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A New Criterion of the Feedback Stabilizability of Two-by-Two Plants

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***Abstract*—In this paper, a criterion of stabilizability of plants without coprime factorization for 2×2 plants is given. This is expressed by two factorizations. We parametrize stabilizing controllers based on the criterion and also present their applications.**

***Index Terms*—linear systems, parametrization of stabilizing controllers coprime factorization, factorization approach, stabilizability.**

I. Introduction

So far, the coprime factorization plays a central role to obtain a stabilizing controller[1]. In this paper, we consider the model in which the plants may not admit coprime factorizations. We generalize the notion of coprime factorization. By using this, we propose a new method to obtain a stabilizing controller of a plant which may not admit coprime factorization. The coprime factorization is based on the factorization approach.

The factorization approach to control systems has the advantage that it includes, within a single framework, numerous linear systems such as continuous-time as well as discrete-time systems, lumped as well as distributed systems, one-dimensional as well as multidimensional systems, etc.[1], [2], [3], [4], [5], [6], [7], [8], [9]. In the factorization approach, when problems such as feedback stabilization are studied, one can focus on the key aspects of the problem under study rather than be distracted by the special features of a particular class of linear systems. This approach leads to conceptually simple and computationally tractable solutions to many important and interesting problems[1], [10]. A transfer matrix of this approach is considered as the ratio of two stable causal transfer matrices. One of the attractive points of the factorization approach is the fact that all stabilizing controllers can be obtained by the Youla-Kučera parametrization with coprime factorization [11], [12], [13]. This Youla-Kučera parametrization has been used in a wide variety of applications for a long time (e.g. [14], [15], [16], [17], [18]).

For a long time, the theory of the factorization approach had been founded on the coprime factorizability of transfer matrices. On the other hand, Anantharam showed in [19] a model that has plants which are stabilizable but do not admit coprime factorization. Mori and Abe also showed such a model in [6]. In [20], we showed that the stabilizability can be given by Bézout identity based on two factorizations.

In this paper, we give an extension of coprime factorization for 2×2 plants, which are of two inputs and two outputs. The coprime factorization is usually used to obtain stabilizing controllers the given plant. By the extension of the coprime factorization, we give a method to obtain a stabilizing controller of plants that may not admit the coprime factorization. The form of stabilizing controller without coprime factorizations is so far based on many coprime factorizations over local rings[5], [6], [21], [22]. On the other hand, the result of this paper is based on just two coprime-like factorizations. Also, we give a method to obtain alternative stabilizing controllers of plants. The result of this paper can be considered as a generalization of [20] if plant is single-input single-output.

This paper is started with preliminaries from Section II to recall the notion of the factorization approach. In Section III, we generalize a notion of coprime factorization. In Section IV, we give examples for the proposed method. First example will be the plants that admit coprime factorizations. Next one will be Anantharam's example[19]. Third one will be the discrete-time systems without the unit-delay element within the framework of the factorization approach.

II. Preliminaries

The stabilization problem considered in this paper follows that of [5], and [6], who consider the feedback system [1, Ch.5, Fig. 5.1] as in Fig. 1. In the figure, u_1 and u_2 are inputs, y_1 and y_2 outputs, and e_1 and e_2 "errors." We employ the symbols used in [5] and [22] in general. For further details, the reader is referred to [1], [4], [5], and [6].

