Classes of Ordinary Differential Equations Obtained for the Probability Functions of Exponentiated Pareto Distribution

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***Abstract*— In this paper, the differential calculus was used to obtain some classes of ordinary differential equations (ODEs) for the probability density function, quantile function, survival function, inverse survival function, hazard function and reversed hazard function of the exponentiated Pareto distribution. The stated necessary conditions required for the existence of the ODEs are consistent with the various parameters that defined the distribution. Solutions of these ODEs by using numerous available methods are new ways of understanding the nature of the probability functions that characterize the distribution. The method can be extended to other probability distributions and can serve as an alternative to approximation.**

***Index Terms*— Exponentiated Pareto distribution, reversed hazard function, calculus, differentiation, inverse survival function**

I. Introduction

THE distribution was introduced by Gupta et al. [1] for modeling failure time data but was proposed formally as a probability model by [2]. Afify [3] obtained the Bayes estimates for the distribution. Ali and Woo [4] obtained the maximum likelihood estimates of the tail probability of the distribution. Nooghabi and Nooghabi [5] derived the improved moments for the distribution. The distribution is a submodel of the extended Pareto distribution proposed by Mead [6] and the new Weibull –Pareto distribution by Tahir et al. [7].

Other examples of exponentiated distributions obtained by different researchers include**:** exponentiated Weibull distribution [8-10], exponentiated generalized inverted exponential distribution [11], exponentiated generalized inverse Gaussian distribution [12], exponentiated generalized inverse Weibull distribution [13-14], gamma-exponentiated exponential distribution [15], exponentiated Gompertz distribution [16-17], beta exponentiated Mukherjii-Islam Distribution [18], transmuted exponentiated Pareto-i distribution [19], gamma exponentiated exponential–Weibull distribution [20], exponentiated gamma distribution [21], exponentiated Gumbel distribution [22], exponentiated uniform distribution [23] and beta exponentiated Weibull distribution [24-25]. Others are: exponentiated log-logistic distribution [26], McDonald exponentiated gamma distribution [27], exponentiated Generalized Weibull Distribution [28], beta exponentiated gamma distribution [29], exponentiated gamma distribution [30], exponentiated Pareto distribution [31], exponentiated Kumaraswamy distribution [32], exponentiated modified Weibull extension distribution [33], exponentiated Weibull-Pareto distribution [34], exponentiated lognormal distribution [35], exponentiated Perks distribution [36], Kumaraswamy-transmuted exponentiated modified Weibull distribution [37], exponentiated power Lindley–Poisson distribution [38] and exponentiated Chen distribution [39]. Also available in scientific are: exponentiated reduced Kies distribution [40], exponentiated inverse Weibull geometric distribution [41], exponentiated geometric distribution [42-43], exponentiated Weibull geometric distribution [44], exponentiated transmuted Weibull geometric distribution [45], exponentiated half logistic distribution [46], transmuted exponentiated Gumbel distribution [47], exponentiated Kumaraswamy-power function distribution [48], exponentiated-log-logistic geometric distribution [49] and bivariate exponentaited generalized Weibull-Gompertz distribution [50].

The aim of this research is to develop ordinary differential equations (ODE) for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of exponentiated Pareto distribution by the use of differential calculus. Calculus is a very key tool in the determination of mode of a given probability distribution and in estimation of parameters of probability distributions, amongst other uses. The research is an extension of the ODE to other probability functions other than the PDF. Similar works done where the PDF of probability distributions was expressed as ODE whose solution is the PDF are available. They include: Laplace distribution [51], beta distribution [52], raised cosine distribution [53], Lomax distribution [54], beta prime distribution or inverted beta distribution [55].

