \quad (101)$$

Survival Function

The survival function (SF) of the truncated exponential distribution is derived from the cumulative distribution function and is given by;

$$S(t) = \frac{\mathrm{e}^{-\frac{t}{\lambda}} - \mathrm{e}^{-\frac{b}{\lambda}}}{1-\mathrm{e}^{-\frac{b}{\lambda}}} \quad (102)$$

Let $C = \mathrm{e}^{-\frac{b}{\lambda}}$ (103)

$$S(t) = \frac{\mathrm{e}^{-\frac{t}{\lambda}} - C}{1-C} \quad (104)$$

The first order differential equation for the SF can be obtained from the differentiation of equation (104);

$$S'(t) = \frac{-\frac{1}{\lambda}\mathrm{e}^{-\frac{t}{\lambda}}}{1-C} \quad (105)$$

The condition necessary for the existence of the equation is $\lambda, b > 0, t \geq 0$.

Some simplifications are done to reduce equation (105) to an ordinary differential equation.
From equation (104);

$$(1-C)S(t) = \mathrm{e}^{-\frac{t}{\lambda}} - C \quad (106)$$

$$\mathrm{e}^{-\frac{t}{\lambda}} = (1-C)S(t) + C \quad (107)$$

Substitute equation (107) into equation (105), to obtain;

$$S'(t) = -\frac{(1-C)S(t) + C}{\lambda(1-C)} \quad (108)$$

$$\lambda(1-C)S'(t) + (1-C)S(t) + C = 0 \quad (109)$$

Let $D = 1-C$ (110)

The first order ordinary differential for the survival function of the truncated exponential distribution is given as;

$$\lambda D S'(t) + D S(t) + 1 - D = 0 \quad (111)$$

$$S(0) = 1 \quad (112)$$

To obtain the second order differential equation, differentiate equation (105) to obtain;

$$S''(t) = \frac{\frac{1}{\lambda^2}\mathrm{e}^{-\frac{t}{\lambda}}}{1-C} \quad (113)$$

The condition necessary for the existence of the equation is $\lambda, b > 0, t \geq 0$.

Simplifying to obtain;

$$S''(t) = -\frac{1}{\lambda}\left[\frac{-\frac{1}{\lambda}\mathrm{e}^{-\frac{t}{\lambda}}}{1-C}\right] \quad (114)$$

$$S''(t) = -\frac{1}{\lambda}S'(t) \quad (115)$$

The second order ordinary differential for the Survival function of the truncated exponential distribution is given as; $\lambda S''(t) + S'(t) = 0$

(116) $S'(0) = -\frac{1}{\lambda(1-C)}$

(117) To obtain the third order differential equation, differentiate equation (113) to obtain;

$$S'''(t) = -\frac{\frac{1}{\lambda^3}\mathrm{e}^{-\frac{t}{\lambda}}}{1-C} \quad (118)$$

The condition necessary for the existence of the equation is $\lambda, b > 0, t \geq 0$.

Simplifying to obtain;

$$S'''(t) = -\frac{1}{\lambda}\left[\frac{\frac{1}{\lambda^2}\mathrm{e}^{-\frac{t}{\lambda}}}{1-C}\right] \quad (119)$$

$$S'''(t) = -\frac{1}{\lambda}S''(t) \quad (120)$$

The third order ordinary differential for the Survival function of the truncated exponential distribution is given as; $\lambda S'''(t) + S''(t) = 0$

(121) $S''(0) = \frac{1}{\lambda^2(1-C)}$

(122) Similarly, higher order ordinary differential equations can be obtained such as;

$$\lambda S^{iv}(t) + S'''(t) = 0 \quad (123)$$

$$\lambda S^{v}(t) + S^{iv}(t) = 0 \quad (124)$$

Hazard Function

The hazard function (HF) of the truncated exponential distribution is the ratio of the probability density function to the survival function. This is given as;

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$$h(t)=\frac{\frac{1}{\lambda}e^{-\frac{t}{\lambda}}}{e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}} \quad (125)$$

The first order differential equation for the HF can be obtained from the differentiation of equation (125);

$$h'(t)=\frac{\frac{1}{\lambda^2}\left(e^{-\frac{t}{\lambda}}\right)^2}{\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)^2}-\frac{\frac{1}{\lambda^2}\left(e^{-\frac{t}{\lambda}}\right)}{\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)} \quad (126)$$

The condition necessary for the existence of the equation is $\lambda, b>0, t\geq 0$.

$$h'(t)=h^2(t)-\lambda h(t) \quad (127)$$

The first order ordinary differential for the Hazard function of the truncated exponential distribution is given as;

$$h'(t)-h^2(t)+\lambda h(t)=0 \quad (128)$$

$$h(0)=\frac{1}{\lambda(1-e^{-\frac{b}{\lambda}})} \quad (129)$$

To obtain the second order differential equation, differentiate equation (125) to obtain;

$$h''(t)=\frac{2}{\lambda^3}\left(e^{-\frac{t}{\lambda}}\right)^3\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)^{-3}$$
$$-\frac{2}{\lambda^3}\left(e^{-\frac{t}{\lambda}}\right)^2\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)^{-2}-\frac{1}{\lambda^3}\left(e^{-\frac{t}{\lambda}}\right)^2\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)^{-2}$$
$$+\frac{1}{\lambda^3}\left(e^{-\frac{t}{\lambda}}\right)\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)^{-1} \quad (130)$$

The condition necessary for the existence of the equation is $\lambda, b>0, t\geq 0$.