We consider that the set of stable causal transfer functions is an integral domain, denoted by $\mathcal{A}$. The total ring of fractions of $\mathcal{A}$ is denoted by $\mathcal{F}$; that is, $\mathcal{F} = \{n/d \,|\, n, d \in \mathcal{A},\ d \neq 0\}$. This $\mathcal{F}$ is considered as the set of all possible transfer functions. Matrices over $\mathcal{F}$ are transfer matrices. Let $\mathcal{Z}$ be a prime ideal of $\mathcal{A}$ with $\mathcal{Z} \neq \mathcal{A}$. Define the subsets $\mathcal{P}$ and $\mathcal{P}_s$ of $\mathcal{F}$ as follows: $\mathcal{P} = \{a/b \in \mathcal{F} \,|\, a \in \mathcal{A},\ b \in \mathcal{A}\backslash\mathcal{Z}\}$, $\mathcal{P}_s = \{a/b \in \mathcal{F} \,|\, a \in \mathcal{Z},\ b \in \mathcal{A}\backslash\mathcal{Z}\}$. Then, a transfer function is called *causal* (*strictly causal*) if it is in $\mathcal{P}$ ($\mathcal{P}_s$). Analogously, a transfer matrix is called *causal* (*strictly causal*) if every entry of the transfer matrix is in $\mathcal{P}$ ($\mathcal{P}_s$), We denote by $\mathcal{A}_\lambda$ the ring of fractions of $\mathcal{A}$ with respect to the multiplicative subset $\{\lambda^k \,|\, \lambda \in \mathcal{A},\ \text{integer } k \geq 0\}$; that is, $\mathcal{A}_\lambda = \{n/d \,|\, n \in \mathcal{A},\ d = \lambda^k,\ \lambda \in \mathcal{A},\ k \geq 0\}$.

Throughout the paper, the plant we consider has 2 inputs and 2 outputs, and its transfer matrix, which is also called a

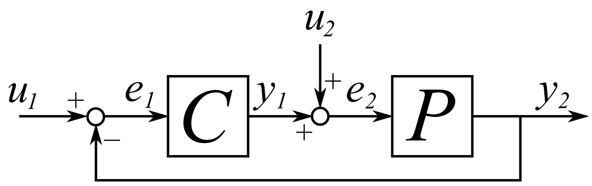

Fig. 1. Feedback System.

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plant itself simply, is denoted by P and in $\mathcal{P}^{2\times 2}$ (that is, P is causal). We can always represent P in the form of a fraction $P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N}$, where N, $\widetilde{N} \in \mathcal{A}^{2\times 2}$, $D \in \mathcal{A}^{2\times 2}$, $\widetilde{D} \in \mathcal{A}^{n\times n}$ with nonsingular D and $\widetilde{D}$.

For $P \in \mathcal{P}^{2\times 2}$ and $C \in \mathcal{F}^{2\times 2}$, a matrix $H(P,C) \in \mathcal{F}^{(m+n)\times(m+n)}$ is defined as

$$H(P,C) := \begin{bmatrix} (I_2+PC)^{-1} & -P(I_2+CP)^{-1} \\ C(I_2+PC)^{-1} & (I_2+CP)^{-1} \end{bmatrix} \qquad (1)$$

provided that $\det(I_2+PC)$ is a nonzero of $\mathcal{A}$. This $H(P,C)$ is the transfer matrix from $[\ u_1\ u_2\ ]^t$ to $[\ e_1\ e_2\ ]^t$ of the feedback system of Fig. 1. If $\det(I_2+PC)$ is a nonzero and $H(P,C) \in \mathcal{A}^{(m+n)\times(m+n)}$, then we say that the plant P is *stabilizable*, P is *stabilized* by C, and C is a *stabilizing controller* of P. In the definition above, we do not mention the causality of the stabilizing controller. However, it is known that if a causal plant is stabilizable, there always exists a causal stabilizing controller of the plant [6].

We also introduce the notion of $\mathcal{A}_\lambda$-stability, where $\lambda \in \mathcal{A}$. If the transfer matrix $H(P,C)$ is over $\mathcal{A}_\lambda$ rather than over $\mathcal{A}$ and $\det(I_2+PC) \neq 0$, then we say that the plant P is *$\mathcal{A}_\lambda$-stabilizable*, P is *$\mathcal{A}_\lambda$-stabilized* by C, and C is a *$\mathcal{A}_\lambda$-stabilizing controller*of P.

It is known that $W(P,C)$ defined below is over $\mathcal{A}$ if and only if $H(P,C)$ is over $\mathcal{A}$:

$$W(P,C) := \begin{bmatrix} C(I_2+PC)^{-1} & -CP(I_2+CP)^{-1} \\ PC(I_2+PC)^{-1} & P(I_2+CP)^{-1} \end{bmatrix}. \qquad (2)$$

This $W(P,C)$ is the transfer matrix from $[\ u_1\ u_2\ ]^t$ to $[\ y_1\ y_2\ ]^t$.