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II. Probability Density Function

The probability density function of the exponentiated Pareto distribution is given as;

$$f(x)=\alpha\theta(1+x)^{-(\alpha+1)}[1-(1+x)^{-\alpha}]^{\theta-1} \quad (1)$$

To obtain the first order ordinary differential equation for the probability density function of the exponentiated Pareto distribution, differentiate equation (1), to obtain;

$$f'(x)=\left\{\begin{array}{l}-\dfrac{(\alpha+1)(1+x)^{-(\alpha+2)}}{(1+x)^{-(\alpha+1)}}\\+\dfrac{\alpha(\theta-1)(1+x)^{-(\alpha+1)}[1-(1+x)^{-\alpha}]^{\theta-2}}{[1-(1+x)^{-\alpha}]^{\theta-1}}\end{array}\right\}f(x) \quad (2)$$

$$f'(x)=\left\{-\frac{(\alpha+1)}{(1+x)}+\frac{\alpha(\theta-1)(1+x)^{-(\alpha+1)}}{1-(1+x)^{-\alpha}}\right\}f(x) \quad (3)$$

The condition necessary to the existence of equation is $\alpha,\theta,x>0$.

The ordinary differential equations can be obtained for given values of the parameters. Some cases are considered and can be seen in Table I.

TABLE I
CLASSES OF DIFFERENTIAL EQUATIONS OBTAINED FOR THE PROBABILITY DENSITY FUNCTION OF EXPONENTIATED PARETO DISTRIBUTION FOR DIFFERENT PARAMETERS

α	θ	Ordinary differential equations
1	1	$(1+x)f'(x)+2f(x)=0$
1	2	$x^3(1+x)f'(x)+(2x^3-(1+x)^2)f(x)=0$
1	3	$x^3(1+x)f'(x)+2(x^3-(1+x)^2)f(x)=0$

To obtain an ordinary differential equation that is independent of the powers of the parameters, differentiate equation (3);

$$f''(x)=\left\{\begin{array}{l}\dfrac{\alpha+1}{(1+x)^2}-\dfrac{(\theta-1)\alpha^2((1+x)^{-(\alpha+1)})^2}{(1-(1+x)^{-\alpha})^2}\\-\dfrac{\alpha(\theta-1)(\alpha+1)(1+x)^{-(\alpha+2)}}{1-(1+x)^{-\alpha}}\end{array}\right\}f(x)+\left\{-\frac{(\alpha+1)}{(1+x)}+\frac{\alpha(\theta-1)(1+x)^{-(\alpha+1)}}{1-(1+x)^{-\alpha}}\right\}f'(x) \quad (4)$$

The following equations obtained from equation (3) are required in the simplification of equation (4). From equation (3);

$$-\frac{(\alpha+1)}{(1+x)}+\frac{\alpha(\theta-1)(1+x)^{-(\alpha+1)}}{1-(1+x)^{-\alpha}}=\frac{f'(x)}{f(x)} \quad (5)$$

$$\frac{\alpha(\theta-1)(1+x)^{-(\alpha+1)}}{1-(1+x)^{-\alpha}}=\frac{f'(x)}{f(x)}+\frac{\alpha+1}{1+x} \quad (6)$$

$$\left(\frac{\alpha(\theta-1)(1+x)^{-(\alpha+1)}}{1-(1+x)^{-\alpha}}\right)^2=\left(\frac{f'(x)}{f(x)}+\frac{\alpha+1}{1+x}\right)^2 \quad (7)$$

$$\frac{(\theta-1)\alpha^2((1+x)^{-(\alpha+1)})^2}{(1-(1+x)^{-\alpha})^2}=\frac{1}{\theta-1}\left(\frac{f'(x)}{f(x)}+\frac{\alpha+1}{1+x}\right)^2 \quad (8)$$

$$\frac{\alpha(\theta-1)(\alpha+1)(1+x)^{-(\alpha+1)}}{1-(1+x)^{-\alpha}}=\alpha+1\left(\frac{f'(x)}{f(x)}+\frac{\alpha+1}{1+x}\right) \quad (9)$$

$$\frac{\alpha(\theta-1)(\alpha+1)(1+x)^{-(\alpha+2)}}{1-(1+x)^{-\alpha}}=\frac{\alpha+1}{x+1}\left(\frac{f'(x)}{f(x)}+\frac{\alpha+1}{1+x}\right) \quad (10)$$

Substituting equations (5), (8) and (10) into equation (4) gives

$$f''(x)=\frac{f'^2(x)}{f(x)}+\left\{\begin{array}{l}\dfrac{\alpha+1}{(1+x)^2}-\dfrac{1}{\theta-1}\left(\dfrac{f'(x)}{f(x)}+\dfrac{\alpha+1}{1+x}\right)^2\\-\dfrac{\alpha+1}{x+1}\left(\dfrac{f'(x)}{f(x)}+\dfrac{\alpha+1}{1+x}\right)\end{array}\right\}f(x) \quad (11)$$

The condition necessary to the existence of equation is $\alpha,x>0,\theta>1.$

The ordinary differential equations can be obtained for the particular values of the parameters.

