Laplace Distribution: Ordinary Differential Equations

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***Abstract*— Differential calculus was used in this paper to obtain the ordinary differential equations (ODE) of the probability functions of the Laplace distribution. The parameters and support that characterized Laplace distribution inevitably determine the behavior, existence, uniqueness and solution of the ODEs. The method is recommended to be applied to other probability distributions and probability functions not considered in this paper. Computer codes and programs can be used for the implementation.**

***Index Terms*— Differential calculus, quantile function, hazard function, reversed hazard function, survival function, inverse survival function, probability density function, Laplace.**

I. Introduction

CALCULUS in general and differential calculus in particular is often used in statistics in parameter and modal estimations. The method of maximum likelihood is an example.

Differential equations often arise from the understanding and modeling of real life problems or some observed physical phenomena. Approximations of probability functions are one of the major areas of application of calculus and ordinary differential equations in mathematical statistics. The approximations are helpful in the recovery of the probability functions of complex distributions [1-10].

Apart from mode estimation, parameter estimation and approximation, probability density function (PDF) of distributions can be transformed as ODE whose solution yields the respective PDF. Some of which are available: see [11-15].

The aim of this paper is to obtain homogenous ODE for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of the Laplace distribution. This will also help to provide the answers as to whether there are discrepancies between the support of the distribution and the conditions necessary for the existence of the various ODE. Similar results for other distributions have been proposed, see [16-29] for details.

Laplace distribution is sometimes refers to as the double exponential distribution because of its identical nature to the exponential distribution. Earlier studied by [30] and extended to the multivariate Laplace distribution by [31] and multivariate generalized Laplace distribution by [32]. Some of the other modifications and extensions of the distribution include: three-parameter asymmetric Laplace distribution [33], skew Laplace distribution [34], beta Laplace distribution [35], truncated skew-Laplace distribution [36], alpha–Skew–Laplace distribution [37]. Others are: normal-Laplace distribution [38] and Semi-α-Laplace distributions [39].
Some of the areas of the distribution have been explored such as: goodness of fit tests and other statistical tests [40-43]; estimation of parameters [44-45].
An important variant of the distribution is the log-Laplace distribution whose logarithm is the Laplace distribution. Notable among the applications are the works of [46] that applied the distribution to the analysis of financial data while [47] applied it to regression analysis.

Differential calculus was used to obtain the results.

II. Probability Density Function

The PDF of the Laplace distribution is given by;

$$f(x)=\frac{1}{2b}\exp\left(-\frac{|x-\mu|}{b}\right) \quad (1)$$

$$f(x)=\frac{1}{2b}\begin{cases}\exp\left(-\dfrac{\mu-x}{b}\right) & \text{if } x<\mu \\ \exp\left(-\dfrac{x-\mu}{b}\right) & \text{if } x\geq\mu\end{cases} \quad (2)$$

Differentiate equation (2), to obtain the first order ODE;
Case I, $x<\mu$:

$$f(x)=\frac{1}{2b}\exp\left(-\frac{\mu-x}{b}\right) \quad (3)$$

$$f'(x)=\frac{1}{2b^2}\exp\left(-\frac{\mu-x}{b}\right) \quad (4)$$

The equation can only exists for $b>0$.
The first order ODE for the PDF of the Laplace distribution for case I is given as;

$$bf'(x)-f(x)=0 \quad (5)$$

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$$f(0)=\frac{1}{2b}\exp\left(-\frac{\mu}{b}\right) \quad , \tag{6}$$

Case II, $x \geq \mu$:

$$f(x)=\frac{1}{2b}\exp\left(-\frac{x-\mu}{b}\right) \tag{7}$$

$$f'(x)=-\frac{1}{2b^2}\exp\left(-\frac{x-\mu}{b}\right) \tag{8}$$

The equation can only exists for $b>0$.

