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Some Fixed Point Results for Contraction Mappings on E-uniform Spaces

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***Abstract*—We introduce the notion of E-uniform spaces and prove some fixed point theorems for contraction maps on E-uniform spaces. Examples are given to support our results. The results generalize and extend some results in literature.**

***Index Terms*—fixed points, contraction maps, metric type space, uniform space, E-uniform space, generalized uniform space.**

I. Introduction and Preliminaries

A Uniform space (X, ϕ) is a nonempty set X equipped with a nonempty family ϕ of subsets of $X \times X$ satisfying the following properties:

(i) Every $U \in \phi$ contains the diagonal $\{(x, x) : x \in R\}$;
(ii) If $U \in \phi$, so does U^{-1};
(iii) If $U \in \phi$, then there exists $V \in \phi$ such that, whenever (x, y) and (y, z) are in V, then (x, z) is in U.

ϕ is called the uniform structure of X. The definition of uniform space is contained in Gottfried Kothe et. al. [1]. Wlodarczyk and Plebaniak [2] generalized the uniform space by introducing the triangle inequality to the notion of uniform spaces.

Definition I:[2] Let X be a nonempty set. The family
$D = \{d_\alpha : X \times X \to [0, \infty], \alpha \in A\}$, A -index set,
is said to be a D-family of generalized pseudometric on X if the following three conditions hold:
(D_1) $\forall\ \alpha \in A \quad \forall x \in X\{d_\alpha(x, x) = 0\}$;
(D_2) $\forall\ \alpha \in A \quad \forall x, y \in X\{d_\alpha(x, y) = d_\alpha(y, x)\}$; and
(D_3) if $\alpha \in A$ and $x, y, z \in X$ and if $d_\alpha(x, z)$ and $d_\alpha(y, z)$ are finite then $d_\alpha(x, y)$ is finite and
$d_\alpha(x, y) \leq d_\alpha(x, z) + d_\alpha(z, y)\}$
is D-family, then the pair (X, D) is called a generalized uniform space.

Cone metric space was introduced in [3] as a generalization of the metric space in which the metric takes values in a Banach space rather than in a Real line. It was proved in [4] that the introduction of metric type structure in cone metric spaces showed that classical proofs related to KKM mappings which was earlier proved in [5] do apply almost identically in these metric type spaces. As a result the extension of known fixed point results to cone metric space and underlying Banach spaces are not necessary.
The following result on cone metric spaces have a metric type structure.

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Theorem I:[6] Let (X, d) be a metric cone over the Banach space E with the cone P which is normal with the normal constant K. The mapping $D : X \times X \to [0, \infty)$ defined by $D(x, y) = ||d(x, y)||$ satisfies the following properties:

(i) $D(x, y) = 0$ if and only if $x = y$;
(ii) $D(x, y) = D(y, x)$, for any $x, y \in X$;
(iii) $D(x, y) \leq k(D(x, z_1) + D(z_1, z_2) + \cdots + D(z_n, y))$, for any points $x, y, z_i \in X, \quad i = 1, 2, \cdots, n$.

In the above result, the classical triangle inequality satisfied by a distance is not true. Also, there are many examples where the triangle inequality fails (see [7] for instance). This led to the introduction of the notion of metric type space.

Definition II:[4] Let X be a set. Let $D : X \times X \to [0, \infty)$ be a function which satisfies

(i) $D(x, y) = 0$ if and only if $x = y$;
(ii) $D(x, y) = D(y, x)$, for any $x, y \in X$;
(iii) $D(x, y) \leq k(D(x, z) + D(z, y))$, for any points $x, y, z \in X$ for some constant $k > 0$.

The pair (X, D) is called a metric type space.
In this paper, we introduce the concept of E-uniform spaces which is an extension of metric-type spaces and generalized uniform spaces.

II. E-uniform Spaces

Motivated by the result of Khamsi and Hussain [4], we introduced the following results:
Definition III: Let X be a nonempty set. The family
$E = \{E_\alpha : X \times X \to [0, \infty], \alpha \in A\}$, A -index set,
is said to be an E-family on $X \times X$ if the following three axioms hold:
(E_1) $\forall\ \alpha \in A \ \ \forall x \in X\{E_\alpha(x, x) = 0\}$;
(E_2) $\forall\ \alpha \in A \ \ \forall x, y \in X\{E_\alpha(x, y) = E_\alpha(y, x)\}$; and
(E_3) $\forall\ \alpha \in A \ \ \forall x, y, z \in X$ and for some constant $\delta > 1$,
$E_\alpha(x, y) \leq \delta(E_\alpha(x, z) + E_\alpha(z, y))$.
The pair (X, E, δ) is called E-uniform spaces. And for the rest of our results we use (X, E) in place of (X, E, δ).

Example I: Let $X = [0, 1]$ be a set. Let $E : X \times X \to R^+$ be defined by $E(x, y) = |x^2 - y^2|$ for all $x, y \in X$. Then $(X, E, 3)$ is an E-uniform space. This does not satisfy the notion of metric type space nor generalized uniform space.

We give the definitions of convergence and completeness in E-uniform space.

Definition IV:

(i) The sequence $\{x_n\}$ converges to $x \in X$ if and only if

$$lim_{n \to \infty} E(x_n, x) = 0.$$

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(ii) The sequence $\{x_n\}$ is Cauchy if and only if

$$lim_{n\to\infty}E(x_n, x_m) = 0.$$

(X, E, α) is complete if and only if any Cauchy sequence in X is convergent.

Definition V: An E-uniform space(X, E) is said to be Hausdorff if and only if the intersection of all $V \in E$ reduces to the diagonal $\{(x, x) : x \in X\}$, i.e if $(x, y) \in V$ for all $V \in E$ implies $x = y$. This guarantees the uniqueness of the limits of sequences.