$$h''(t)=\frac{2}{\lambda^3}\left(e^{-\frac{t}{\lambda}}\right)^3\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)^{-3}$$
$$-\frac{3}{\lambda^3}\left(e^{-\frac{t}{\lambda}}\right)^2\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)^{-2}-\frac{1}{\lambda^3}\left(e^{-\frac{t}{\lambda}}\right)\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)^{-1} \quad (131)$$

$$h''(t)=\frac{\frac{2}{\lambda^3}\left(e^{-\frac{t}{\lambda}}\right)^3}{\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)^3}-\frac{\frac{3}{\lambda^3}\left(e^{-\frac{t}{\lambda}}\right)^2}{\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)^2}+\frac{\frac{1}{\lambda^3}\left(e^{-\frac{t}{\lambda}}\right)}{\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)} \quad (132)$$

$$h''(t)=2h^3(t)-3\lambda h^2(t)+\lambda^2 h(t) \quad (133)$$

The second order ordinary differential for the Hazard function of the truncated exponential distribution is given as;

$$h''(t)-2h^3(t)+3\lambda h^2(t)-\lambda^2 h(t)=0 \quad (134)$$

$$h'(0)=\frac{e^{-\frac{b}{\lambda}}}{\lambda^2(1-e^{-\frac{b}{\lambda}})^2} \quad (135)$$

To obtain the third order differential equation, differentiate equation (131) to obtain;

$$h'''(t)=\frac{6}{\lambda^4}\left(e^{-\frac{t}{\lambda}}\right)^4\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)^{-4}-\frac{6}{\lambda^4}\left(e^{-\frac{t}{\lambda}}\right)^3\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)^{-3}$$
$$-\frac{6}{\lambda^4}\left(e^{-\frac{t}{\lambda}}\right)^3\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)^{-3}+\frac{6}{\lambda^4}\left(e^{-\frac{t}{\lambda}}\right)^2\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)^{-2}$$
$$+\frac{1}{\lambda^4}\left(e^{-\frac{t}{\lambda}}\right)^2\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)^{-2}-\frac{1}{\lambda^4}\left(e^{-\frac{t}{\lambda}}\right)\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)^{-1} \quad (136)$$

The condition necessary for the existence of the equation is $\lambda, b>0, t\geq 0$.

$$h'''(t)=\frac{6}{\lambda^4}\left(e^{-\frac{t}{\lambda}}\right)^4\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)^{-4}-\frac{12}{\lambda^4}\left(e^{-\frac{t}{\lambda}}\right)^3\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)^{-3}$$
$$+\frac{7}{\lambda^4}\left(e^{-\frac{t}{\lambda}}\right)^2\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)^{-2}-\frac{1}{\lambda^4}\left(e^{-\frac{t}{\lambda}}\right)\left(e^{-\frac{t}{\lambda}}-e^{-\frac{b}{\lambda}}\right)^{-1} \quad (137)$$

$$h'''(t)=6h^4(t)-12\lambda h^3(t)+7\lambda^2 h^2(t)-\lambda^3 h(t) \quad (138)$$

The third order ordinary differential for the Hazard function of the truncated exponential distribution is given as;

$$h'''(t)-6h^4(t)+12\lambda h^3(t)-7\lambda^2 h^2(t)+\lambda^3 h(t)=0 \quad (139)$$

$$h''(0)=\frac{e^{-\frac{b}{\lambda}}(1-e^{-\frac{b}{\lambda}})}{\lambda^3(1-e^{-\frac{b}{\lambda}})^3} \quad (140)$$

Reversed Hazard Function

The reversed hazard function (RSF) of the truncated exponential distribution is given by;

$$j(t)=\frac{e^{-\frac{t}{\lambda}}}{\lambda(1-e^{-\frac{t}{\lambda}})} \quad (141)$$

The first order differential equation for the RHF can be obtained from the differentiation of equation (141);

$$j'(t)=\left\{-\frac{\frac{1}{\lambda}e^{-\frac{t}{\lambda}}}{e^{-\frac{t}{\lambda}}}-\frac{\frac{1}{\lambda}e^{-\frac{t}{\lambda}}(1-e^{-\frac{t}{\lambda}})^{-2}}{(1-e^{-\frac{t}{\lambda}})^{-1}}\right\}j(t) \quad (142)$$

The condition necessary for the existence of the equation is $\lambda, b>0, t\geq 0$.

$$j'(t) = -\left(\frac{1}{\lambda} + \frac{e^{-\frac{t}{\lambda}}}{\lambda(1-e^{-\frac{t}{\lambda}})} \right) j(t) \tag{143}$$

$$j'(t) = -\left(\frac{1}{\lambda} + j(t) \right) j(t) \tag{144}$$

The first order ordinary differential for the reverse hazard function of the truncated exponential distribution is given as; $\lambda j'(t) + \lambda j^2(t) + j(t) = 0$

(145) $$j(1) = \frac{e^{-\frac{1}{\lambda}}}{\lambda(1-e^{-\frac{1}{\lambda}})} = \frac{1}{\lambda(e^{\frac{1}{\lambda}}-1)}$$

(146) The ODEs of all the probability functions considered can be obtained for the particular values of the distribution. Several analytic, semi-analytic and numerical methods can be applied to obtain the solutions of the respective differential equations [29-32]. Also comparison with two or more solution methods is useful in understanding the link between ODEs and the probability distributions.

III. Concluding Remarks

In this work, differentiation was used to obtain some classes of ordinary differential equations for the probability density function (PDF), quantile function (QF), survival function (SF), inverse survival function (ISF) and reversed hazard function (RHF) of the exponential and truncated exponential distributions. The nature of exponential distribution prohibits the availability of the ODE of the hazard function. However, that was not the case with the truncated exponential distribution. In all, the parameters that define the respective distributions determine the nature of the respective ODEs and the range determines the existence of the ODEs.

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