The first order ODE for the PDF of the Laplace distribution for case II is given as;

$$bf'(x)+f(x)=0 \tag{9}$$

$$f(0)=\frac{1}{2b}\exp\left(\frac{\mu}{b}\right) \quad , \tag{10}$$

Differentiate equations (4) and (8), to obtain the second order ODE;

Case I, $x<\mu$:

$$f''(x)=\frac{1}{2b^3}\exp\left(-\frac{\mu-x}{b}\right) \tag{11}$$

The equation can only exists for $b>0$.

Two ODEs can be acquired from the further breaking down of equation (11). These are listed as follows; ODE 1;

$$f''(x)=\frac{1}{2b^3}\exp\left(-\frac{\mu-x}{b}\right)= \left(\frac{1}{b^2}\right)\left(\frac{1}{2b}\right)\exp\left(-\frac{\mu-x}{b}\right) \tag{12}$$

$$f''(x)=\frac{1}{b^2}f(x) \tag{13}$$

$$b^2 f''(x)-f(x)=0 \tag{14}$$

ODE 2;

$$f''(x)=\frac{1}{2b^3}\exp\left(-\frac{\mu-x}{b}\right) =\left(\frac{1}{b}\right)\left(\frac{1}{2b^2}\right)\exp\left(-\frac{\mu-x}{b}\right) \tag{15}$$

$$f''(x)=\frac{1}{b}f'(x) \tag{16}$$

$$bf''(x)-f'(x)=0 \tag{17}$$

$$f'(0)=\frac{1}{2b^2}\exp\left(-\frac{\mu}{b}\right) \tag{18}$$

Case II, $x \geq \mu$:

$$f''(x)=\frac{1}{2b^3}\exp\left(-\frac{x-\mu}{b}\right) \tag{19}$$

The equation can only exists for $b>0$.

Two ODEs can be acquired from the further breaking down of equation (19). These are listed as follows;

ODE 1;

$$f''(x)=\frac{1}{2b^3}\exp\left(-\frac{x-\mu}{b}\right) =\left(\frac{1}{b^2}\right)\left(\frac{1}{2b}\right)\exp\left(-\frac{x-\mu}{b}\right) \tag{20}$$

$$f''(x)=\frac{1}{b^2}f(x) \tag{21}$$

$$b^2 f''(x)-f(x)=0 \tag{22}$$

ODE 2;

$$f''(x)=\frac{1}{2b^3}\exp\left(-\frac{x-\mu}{b}\right) =\left(-\frac{1}{b}\right)\left(-\frac{1}{2b^2}\right)\exp\left(-\frac{x-\mu}{b}\right) \tag{23}$$

$$f''(x)=-\frac{1}{b}f'(x) \tag{24}$$

$$bf''(x)+f'(x)=0 \tag{25}$$

$$f'(0)=-\frac{1}{2b^2}\exp\left(\frac{\mu}{b}\right) \tag{26}$$

Differentiate equations (11) and (19) to obtain the third order ODE;

Case I, $x<\mu$:

$$f'''(x)=\frac{1}{2b^4}\exp\left(-\frac{\mu-x}{b}\right) \tag{27}$$

The equation can only exists for $b>0$.

Three ODEs can be acquired from the further breaking down of equation (27). These are listed as follows;

ODE 1;

$$f'''(x)=\frac{1}{2b^4}\exp\left(-\frac{\mu-x}{b}\right) =\left(\frac{1}{b^3}\right)\left(\frac{1}{2b}\right)\exp\left(-\frac{\mu-x}{b}\right) \tag{28}$$

$$f'''(x)=\frac{1}{b^3}f(x) \tag{29}$$

$$b^3 f'''(x)-f(x)=0 \tag{30}$$

ODE 2;

$$f'''(x)=\frac{1}{2b^4}\exp\left(-\frac{\mu-x}{b}\right) =\left(\frac{1}{b^2}\right)\left(\frac{1}{2b^2}\right)\exp\left(-\frac{\mu-x}{b}\right) \tag{31}$$

$$f'''(x)=\frac{1}{b^2}f'(x) \tag{32}$$

$$b^2 f'''(x)-f'(x)=0 \tag{33}$$

ODE 3;