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III. Extension of Coprime Factorization

In this section, we give an extension of coprime factorization, which leads a criterion of stabilizability and parametrization of stabilizing controllers. Hereafter, we assume that numbers of inputs and outputs are equal to each other, denoted by n (that is, $m = n$).

The following is the first mail result of this paper.

Theorem 1: Let P be a causal plant of $\mathcal{P}^{n\times n}$. Assume that $\det(P)$ is nonzero (the inverse of P exists). Then P is stabilizable if and only if there exist matrices N, D, $\widetilde{N}$, $\widetilde{D}$, $\widetilde{Y}$, $\widetilde{X}$, Y, X, N_2 of $\mathcal{A}^{n\times n}$ such that

$$D\widetilde{X}D + D\widetilde{Y}N + \widetilde{D}XD + \widetilde{D}YN = D, \qquad (3)$$

$$P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N} = N_2\widetilde{D}^{-1}. \qquad (4)$$

If P is stabilizable, with the matrices above,

$$(D\widetilde{X} + \widetilde{D}X)^{-1}(D\widetilde{Y} + \widetilde{D}Y). \qquad (5)$$

is a stabilizing controller of P provided the non-singularity of its denominator matrix.

Note 1: Equation (3) seems to contains both right- and left-coprime factorizations. The first two terms of (3) are relatively corresponding to right-coprime factorization, and the remaining terms left-coprime factorization. ▪

Proof: (Brief Proof of Theorem 1) (Only If) Suppose that P is stabilizable. Then, there exists a stabilizing controller C.

Let

$$\begin{aligned} &N = P(I_2+CP)^{-1}, && D = (I_2+CP)^{-1}, \\ &N_2 = P(I_2+CP)^{-1}C, && \\ &\widetilde{N} = (I_2+CP)^{-1}CP, && \widetilde{D} = (I_2+CP)^{-1}C, \\ &\widetilde{Y} = O_{2\times 2}, && \widetilde{X} = I_2, \\ &Y = I_2, && X = O_{2\times 2}. \end{aligned}$$

Then, we have (3) and (4).

(If) Suppose that there exist matrices N, D, N_2, $\widetilde{Y}$, $\widetilde{X}$, $\widetilde{N}$, $\widetilde{D}$, Y, X of $\mathcal{A}^{n\times n}$ with which (3) to (4) hold.

In this brief proof, we omit to mention the non-singularity of inverse matrices in order to focus the main idea of the proof and due to the space limitation.

We let

$$C_{new} = (D\widetilde{X} + \widetilde{D}X)^{-1}(D\widetilde{Y} + \widetilde{D}Y). \qquad (6)$$

From now, we show that $H(P, C_{new})$ is over $\mathcal{A}$. To do so, we show that the matrices $(I_2 + PC_{new})^{-1}$, $P(I_2 + C_{new}P)^{-1}$, $C_{new}(I_2 + PC_{new})^{-1}$, $(I_2 + C_{new}P)^{-1}$ are over $\mathcal{A}$. They are as follows:

$$\begin{aligned} &(I_2 + C_{new}P)^{-1} \\ &= (I_2 + (D\widetilde{X} + \widetilde{D}X)^{-1}(D\widetilde{Y} + \widetilde{D}Y)P)^{-1} \\ &= ((D\widetilde{X} + \widetilde{D}X)^{-1} \\ &\qquad ((D\widetilde{X} + \widetilde{D}X) + (D\widetilde{Y} + \widetilde{D}Y)P))^{-1} \\ &= ((D\widetilde{X} + \widetilde{D}X)^{-1} \\ &\qquad (D\widetilde{X} + \widetilde{D}X + D\widetilde{Y}P + \widetilde{D}YP))^{-1} \\ &= ((D\widetilde{X} + \widetilde{D}X)^{-1} \\ &\qquad (D\widetilde{X}D + \widetilde{D}XD + D\widetilde{Y}N + \widetilde{D}YN)D^{-1})^{-1} \\ &= ((D\widetilde{X} + \widetilde{D}X)^{-1}(D)D^{-1})^{-1} \\ &= D\widetilde{X} + \widetilde{D}X, \end{aligned}$$