III. Quantile Function

The Quantile function of the exponentiated Pareto distribution is given as;

$$Q(p)=\frac{1}{(1-p^{\frac{1}{\theta}})^{\frac{1}{\alpha}}}-1 \quad (12)$$

To obtain the first order ordinary differential equation for the quantile function of the exponentiated Pareto distribution, differentiate equation (12), to obtain;

$$Q'(p)=\frac{1}{\alpha\theta}p^{\frac{1}{\theta}-1}(1-p^{\frac{1}{\theta}})^{-(\frac{1}{\alpha}+1)} \quad (13)$$

The condition necessary to the existence of equation is $\alpha,\theta>0,0<p<1.$

Equation (12) can be written as;

$$\frac{1}{(1-p^{\frac{1}{\theta}})^{\frac{1}{\alpha}}}=Q(p)+1 \quad (14)$$

Substituting equation (14) into equation (13) yields

$$Q'(p)=\frac{p^{\frac{1}{\theta}}(Q(p)+1)}{\alpha\theta p(1-p^{\frac{1}{\theta}})} \quad (15)$$

$$\alpha\theta p(1-p^{\frac{1}{\theta}})Q'(p)-p^{\frac{1}{\theta}}(Q(p)+1)=0 \quad (16)$$

The ordinary differential equations can be obtained for given values of the parameters. Some cases are considered and can be seen in Table II.

TABLE II
CLASSES OF DIFFERENTIAL EQUATIONS OBTAINED FOR THE QUANTILE FUNCTION OF EXPONENTIATED PARETO DISTRIBUTION FOR DIFFERENT PARAMETERS

α	θ	ordinary differential equations
1	1	$(1-p)Q'(p)-Q(p)-1=0$
1	2	$2(1-p)Q'(p)-Q(p)-1=0$
1	3	$3(1-p)Q'(p)-Q(p)-1=0$
2	1	$2p(1-\sqrt{p})Q'(p)-\sqrt{p}Q(p)-\sqrt{p}=0$
2	2	$4p(1-\sqrt{p})Q'(p)-\sqrt{p}Q(p)-\sqrt{p}=0$

IV. SURVIVAL FUNCTION

The survival function of the exponentiated Pareto distribution is given as;

$$S(t)=1-[1-(1+t)^{-\alpha}]^{\theta} \tag{17}$$

To obtain the first order ordinary differential equation for the survival function of the exponentiated Pareto distribution, differentiate equation (17), to obtain;

$$S'(t)=-\alpha\theta(1+t)^{-(\alpha+1)}[1-(1+t)^{-\alpha}]^{\theta-1} \tag{18}$$

$$S'(t)=-\frac{\alpha\theta(1+t)^{-\alpha}[1-(1+t)^{-\alpha}]^{\theta}}{(1+t)[1-(1+t)^{-\alpha}]} \tag{19}$$

The condition necessary to the existence of equation is $\alpha,\theta,t>0$.

Equation (17) can be written as;

$$[1-(1+t)^{-\alpha}]^{\theta}=1-S(t) \tag{20}$$

Substituting equation (20) into equation (19) one has

$$S'(t)=-\frac{\alpha\theta(1+t)^{-\alpha}(1-S(t))}{(1+t)[1-(1+t)^{-\alpha}]} \tag{21}$$

$$(1+t)(1-(1+t)^{-\alpha})S'(t)+\alpha\theta(1+t)^{-\alpha}(1-S(t))=0 \tag{22}$$

The ordinary differential equations can be obtained for given values of the parameters. Some cases are considered and are shown in Table III.

TABLE III
CLASSES OF DIFFERENTIAL EQUATIONS OBTAINED FOR THE SURVIVAL FUNCTION OF EXPONENTIATED PARETO DISTRIBUTION FOR DIFFERENT PARAMETERS .