$$f'''(x) = \frac{1}{2b^4}\exp\left(-\frac{\mu - x}{b}\right) = \left(\frac{1}{b}\right)\left(\frac{1}{2b^3}\right)\exp\left(-\frac{\mu - x}{b}\right) \quad (34)$$

$$f'''(x) = \frac{1}{b} f''(x) \quad (35)$$

$$bf'''(x) - f''(x) = 0 \quad (36)$$

$$f''(0) = \frac{1}{2b^3}\exp\left(-\frac{\mu}{b}\right) \quad (37)$$

Case II, $x \geq \mu$:

$$f'''(x) = -\frac{1}{2b^4}\exp\left(-\frac{x-\mu}{b}\right) \quad (38)$$

The equation can only exists for $b > 0$.

Three ODEs can be acquired from the breaking down of equation (39). These are listed as follows; ODE 1;

$$f'''(x) = -\frac{1}{2b^4}\exp\left(-\frac{x-\mu}{b}\right) = \left(-\frac{1}{b^3}\right)\left(\frac{1}{2b}\right)\exp\left(-\frac{x-\mu}{b}\right) \quad (39)$$

$$f'''(x) = -\frac{1}{b^3} f(x) \quad (40)$$

$$b^3 f'''(x) + f(x) = 0 \quad (41)$$

ODE 2;

$$f'''(x) = -\frac{1}{2b^4}\exp\left(-\frac{x-\mu}{b}\right) = \left(\frac{1}{b^2}\right)\left(-\frac{1}{2b^2}\right)\exp\left(-\frac{x-\mu}{b}\right) \quad (42)$$

$$f'''(x) = \frac{1}{b^2} f'(x) \quad (43)$$

$$b^2 f'''(x) - f'(x) = 0 \quad (44)$$

ODE 3;

$$f'''(x) = -\frac{1}{2b^4}\exp\left(-\frac{x-\mu}{b}\right) = \left(-\frac{1}{b}\right)\left(-\frac{1}{2b^3}\right)\exp\left(-\frac{x-\mu}{b}\right) \quad (45)$$

$$f'''(x) = -\frac{1}{b} f''(x) \quad (46)$$

$$bf'''(x) + f''(x) = 0 \quad (47)$$

$$f''(0) = \frac{1}{2b^3}\exp\left(\frac{\mu}{b}\right) \quad (48)$$

The result from first order to fifth order ODE are listed in Table (1);

Table 1: First to fifth order ODE for the PDF

Case I, $x < \mu$:	Case II, $x \geq \mu$:
$bf'(x) - f(x) = 0$	$bf'(x) + f(x) = 0$
$b^2 f''(x) - f(x) = 0$ $b^2 f''(x) - f'(x) = 0$	$b^2 f''(x) - f(x) = 0$ $b^2 f''(x) + f'(x) = 0$
$b^3 f'''(x) - f(x) = 0$ $b^2 f''(x) - f'(x) = 0$ $bf'''(x) - f''(x) = 0$	$b^3 f'''(x) + f(x) = 0$ $b^2 f''(x) - f'(x) = 0$ $bf'''(x) + f''(x) = 0$
$b^4 f^{iv}(x) - f(x) = 0$ $b^3 f^{iv}(x) - f'(x) = 0$ $b^2 f^{iv}(x) - f''(x) = 0$ $bf^{iv}(x) - f'''(x) = 0$	$b^4 f^{iv}(x) - f(x) = 0$ $b^3 f^{iv}(x) + f'(x) = 0$ $b^2 f^{iv}(x) - f''(x) = 0$ $bf^{iv}(x) + f'''(x) = 0$
$b^5 f^{v}(x) - f(x) = 0$ $b^4 f^{v}(x) - f'(x) = 0$ $b^3 f^{v}(x) - f''(x) = 0$ $b^2 f^{v}(x) - f'''(x) = 0$ $bf^{v}(x) - f^{iv}(x) = 0$	$b^5 f^{v}(x) + f(x) = 0$ $b^4 f^{v}(x) - f'(x) = 0$ $b^3 f^{v}(x) + f''(x) = 0$ $b^2 f^{v}(x) - f'''(x) = 0$ $bf^{v}(x) + f^{iv}(x) = 0$

The results are similar to the ones proposed in [16-29].