$$\begin{aligned} P(I_2 + C_{new}P)^{-1} &= P(D\widetilde{X} + \widetilde{D}X) \\ &= N\widetilde{X} + N_2X, \end{aligned}$$

$$\begin{aligned} &(I_2 + C_{new}P)^{-1}C_{new} \\ &= (D\widetilde{X} + \widetilde{D}X)(D\widetilde{X} + \widetilde{D}X)^{-1}(D\widetilde{Y} + \widetilde{D}Y) \\ &= (D\widetilde{Y} + \widetilde{D}Y), \end{aligned}$$

$$\begin{aligned} &(I_2 + PC_{new})^{-1} \\ &= I_2 - PC_{new}(I_2 + PC_{new})^{-1} \\ &= I_2 - P(I_2 + C_{new}P)^{-1}C_{new} \\ &= I_2 - (N\widetilde{Y} + N_2Y). \end{aligned}$$

The four matrices above are over $\mathcal{A}$. Hence, C_{new} is a stabilizing controller of P, so that P is stabilizable. ■

From Theorem 1, we can parametrize stabilizing controllers of plant, the second main result of this paper, as follows.

Theorem 2: Employ again the symbols of Theorem 1, satisfying (3) and (4). Let R_1 and R_2 be parameters of $\mathcal{A}^{n\times n}$. Then,

$$(D(\widetilde{X} - R_1\widetilde{N}) + \widetilde{D}(X - R_2\widetilde{N}))^{-1} \\ (D(\widetilde{Y} + R_1\widetilde{D}) + \widetilde{D}(Y + R_2\widetilde{D})). \qquad (7)$$

is a stabilizing controller of P provided the non-singularity of its denominator matrix.

Proof: From (3), the following equation holds for any R_1 and R_2 of $\mathcal{A}^{n\times n}$:

$$D(\widetilde{X}-R_1\widetilde{N})D+D(\widetilde{Y}+R_1\widetilde{D})N \\ +\widetilde{D}(X-R_2\widetilde{N})D+\widetilde{D}(Y+R_2\widetilde{D})N=D. \qquad (8)$$

We still have (4), so that Theorem 1 holds. Now, (5) is corresponding to (7). ■

IV. Examples

Example 1: First let us consider an example given by Anantharam in [19]. Anantharam considered the case $\mathcal{A}=\mathbb{Z}[\sqrt{-5}]=\{u+v\sqrt{-5}\,|\,u,v\in\mathbb{Z}\}$, where $\mathbb{Z}$ denotes the set of integers. This ring [23, pp.134–135] is isomorphic to $\mathbb{Z}[x]/(x^2+5)$ and is an integral domain but not a unique factorization domain. In fact, $6\in\mathbb{Z}[\sqrt{-5}]$ has two factorizations, $2\cdot 3$ and $(1+\sqrt{-5})\cdot(1-\sqrt{-5})$. He showed that a single-input single-output plant $(1+\sqrt{-5})/2$ does not admit a coprime factorization but is stabilizable and $(1-\sqrt{-5})/(-2)$ is a stabilizing controller.

Now, we consider the following plant P:

$$P=\begin{bmatrix}(1+\sqrt{-5})/2 & 0\\ 0 & (1+2\sqrt{-5})/7\end{bmatrix}, \qquad (9)$$

which has alternative fractional representation

$$P=\begin{bmatrix}3/(1-\sqrt{-5}) & 0\\ 0 & 3/(1-2\sqrt{-5})\end{bmatrix}. \qquad (10)$$

We now consider

$$\begin{aligned}
N&=\begin{bmatrix}-(1+\sqrt{-5}) & 0\\ 0 & -(1+2\sqrt{-5})\end{bmatrix},\\
D&=\begin{bmatrix}-2 & 0\\ 0 & -7\end{bmatrix},\\
\widetilde{N}&=N_2=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix},\\
\widetilde{D}&=\begin{bmatrix}(1-\sqrt{-5}) & 0\\ 0 & (1-2\sqrt{-5})\end{bmatrix},\\
\widetilde{Y}&=O_{2\times 2},\\
\widetilde{X}&=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\\
Y&=\begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix},\\
X&=O_{2\times 2}.
\end{aligned}$$

Then, we have (3) and (4), so that the plant P is stabilizable.