α	θ	ordinary differential equations
1	1	$t(1+t)S'(t)-S(t)+1=0$
1	2	$t(1+t)S'(t)-2S(t)+2=0$
1	3	$t(1+t)S'(t)-3S(t)+3=0$
2	1	$t(1+t)(2+t)S'(t)-2S(t)+2=0$
2	2	$t(1+t)(2+t)S'(t)-4S(t)+4=0$
2	3	$t(1+t)(2+t)S'(t)-6S(t)+6=0$

Using equation (3) and (18), it can be seen that;

$$S'(t)=-f(t)\Rightarrow S''(t)=-f'(t) \tag{23}$$

$$S''(t)=\left\{-\frac{(\alpha+1)}{(1+t)}+\frac{\alpha(\theta-1)(1+t)^{-(\alpha+1)}}{1-(1+t)^{-\alpha}}\right\}S'(t) \tag{24}$$

The condition necessary to the existence of equation is $\alpha,\theta,t>0$.

The ordinary differential equations can be obtained for given values of the parameters. Some cases are considered and shown in Table IV.

TABLE IV:
CLASSES OF SECOND ORDER DIFFERENTIAL EQUATIONS OBTAINED FOR THE PROBABILITY SURVIVAL FUNCTION OF EXPONENTIATED PARETO DISTRIBUTION FOR DIFFERENT PARAMETERS

α	θ	ordinary differential equations
1	1	$(1+t)S''(t)+2S'(t)=0$
1	2	$t^3(1+t)S''(t)+(2t^3-(1+t)^2)S'(t)=0$
1	3	$t^3(1+t)S''(t)+2(t^3-(1+t)^2)S'(t)=0$

V. INVERSE SURVIVAL FUNCTION

The inverse survival function of the exponentiated Pareto distribution is given as;

$$Q(p)=\frac{1}{(1-(1-p)^{\frac{1}{\theta}})^{\frac{1}{\alpha}}}-1 \tag{25}$$

To obtain the first order ordinary differential equation for the inverse survival function of the exponentiated Pareto distribution, differentiate equation (25), to obtain;

$$Q'(p)=-\frac{(1-p)^{\frac{1}{\theta}-1}(1-(1-p)^{\frac{1}{\theta}})^{-(\frac{1}{\alpha}+1)}}{\alpha\theta} \tag{26}$$

The condition necessary to the existence of equation is $\alpha,\theta>0, 0<p<1$.

$$Q'(p)=-\frac{(1-p)^{\frac{1}{\theta}}(1-(1-p)^{\frac{1}{\theta}})^{-\frac{1}{\alpha}}}{\alpha\theta(1-p)(1-(1-p)^{\frac{1}{\theta}})} \tag{27}$$

Equation (25) can also be written as;

$$\frac{1}{(1-(1-p)^{\frac{1}{\theta}})^{\frac{1}{\alpha}}}=Q(p)+1 \tag{28}$$

Substituting equation (28) into equation (27), one obtains

$$Q'(p)=-\frac{(1-p)^{\frac{1}{\theta}}(Q(p)+1)}{\alpha\theta(1-p)(1-(1-p)^{\frac{1}{\theta}})} \tag{29}$$

$$\alpha\theta(1-p)(1-(1-p)^{\frac{1}{\theta}})Q'(p)+(1-p)^{\frac{1}{\theta}}(Q(p)+1)=0 \tag{30}$$

The ordinary differential equations can be obtained for given values of the parameters. Some cases are considered and shown in Table V.

TABLE V
CLASSES OF DIFFERENTIAL EQUATIONS OBTAINED FOR THE INVERSE SURVIVAL FUNCTION OF EXPONENTIATED PARETO DISTRIBUTION FOR DIFFERENT PARAMETERS

θ	α	Ordinary differential equations
1	1	$pQ'(p)+Q(p)+1=0$
1	2	$2pQ'(p)+Q(p)+1=0$
1	3	$3pQ'(p)+Q(p)+1=0$
2	1	$2(1-p)\left(1-\sqrt{1-p}\right)Q'(p)+\sqrt{1-p}\left(Q'(p)+1\right)=0$
2	2	$4(1-p)\left(1-\sqrt{1-p}\right)Q'(p)+\sqrt{1-p}\left(Q'(p)+1\right)=0$
2	3	$6(1-p)\left(1-\sqrt{1-p}\right)Q'(p)+\sqrt{1-p}\left(Q'(p)+1\right)=0$