III. Quantile Function

The QF of the Laplace distribution is given by;

$$Q(p) = \begin{cases} \mu + \text{b ln(2p)} & \text{derived if } x < \mu \\ \mu - \text{b ln(2(1-p))} & \text{derived if } x \geq \mu \end{cases} \quad (49)$$

Differentiate equation (49), to obtain the first order ODE;

Case I, $x < \mu$:

$$Q(p) = \mu + b\ln(2p) \quad (50)$$

$$Q'(p) = \frac{b}{p} \quad (51)$$

The equation can only exists for is $b > 0, 0 < p < 1$.

The first order ODE for the QF of the Laplace distribution for case I is given as;

$$pQ'(p) - b = 0 \quad (52)$$

$$Q(0.1) = \mu - 1.60944b \quad (53)$$

Case II, $x \geq \mu$:

$$Q(p) = \mu - b\ln(2(1-p)) \quad (54)$$

$$Q'(p) = \frac{b}{1-p} \quad (55)$$

The equation can only exists for is $b > 0, 0 < p < 1$.

The first order ODE for the QF of the Laplace distribution for case II is given as;

$$(1-p)Q'(p) - b = 0 \quad (56)$$

$$Q(0.1) = \mu - 0.5878b \quad (57)$$

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Differentiate equations (51) and (55), to obtain the second order ODEs;
Case I, $x < \mu$:

$$Q''(p) = -\frac{b}{p^2} \quad (58)$$

$$Q''(p) = -\frac{1}{p}Q'(p) \quad (59)$$

The equation can only exists for $b > 0, 0 < p < 1$.

The second order ODE for the QF of the Laplace distribution for case I is given as;

$$pQ''(p) + Q'(p) = 0 \quad (60)$$

$$Q'(0.1) = 10b \quad (61)$$

Case II, $x \geq \mu$:

$$Q''(p) = \frac{b}{(1-p)^2} \quad (62)$$

$$Q''(p) = \frac{1}{1-p}Q'(p) \quad (63)$$

The equation can only exists for $b > 0, 0 < p < 1$.

The second order ODE for the QF of the Laplace distribution for case II is given as;

$$(1-p)Q''(p) - Q'(p) = 0 \quad (64)$$

$$Q'(0.1) = \frac{10b}{9} \quad (65)$$

Differentiate equations (58) and (62), to obtain the third order ODEs;
Case I, $x < \mu$:

$$Q'''(p) = \frac{2b}{p^3} \quad (66)$$

The equation can only exists for $b > 0, 0 < p < 1$.

Two ODEs can be acquired from the further breaking down of equation (66). These are listed as follows;

ODE 1;

$$Q'''(p) = \frac{2b}{p^3} = \left(\frac{b}{p}\right)\left(\frac{2}{p^2}\right) \quad (67)$$

$$Q'''(p) = \frac{2}{p^2}Q'(p) \quad (68)$$

$$p^2 Q'''(p) - 2Q'(p) = 0 \quad (69)$$

ODE 2;

$$Q'''(p) = \frac{2b}{p^3} = \left(-\frac{b}{p^2}\right)\left(-\frac{2}{p}\right) \quad (70)$$

$$Q'''(p) = -\frac{2}{p}Q''(p) \quad (71)$$

$$pQ'''(p) + 2Q''(p) = 0 \quad (72)$$

$$Q''(0.1) = -100b \quad (73)$$

Case II, $x \geq \mu$:

$$Q'''(p) = \frac{2b}{(1-p)^3} \quad (74)$$

The equation can only exists for $b > 0, 0 < p < 1$.