Now the stabilizing controller C of (6) is

$$\begin{aligned}
C&=(D\widetilde{X}+\widetilde{D}X)^{-1}(D\widetilde{Y}+\widetilde{D}Y)\\
&=\begin{bmatrix}-2 & 0\\ 0 & -7\end{bmatrix}^{-1}\begin{bmatrix}1-\sqrt{-5} & 0\\ 0 & -2+4\sqrt{-5}\end{bmatrix}^{-1}\\
&=\begin{bmatrix}-\frac{1}{2}+\frac{1}{2}\sqrt{-5} & 0\\ 0 & -\frac{2}{7}+\frac{4}{7}\sqrt{-5}\end{bmatrix}^{-1}
\end{aligned}$$

which is a stabilizing controller of (9). In fact, $H(P,C)$ is as follows:

$$H(P,C)=\begin{bmatrix}-2 & 0 & 1+\sqrt{-5} & 0\\ 0 & 7 & 0 & -1-2\sqrt{-5}\\ 1-\sqrt{-5} & 0 & -2 & 0\\ 0 & -2+4\sqrt{-5} & 0 & 7\end{bmatrix},$$

which is over $\mathcal{A}$ $(=\mathbb{Z}[\sqrt{-5}])$. ▪

Example 2: Consider the discrete-time systems without unit-delay element. Mori [6] considered the case $\mathcal{A}=\mathbb{R}[z^2,z^3]$, where $\mathbb{R}$ denotes the set of real numbers. This ring is an integral domain but not a unique factorization domain. In fact, $z^6\in\mathcal{A}$ has two factorizations, $z^2\cdot z^2\cdot z^2$ and $z^3\cdot z^3$. He showed that the plant

$$P:=\begin{bmatrix}(1-z^3)/(1-z^2)\\ (1-8z^3)/(1-4z^2)\end{bmatrix}\in\mathcal{P}^{2\times 1}$$

does not admit a coprime factorization but is stabilizable.

Now we consider that plant

$$P=\begin{bmatrix}\dfrac{1-z^2}{1-z^3} & 0\\ 0 & \dfrac{1-8z^3}{1-4z^2}\end{bmatrix}. \qquad (11)$$

Then, we have

$$\begin{aligned}
N&=\begin{bmatrix}\frac{1}{3}(1+z^3) & 0\\ 0 & 1+z^2-6z^3+4z^4\end{bmatrix},\\
D&=\begin{bmatrix}\frac{1}{3}(1+z^2+z^4) & 0\\ 0 & 1-3z^2+2z^3\end{bmatrix},\\
\widetilde{N}&=N_2=\begin{bmatrix}\frac{1}{3}(1-z^2)(2+z^2) & 0\\ 0 & -(1-8z^3)\end{bmatrix},\\
\widetilde{D}&=\begin{bmatrix}\frac{1}{3}(1-z^3)(2+z^2) & 0\\ 0 & -(1-4z^2)\end{bmatrix},\\
\widetilde{Y}&=O_{2\times 2},\\
\widetilde{X}&=\begin{bmatrix}1 & 0\\ 0 & \frac{1}{49}(50+120z^2+288z^4)\end{bmatrix},\\
Y&=\begin{bmatrix}1 & 0\\ 0 & \frac{1}{49}(1-30z^2+108z^3-72z^4)\end{bmatrix},\\
X&=O_{2\times 2}.
\end{aligned}$$

They satisfy (3) to (4).