VI. HAZARD FUNCTION

The hazard function of the exponentiated Pareto distribution is:

$$h(t)=\frac{\alpha\theta(1+t)^{-(\alpha+1)}[1-(1+t)^{-\alpha}]^{\theta-1}}{1-[1-(1+t)^{-\alpha}]^{\theta}} \quad (31)$$

To obtain the first order ordinary differential equation for the hazard function of the exponentiated Pareto distribution, differentiate equation (31), to obtain;

$$h'(t)=-\frac{(\alpha+1)(1+t)^{-(\alpha+2)}}{(1+t)^{-(\alpha+1)}}h(t)+\frac{\alpha(\theta-1)(1+t)^{-(\alpha+1)}[1-(1+t)^{-\alpha}]^{\theta-2}}{[1-(1+t)^{-\alpha}]^{\theta-1}}h(t)+\frac{\alpha\theta(1+t)^{-(\alpha+1)}[1-(1+t)^{-\alpha}]^{\theta-1}(1-[1-(1+t)^{-\alpha}]^{\theta})^{-2}h(t)}{(1-[1-(1+t)^{-\alpha}]^{\theta})^{-1}} \quad (32)$$

The condition necessary to the existence of equation is $\alpha,\theta,t>0$.

$$h'(t)=\left\{\begin{matrix}-\dfrac{(\alpha+1)}{1+t}+\dfrac{\alpha(\theta-1)(1+t)^{-(\alpha+1)}}{1-(1+t)^{-\alpha}}\\+\dfrac{\alpha\theta(1+t)^{-(\alpha+1)}[1-(1+t)^{-\alpha}]^{\theta-1}}{(1-[1-(1+t)^{-\alpha}]^{\theta})}\end{matrix}\right\}h(t) \quad (33)$$

$$h'(t)=\left\{-\frac{(\alpha+1)}{1+t}+\frac{\alpha(\theta-1)(1+t)^{-(\alpha+1)}}{1-(1+t)^{-\alpha}}+h(t)\right\}h(t) \quad (34)$$

The ordinary differential equations can be obtained for given values of the parameters. To obtain an ordinary differential equation that is independent of the powers of the parameters, differentiate equation (34);

$$h''(t)=\left\{\begin{matrix}\dfrac{\alpha+1}{(1+t)^2}-\dfrac{(\theta-1)\alpha^2((1+t)^{-(\alpha+1)})^2}{(1-(1+t)^{-\alpha})^2}\\-\dfrac{\alpha(\theta-1)(\alpha+1)(1+t)^{-(\alpha+2)}}{1-(1+t)^{-\alpha}}+h'(t)\end{matrix}\right\}h(t)+\left\{-\frac{(\alpha+1)}{(1+t)}+\frac{\alpha(\theta-1)(1+t)^{-(\alpha+1)}}{1-(1+t)^{-\alpha}}+h(t)\right\}h'(t) \quad (35)$$

The condition necessary to the existence of equation is $\alpha,\theta,t>0$.

The following equations obtained from equation (34) are required in the simplification of equation (35). From equation (34), we have

$$-\frac{(\alpha+1)}{(1+t)}+\frac{\alpha(\theta-1)(1+t)^{-(\alpha+1)}}{1-(1+t)^{-\alpha}}+h(t)=\frac{h'(t)}{h(t)} \quad (36)$$

$$\frac{\alpha(\theta-1)(1+t)^{-(\alpha+1)}}{1-(1+t)^{-\alpha}}=\frac{h'(t)}{h(t)}+\frac{\alpha+1}{1+t}-h(t) \quad (37)$$

$$\left(\frac{\alpha(\theta-1)(1+t)^{-(\alpha+1)}}{1-(1+t)^{-\alpha}}\right)^2=\left(\frac{h'(t)}{h(t)}+\frac{\alpha+1}{1+t}-h(t)\right)^2 \quad (38)$$