Two ODEs can be acquired from the further breaking down of equation (74). These are listed as follows;

ODE 1;

$$Q'''(p) = \frac{2b}{(1-p)^3} = \left(\frac{b}{1-p}\right)\left(\frac{2}{(1-p)^2}\right) \quad (75)$$

$$Q'''(p) = \frac{2}{(1-p)^2}Q'(p) \quad (76)$$

$$(1-p)^2 Q'''(p) - 2Q'(p) = 0 \quad (77)$$

ODE 2;

$$Q'''(p) = \frac{2b}{(1-p)^3} = \left(\frac{b}{(1-p)^2}\right)\left(\frac{2}{1-p}\right) \quad (78)$$

$$Q'''(p) = \frac{2}{(1-p)}Q''(p) \quad (79)$$

$$(1-p)Q'''(p) - 2Q''(p) = 0 \quad (80)$$

$$Q''(0.1) = \frac{100b}{81} \quad (81)$$

The result from first order to fifth order ODEs re listed in Table (2);

Table 2: First to fifth order ODE for the QF

Case I, $x < \mu$:
$pQ''(p) + Q'(p) = 0$
$p^2 Q'''(p) - 2Q'(p) = 0$ $pQ'''(p) + 2Q''(p) = 0$
$p^3 Q^{iv}(p) + 6Q'(p) = 0$ $p^2 Q^{iv}(p) - 6Q''(p) = 0$ $pQ^{iv}(p) + 3Q'''(p) = 0$
$p^4 Q^{v}(p) - 24Q'(p) = 0$ $p^3 Q^{v}(p) + 24Q''(p) = 0$ $p^2 Q^{v}(p) - 12Q'''(p) = 0$ $pQ^{v}(p) + 4Q^{iv}(p) = 0$
Case II, $x \geq \mu$:
$(1-p)Q''(p) + Q'(p) = 0$
$(1-p)^2 Q'''(p) - 2Q'(p) = 0$ $(1-p)Q'''(p) - 2Q''(p) = 0$
$(1-p)^3 Q^{iv}(p) - 6Q'(p) = 0$ $(1-p)^2 Q^{iv}(p) - 6Q''(p) = 0$ $(1-p)Q^{iv}(p) - 3Q'''(p) = 0$

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$$\begin{array}{|l|}\hline (1-p)^4 Q^{v}(p) - 24Q'(p) = 0 \\ (1-p)^3 Q^{v}(p) - 24Q''(p) = 0 \\ (1-p)^2 Q^{v}(p) - 12Q'''(p) = 0 \\ (1-p) Q^{v}(p) - 4Q^{iv}(p) = 0 \\ \hline \end{array}$$

The results are similar to the ones proposed in [16-29].

IV. Survival Function

The SF of the Laplace distribution is given by;

$$S(t) = \begin{cases} 1 - \frac{1}{2}\exp\left(\frac{t-\mu}{b}\right) & \text{if } t < \mu \\ \frac{1}{2}\exp\left(-\frac{t-\mu}{b}\right) & \text{if } t \ge \mu \end{cases} \quad (82)$$

Differentiate equation (82), to obtain the first order ODE;
Case I, $t < \mu$:

$$S(t) = 1 - \frac{1}{2}\exp\left(\frac{t-\mu}{b}\right) \quad (83)$$

$$S'(t) = -\frac{1}{2b}\exp\left(\frac{t-\mu}{b}\right) \quad (84)$$

The equation can only exists for $b > 0$.
Equation (83) can also be written as;

$$\frac{1}{2}\exp\left(\frac{t-\mu}{b}\right) = 1 - S(t) \quad (85)$$

Simplify equation (84) using equation (85) to obtain;

$$S'(t) = -\frac{1}{b}(1 - S(t))$$

(86) The first order ODE for the SF of the Laplace distribution for case I is given as;

$$bS'(t) - S(t) + 1 = 0 \quad (87)$$

$$S(0) = 1 - \frac{1}{2}\exp\left(\frac{-\mu}{b}\right) \quad (88)$$

Case II, $t \ge \mu$:

$$S(t) = \frac{1}{2}\exp\left(-\frac{t-\mu}{b}\right) \quad (89)$$

$$S'(t) = -\frac{1}{2b}\exp\left(-\frac{t-\mu}{b}\right) \quad (90)$$

The equation can only exists for $b > 0$.