Now the stabilizing controller C of (6) is

$$\begin{aligned}
C&=(D_1\widetilde{X}_1+\widetilde{D}_1X_1)^{-1}(D_1\widetilde{Y}_1+\widetilde{D}_1Y_1)\\
&=\begin{bmatrix}d_1 & 0\\ 0 & d_2\end{bmatrix}^{-1}\begin{bmatrix}n_1 & 0\\ 0 & n_2\end{bmatrix},
\end{aligned}$$

where

$$\begin{aligned}
d_1&=\frac{-1}{3}(2+z^2)(-1+z^3),\\
d_2&=\frac{-1}{49}(-1+4z^2)(-1+30z^2-108z^3+72z^4),\\
n_1&=\frac{1}{3}(1+z^2+z^4),\\
n_2&=\frac{2}{49}(1-3z^2+2z^3)(25+60z^2+144z^4).
\end{aligned}$$

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This C is a stabilizing controller of (11). Then, $H(P,C)$ is as follows:

$$H(P,C) = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix},$$

where

$$\begin{aligned} h_{11} = h_{33} &= \frac{1}{3}(1 + z^2 + z^4), \\ h_{22} = h_{44} &= \frac{2}{49}(25 - 15z^2 + 50z^3 - 36z^4 \\ &\quad + 120z^5 - 432z^6 + 288z^7), \\ h_{13} &= \frac{1}{3}(-1 - z^3), \\ h_{24} &= \frac{-2}{49}(25 + 85z^2 - 150z^3 + 304z^4 \\ &\quad - 360z^5 + 384z^6 - 864z^7 + 576z^8), \\ h_{31} &= \frac{1}{3}(2 + z^2 - 2z^3 - z^5), \\ h_{42} &= \frac{1}{49}(-1 + 34z^2 - 108z^3 - 48z^4 \\ &\quad + 432z^5 - 288z^6), \end{aligned}$$

and other h_{ij}'s are zero. This C is over $\mathcal{A}$. ▪

Example 3: Let us apply Theorem 2 to Examples 1 and 2. Consider again Example 1. Let

$$R_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \tag{12}$$

as an example. Then, alternative stabilizing controller is (7) is given as follows:

$$\begin{aligned} &(D(\widetilde{X} - R_1\widetilde{N}) + \widetilde{D}(X - R_2\widetilde{N}))^{-1} \\ &\qquad (D(\widetilde{Y} + R_1\widetilde{D}) + \widetilde{D}(Y + R_2\widetilde{D})) \\ &= \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}^{-1} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}, \end{aligned}$$

where

$$\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} -21 - 9\sqrt{-5} & -58 - 10\sqrt{-5} \\ -84 & -182 + 28\sqrt{-5} \end{bmatrix},$$

$$\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} = \begin{bmatrix} -11 + 15\sqrt{-5} & -6 + 18\sqrt{-5} \\ 42 + 42\sqrt{-5} & 67 + 48\sqrt{-5} \end{bmatrix}.$$

Then $H(P,C)$ is as follows:

$$\begin{bmatrix} -11 + 15\sqrt{-5} & 4 + 34\sqrt{-5} & 43 - 2\sqrt{-5} & 48 - 6\sqrt{-5} \\ 12 + 24\sqrt{-5} & 67 + 48\sqrt{-5} & 54 - 18\sqrt{-5} & 59 - 26\sqrt{-5} \\ -21 - 9\sqrt{-5} & -58 - 10\sqrt{-5} & -11 + 15\sqrt{-5} & -6 + 18\sqrt{-5} \\ -84 & -182 + 28\sqrt{-5} & 42 + 42\sqrt{-5} & 67 + 48\sqrt{-5} \end{bmatrix},$$

which is over $\mathcal{A}$.

Next consider Example 2. Let R_1 and R_2 be as in (12) again. Then, alternative stabilizing controller is (7) is given as follows:

$$\begin{aligned} &(D(\widetilde{X} - R_1\widetilde{N}) + \widetilde{D}(X - R_2\widetilde{N}))^{-1} \\ &\qquad (D(\widetilde{Y} + R_1\widetilde{D}) + \widetilde{D}(Y + R_2\widetilde{D})) \\ &= \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}^{-1} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} n_{11} &= \frac{1}{9}(2 + z^2)(-1 + z^3)(-14 - 6z^2 + 10z^3 - z^4 + 5z^5), \\ n_{12} &= \frac{-2}{3}(7 - 24z^2 - 6z^3 - 15z^4 + 21z^5 - 4z^6 + 12z^7), \\ n_{21} &= \frac{1}{3}(-8 + 34z^2 + 20z^3 + 19z^4 \\ &\quad - 28z^5 - 12z^6 - 19z^7 - 6z^8), \\ n_{22} &= \frac{1}{49}(195 - 1730z^2 - 500z^3 \\ &\quad + 3872z^4 + 2000z^5 - 288z^6), \\ d_{11} &= \frac{1}{9}(-19 + 2z^2 + 20z^3 + 18z^4 \\ &\quad + 7z^6 - 15z^7 + z^8 - 5z^9), \\ d_{12} &= \frac{2}{3}(-1 + 8z^3)(-7 - 4z^2 + 6z^3 - z^4 + 3z^5), \\ d_{21} &= \frac{1}{3}(8 - 42z^2 - 12z^3 + 15z^4 + 6z^5 + 19z^6 + 6z^7), \\ d_{22} &= \frac{2}{49}(-73 + 475z^2 + 1030z^3 - 36z^4 \\ &\quad - 3800z^5 - 2000z^6 + 288z^7). \end{aligned}$$