$$\frac{(\theta-1)\alpha^2((1+t)^{-(\alpha+1)})^2}{(1-(1+t)^{-\alpha})^2}=\frac{1}{\theta-1}\left(\frac{h'(t)}{h(t)}+\frac{\alpha+1}{1+t}-h(t)\right)^2 \quad (39)$$

$$\frac{\alpha(\theta-1)(\alpha+1)(1+t)^{-(\alpha+1)}}{1-(1+t)^{-\alpha}}=\alpha+1\left(\frac{h'(t)}{h(t)}+\frac{\alpha+1}{1+t}-h(t)\right) \quad (40)$$

$$\frac{\alpha(\theta-1)(\alpha+1)(1+t)^{-(\alpha+2)}}{1-(1+t)^{-\alpha}}=\frac{\alpha+1}{t+1}\left(\frac{h'(t)}{h(t)}+\frac{\alpha+1}{1+t}-h(t)\right) \quad (41)$$

Substitute equations (36), (39) and (41) into equation (35) to get

$$h''(t)=\frac{h'^2(t)}{h(t)}+\left\{\frac{\alpha+1}{(1+t)^2}-\frac{1}{\theta-1}\left(\frac{h'(t)}{h(t)}+\frac{\alpha+1}{1+t}-h(t)\right)^2\right\}h(t)\left\{-\frac{\alpha+1}{t+1}\left(\frac{h'(t)}{h(t)}+\frac{\alpha+1}{1+t}-h(t)\right)+h'(t)\right\}h(t) \quad (42)$$

The condition necessary to the existence of equation (42) is $\alpha,x>0,\theta>1$.

The ordinary differential equations can be obtained for the particular values of the parameters.

VII. REVERSED HAZARD FUNCTION

The reversed hazard function of the exponentiated Pareto distribution is given as;

$$j(t)=\frac{\alpha\theta(1+t)^{-(\alpha+1)}}{1-(1+t)^{-\alpha}} \quad (43)$$

To obtain the first order ordinary differential equation for the reversed hazard function of the exponentiated Pareto distribution, differentiate equation (43), to obtain;

$$j'(t)=\begin{Bmatrix} -\dfrac{(\alpha+1)(1+t)^{-(\alpha+2)}}{(1+t)^{-(\alpha+1)}} \\ -\dfrac{\alpha(1+t)^{-(\alpha+1)}(1-(1+t)^{-\alpha})^{-2}}{(1-(1+t)^{-\alpha})^{-1}} \end{Bmatrix} j(t) \quad (44)$$

The condition necessary to the existence of equation is $\alpha,\theta,t>0$.

$$j'(t)=-\left\{\frac{\alpha+1}{1+t}+\frac{\alpha(1+t)^{-(\alpha+1)}}{1-(1+t)^{-\alpha}}\right\}j(t) \quad (45)$$

$$j'(t)=-\left\{\frac{\alpha+1}{1+t}+\frac{j(t)}{\theta}\right\}j(t) \quad (46)$$

The first order ordinary differential equations for the reversed hazard function of the exponentiated Pareto distribution is given by;

$$\theta(t+1)j'(t)+\theta(\alpha+1)j(t)+(t+1)j^2(t)=0 \quad (47)$$

$$j(1)=\frac{\alpha\theta 2^{-(\alpha+1)}}{1-2^{-\alpha}}=\frac{\alpha\theta 2^{-\alpha}}{2(1-2^{-\alpha})}=\frac{\alpha\theta}{2(2^{\alpha}-1)} \quad (48)$$

The ODEs of all the probability functions considered can be obtained for the particular values of the distribution. Several analytic, semi-analytic and numerical methods can be applied to obtain the solutions of the respective differential equations [56-69]. Also, comparison with two or more solution methods is useful in understanding the link between ODEs and the probability distributions.

VIII. Concluding Remarks

In this work, differentiation was used to obtain some classes of ordinary differential equations for the probability density function (PDF), quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of the exponentiated Pareto distribution. In all, the parameters that define the distribution determine the nature of the respective ODEs and the range determines the existence of the ODEs. Furthermore, the complexity of the ODEs depends largely on the values of the parameters.

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