$$S'(t) = -\frac{1}{b}S(t) \quad (91)$$

The first order ODE for the SF of the Laplace distribution for case II is given as;

$$bS'(t) + S(t) = 0 \quad (92)$$

$$S(t) = \frac{1}{2}\exp\left(\frac{\mu}{b}\right) \quad (93)$$

Differentiate equations (84) and (90), to obtain the second order ODE;
Case I, $t < \mu$:

$$S''(t) = -\frac{1}{2b^2}\exp\left(\frac{t-\mu}{b}\right) \quad (94)$$

The equation can only exists for is $b > 0$.
Two ODEs can be acquired from the further breaking down of equation (94). These are listed as follows;
ODE 1;
Using equation (85) in equation (94);

$$S''(t) = -\frac{1}{b^2}(1 - S(t)) \quad (95)$$

$$b^2 S''(t) - S(t) + 1 = 0 \quad (96)$$

ODE 2;

$$S''(t) = -\frac{1}{2b^2}\exp\left(\frac{t-\mu}{b}\right) = \left(\frac{1}{b}\right)\left(-\frac{1}{2b}\right)\exp\left(\frac{t-\mu}{b}\right) \quad (97)$$

$$S''(t) = \frac{1}{b}S'(t) \quad (98)$$

$$bS''(t) - S'(t) = 0 \quad (99)$$

$$S'(0) = -\frac{1}{2b}\exp\left(-\frac{\mu}{b}\right) \quad (100)$$

Case II, $t \ge \mu$:

$$S''(t) = \frac{1}{2b^2}\exp\left(-\frac{t-\mu}{b}\right) \quad (101)$$

The equation can only exists for $b > 0$.
Two ODEs can be acquired from the further breaking down of equation (94). These are listed as follows;

ODE 1;

$$S''(t) = \frac{1}{2b^2}\exp\left(-\frac{t-\mu}{b}\right) = \left(\frac{1}{b^2}\right)\left(\frac{1}{2}\right)\exp\left(-\frac{t-\mu}{b}\right) \quad (102)$$

$$S''(t) = \frac{1}{b^2}S(t) \quad (103)$$

$$b^2 S''(t) - S(t) = 0 \quad (104)$$

ODE 2;

$$S''(t) = \frac{1}{2b^2}\exp\left(-\frac{t-\mu}{b}\right) = \left(-\frac{1}{b}\right)\left(-\frac{1}{2b}\right)\exp\left(-\frac{t-\mu}{b}\right) \quad (105)$$

$$S''(t) = -\frac{1}{b}S'(t) \quad (106)$$

$$bS''(t) + S'(t) = 0 \quad (107)$$

$$S'(0) = -\frac{1}{2b}\exp\left(\frac{\mu}{b}\right) \quad (108)$$

The result from first order to fifth order is listed in Table (3);

Table 3: First to fifth order ODE for the SF

Case I, $x<\mu$:	Case II, $x \geq \mu$:
$bS'(t)-S(t)+1=0$	$bS'(t)+S(t)=0$
$b^2S''(t)-S(t)+1=0$ $bS''(t)-S'(t)=0$	$b^2S''(t)-S(t)=0$ $bS''(t)+S'(t)=0$
$b^3S'''(t)-S(t)+1=0$ $b^2S'''(t)-S'(t)=0$ $bS'''(t)-S''(t)=0$	$b^3S'''(t)+S(t)=0$ $b^2S'''(t)-S'(t)=0$ $bS'''(t)+S''(t)=0$
$b^4S^{iv}(t)-S(t)+1=0$ $b^3S^{iv}(t)-S'(t)=0$ $b^2S^{iv}(t)-S''(t)=0$ $bS^{iv}(t)-S'''(t)=0$	$b^4S^{iv}(t)-S(t)=0$ $b^3S^{iv}(t)+S'(t)=0$ $b^2S^{iv}(t)-S''(t)=0$ $bS^{iv}(t)+S'''(t)=0$
$b^5S^{v}(t)-S(t)+1=0$ $b^4S^{v}(t)-S'(t)=0$ $b^3S^{v}(t)-S''(t)=0$ $b^2S^{v}(t)-S'''(t)=0$ $bS^{v}(t)-S^{iv}(t)=0$	$b^5S^{v}(t)+S(t)=0$ $b^4S^{v}(t)-S'(t)=0$ $b^3S^{v}(t)+S''(t)=0$ $b^2S^{v}(t)-S'''(t)=0$ $bS^{v}(t)+S^{iv}(t)=0$