Then $H(P,C)$ is as follows:

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix},$$

where

$$\begin{aligned} h_{11} &= \frac{1}{9}(-19 + 2z^2 + 20z^3 + 18z^4 + 7z^6 - 15z^7 + z^8 - 5z^9), \\ h_{12} &= \frac{2}{3}(7 - 31z^2 + z^3 + 9z^4 - 4z^5 + 12z^6), \\ h_{13} &= \frac{1}{9}(19 - 21z^2 - z^3 - 16z^4 - z^5 + 10z^6 - z^7 + 5z^8), \\ h_{14} &= \frac{-2}{3}(7 - 3z^2 - 55z^3 - 3z^4 + 24z^5 - 8z^6 + 24z^7), \\ h_{21} &= \frac{1}{3}(8 - 2z^2 - 84z^3 - 27z^4 - 36z^5 + 64z^6 \\ &\quad + 27z^7 + 38z^8 + 12z^9), \\ h_{22} &= \frac{2}{49}(-73 + 475z^2 + 1030z^3 - 36z^4 - 3800z^5 \\ &\quad - 2000z^6 + 288z^7), \\ h_{23} &= \frac{1}{3}(-8 + 10z^2 + 76z^3 + 25z^4 - 38z^5 - 15z^6 \\ &\quad - 38z^7 - 12z^8), \\ h_{24} &= \frac{-2}{49}(-73 + 183z^2 + 1614z^3 + 696z^4 - 1144z^5 \\ &\quad - 7456z^6 - 4000z^7 + 576z^8), \end{aligned}$$

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$$h_{31} = \frac{1}{9}(28 + 26z^2 - 48z^3 + 8z^4 - 46z^5 + 21z^6 - 13z^7 + 20z^8 - z^9 + 5z^10),$$

$$h_{32} = \frac{-2}{3}(7 - 24z^2 - 6z^3 - 15z^4 + 21z^5 - 4z^6 + 12z^7),$$

$$h_{33} = \frac{1}{9}(-19 + 2z^2 + 20z^3 + 18z^4 + 7z^6 - 15z^7 + z^8 - 5z^9),$$

$$h_{34} = \frac{1}{3}(2(7 + 4z^2 - 62z^3 + z^4 - 35z^5 + 48z^6 - 8z^7 + 24z^8),$$

$$h_{41} = \frac{1}{3}(-8 + 34z^2 + 20z^3 + 19z^4 - 28z^5 - 12z^6 - 19z^7 - 6z^8),$$

$$h_{42} = \frac{1}{49}(195 - 1730z^2 - 500z^3 + 3872z^4 + 2000z^5 - 288z^6),$$

$$h_{43} = \frac{1}{3}(8 - 42z^2 - 12z^3 + 15z^4 + 6z^5 + 19z^6 + 6z^7),$$

$$h_{44} = \frac{2}{49}(-73 + 475z^2 + 1030z^3 - 36z^4 - 3800z^5 - 2000z^6 + 288z^7),$$

which is also over $\mathcal{A}$. ▪

V. Conclusions

In this paper, we consider the plant with two inputs and two outputs. Stabilizing controllers based on precisely two coprime-like factorizations are shown. This can lead a new method to obtain a stabilizing controller. Some examples are also presented.

As future work, we will investigate the case for any sized plants.

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