The results are similar to the ones proposed in [16-29].

V. Inverse Survival Function

The ISF of the Laplace distribution is given by;

$$Q(p)=\begin{cases}\mu+b\ \ln(2(1-p)) & \text{derived if } x<\mu \\ \mu-b\ \ln(2p) & \text{derived if } x\geq\mu\end{cases} \tag{109}$$

Differentiate equation (109), to obtain the first order ODE;
Case I, $x<\mu$:

$$Q(p)=\mu+b\ln(2(1-p)) \tag{110}$$

$$Q'(p)=-\frac{b}{1-p} \tag{111}$$

The equation can only exists for $b>0, 0<p<1$.

The first order ODE for the ISF of the Laplace distribution for case I is given as;

$$(1-p)Q'(p)+b=0 \tag{112}$$

$$Q(0.1)=\mu+0.5878b \tag{113}$$

Case II, $x\geq\mu$:

$$Q(p)=\mu-b\ln(2p) \tag{114}$$

$$Q'(p)=-\frac{b}{p} \tag{115}$$

The equation can only exists for $b>0, 0<p<1$.

The first order ODE for the ISF of the Laplace distribution for case II is given as;

$$pQ'(p)+b=0 \tag{116}$$

$$Q(0.1)=\mu+1.6094b \tag{117}$$

The results are similar to the ones proposed in [16-29].

VI. Hazard Function

The HF of the Laplace distribution is given by;

$$h(t)=\begin{cases}\dfrac{\frac{1}{2b}\exp\left(\frac{t-\mu}{b}\right)}{1-\frac{1}{2}\exp\left(\frac{t-\mu}{b}\right)} & \text{if t} <\mu \\ \dfrac{1}{b} & \text{if t} \geq\mu\end{cases} \tag{118}$$

Differentiate equation (118), to obtain the first order ODE;
Case I, $t<\mu$:

$$h(t)=\frac{\frac{1}{2b}\exp\left(\frac{t-\mu}{b}\right)}{1-\frac{1}{2}\exp\left(\frac{t-\mu}{b}\right)} =\frac{1}{2b}\exp\left(\frac{t-\mu}{b}\right)\left(1-\frac{1}{2}\exp\left(\frac{t-\mu}{b}\right)\right)^{-1} \tag{119}$$

$$h'(t)=\frac{1}{2b}\left\{\begin{matrix}\left(\frac{1}{2b}\right)\left(\exp\left(\frac{t-\mu}{b}\right)\right)^2 \\ \left(1-\frac{1}{2}\exp\left(\frac{t-\mu}{b}\right)\right)^{-2} \\ +\frac{1}{b}\exp\left(\frac{t-\mu}{b}\right)\left(1-\frac{1}{2}\exp\left(\frac{t-\mu}{b}\right)\right)^{-1}\end{matrix}\right\} \tag{120}$$

The equation can only exists for $b>0$.

$$h'(t)=\frac{1}{2b}\exp\left(\frac{t-\mu}{b}\right) \left\{\left(\frac{1}{2b}\right)\exp\left(\frac{t-\mu}{b}\right)\left(1-\frac{1}{2}\exp\left(\frac{t-\mu}{b}\right)\right)^{-1}+\frac{1}{b}\right\} \tag{121}$$

$$h'(t)=h(t)\left\{h(t)+\frac{1}{b}\right\} \tag{122}$$

The first order ODE for the HF of the Laplace distribution for case I is given as;

$$bh'(t)-bh^2(t)-h(t)=0 \tag{123}$$

$$h(0)=\frac{\frac{1}{2b}\exp\left(-\frac{\mu}{b}\right)}{1-\frac{1}{2}\exp\left(-\frac{\mu}{b}\right)} \tag{124}$$

Subsequently, the other higher order differential equations are given;

$$bh''(t)-2bh(t)h'(t)-h'(t)=0 \tag{125}$$

$$bh'''(t)-2bh(t)h''(t)-2bh'^2(t)-h''(t)=0 \tag{126}$$

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The result for case II is zero since the hazard function is constant and as such the ODE is zero. Meanwhile, the results are similar to the ones proposed in [16-29].

VII. Reversed Hazard Function

The RHF of the Laplace distribution is given by;

$$j(t)=\begin{cases}\dfrac{1}{b} & \text{if } t<\mu\\ \dfrac{\frac{1}{2b}\exp\left(-\frac{t-\mu}{b}\right)}{1-\frac{1}{2}\exp\left(-\frac{t-\mu}{b}\right)} & \text{if } t\geq\mu\end{cases} \quad (127)$$

Differentiate equation (127), to obtain the first order ODE; Case II, $t \geq \mu$:

$$j(t)=\frac{\frac{1}{2b}\exp\left(-\frac{t-\mu}{b}\right)}{1-\frac{1}{2}\exp\left(-\frac{t-\mu}{b}\right)} \quad (128)$$

$$=\frac{1}{2b}\exp\left(-\frac{t-\mu}{b}\right)\left(1-\frac{1}{2}\exp\left(-\frac{t-\mu}{b}\right)\right)^{-1}$$

$$j'(t)=\frac{1}{2b}\left\{\begin{array}{l}-\left(\frac{1}{2b}\right)\left(\exp\left(-\frac{t-\mu}{b}\right)\right)^{2}\\ \left(1-\frac{1}{2}\exp\left(-\frac{t-\mu}{b}\right)\right)^{-2}\\ -\frac{1}{b}\exp\left(-\frac{t-\mu}{b}\right)\\ \left(1-\frac{1}{2}\exp\left(-\frac{t-\mu}{b}\right)\right)^{-1}\end{array}\right\} \quad (129)$$

The equation can only exists for $b > 0$.

$$j'(t)=-\frac{1}{2b}\exp\left(-\frac{t-\mu}{b}\right)\left(1-\frac{1}{2}\exp\left(-\frac{t-\mu}{b}\right)\right)^{-1}$$
$$\left\{\left(\frac{1}{2b}\right)\exp\left(-\frac{t-\mu}{b}\right)\left(1-\frac{1}{2}\exp\left(-\frac{t-\mu}{b}\right)\right)^{-1}+\frac{1}{b}\right\} \quad (130)$$

$$j'(t)=-j(t)\left(j(t)+\frac{1}{b}\right) \quad (131)$$

The first order ODE for the RHF of the Laplace distribution for case II is given as;

$$bj'(t)+bj^{2}(t)+j(t)=0 \quad (132)$$

$$j(0)=\frac{\frac{1}{2b}\exp\left(\frac{\mu}{b}\right)}{1-\frac{1}{2}\exp\left(\frac{\mu}{b}\right)}=\frac{\frac{1}{2b}}{\exp\left(-\frac{\mu}{b}\right)-\frac{1}{2}} \quad (133)$$

The results are similar to the ones proposed in [16-29].

VIII. Concluding Remarks

Ordinary differential equations (ODEs) have been obtained for the probability functions of Laplace distribution. This differential calculus and efficient algebraic simplifications were used to derive the various classes of the ODEs. The parameter and the supports that characterize the distribution determine the nature, existence, orientation and uniqueness of the ODEs. The results are in agreement with those available in scientific literature. Furthermore several methods can be used to obtain desirable solutions to the ODEs [48-60]. This method of characterizing distributions cannot be applied to distributions whose PDF or CDF are either not differentiable or the domain of the support of the distribution contains singular points.

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