The proof of the following Lemma follows directly from the definitions.

Lemma: Let (X, E) be an E-uniform space. Let $\{x_n\}$ be arbitrary sequences in X and $\{\alpha_n\}, \{\beta_n\}$ be sequences in R^+ converging to 0. Then for $x, y, z \in X$, if $E(x_n, y) \le \alpha_n$ and $E(x_n, z) \le \beta_n,\ \ \forall n \in N$, then $y = z$. In particular, if $E(x, y) = 0$ and $E(x, z) = 0$, then $y = z$.

Many authors had proved existence and uniqueness of common fixed point of contractive mappings in uniform space (see; [8],[9]). In this paper, adopting the idea of Olatinwo [10], we prove the existence and uniqueness of Banach contraction and generalized contraction mappings in this new setting. Our results simplify some gaps that exist in proving some results in uniform spaces.

III. Fixed Point Results

Theorem II: Let (X, E) be a Hausdorff E-uniform space. Suppose X is complete. Suppose that the sequence $\{x_n\}$ is defined by
$x_{n+1} = Tx_n, n = 1, 2, \cdots$
with $x_0 \in X$. Let $T : X \to X$ be a contraction mapping and there is $0 \le k < \frac{1}{\delta}$ such that

$$E(Tx, Ty) \le kE(x, y) \tag{1}$$

for all $x, y \in X$ and $k\delta < 1$. Then T has a unique fixed point and the sequence $\{x_n\}$ converges to the fixed point of T.
Proof: Let $x_0 \in X$. Choose x_1 such that $x_1 = Tx_0$. In general, we can choose $x_n \in X$ such that $x_n = Tx_{n-1}$. Using (1) and letting $x = x_n$ and $y = x_{n+1}$ we have

$$\begin{aligned} E(x_n, x_{n+1}) &= E(Tx_{n-1}, Tx_n) \\ &\le kE(x_{n-1}, x_n) \\ &\le k^2E(x_{n-2}, x_{n-1}) \\ &\ \vdots \\ &\le k^nE(x_0, x_1) \end{aligned}$$

For $n > m$

$$\begin{aligned} E(x_m, x_n) &\le \delta[E(x_m, x_{m+1}) + E(x_{m+1}, x_n)] \\ &\le \delta E(x_m, x_{m+1}) \\ &+\delta\left[\delta\left[E(x_{m+1}, x_{m+2}) + E(x_{m+2}, x_n)\right]\right] \\ &\le \delta E(x_m, x_{m+1}) + \delta^2E(x_{m+1}, x_{m+2}) \\ &+\delta^3E(x_{m+2}, x_{m+3}) + \cdots \\ &+\delta^nE(x_{n-1}, x_n) \\ &\le \left(k^m\delta + k^{m+1}\delta^2 + k^{m+3}\delta^3 + \cdots \right. \\ &\left. +k^{n-m}\delta^{n-1}\right)E(x_0, x_1) \\ &\le k^m\delta(1 + k\delta + \cdots \\ &+(k\delta)^{n-m-1})E(x_0, x_1) \\ &\le k^m\delta(1 + k\delta + \cdots)E(x_0, x_1) \\ &\le \frac{k^m\delta}{1 - k\delta}E(x_0, x_1). \end{aligned}$$

Letting $m, n \to \infty$ implies that $\{x_n\}$ is a Cauchy sequence. Since (X, E) is complete then there is $x^* \in X$ such that

$$lim_{n\to\infty}x_n = x^*.$$

Now we show that $Tx^* = x^*$. Using (1), we have

$$E(Tx^*, Tx_n) \le kE(x^*, x_n)$$

. As $n \to \infty$, we obtain

$$E(Tx^*, x^*) \le kE(x^*, x^*)$$

. By (E_1) we have

$$E(Tx^*, x^*) \le 0$$

But $E(Tx^*, x^*) \ge 0$. Hence, $E(Tx^*, x^*) = 0$. Since X is Hausdorff, we have $Tx^* = x^*$.
Suppose y* is another fixed point of T such that $y^* = Ty^*$ and $x^* \neq y^*$, then

$$E(x^*, y^*) = E(Tx^*, Ty^*) \le kE(x^*, y^*)$$

a contradiction since $k < 1$, hence $E(x^*, y^*) = 0$.
By carrying out a similar process, we also have $E(y^*, x^*) = 0$. Using the triangle inequality, we have

$$E(x^*, x^*) \le \delta[E(x^*, y^*) + E(y^*, x^*)] = 0$$

This gives $E(x^*, x^*) = 0$ and since $E(x^*, y^*) = 0$. By the Lemma, we have $x^* = y^*$.

The next theorem proves the existence and uniqueness of the fixed point for generalized contraction mappings.

Theorem III: Let (X, E) be a complete E-uniform space and let $T : X \to X$ be a generalized contraction mapping satisfying

$$E(Tx, Ty) \le aE(x, y) + \beta E(Tx, y) \tag{2}$$

for all $x, y \in X$, $a + \beta < 1$ and $a\delta < 1$. Then, T has a unique fixed point.
Proof: Let $x_0 \in X$ be arbitrary chosen. Choose $x_1 \in X$ such that $x_1 = Tx_0$. In general, we can choose $x_n \in X$ such that $x_n = Tx_{n-1}$. Using (2), letting $x = x_n$ and $y = x_{n+1}$ and

assuming $x_n \neq x_{n+1}$, we have

$$\begin{aligned} E(x_n, x_{n+1}) &= E(Tx_{n-1}, Tx_n) \\ &\leq aE(x_{n-1}, x_n) + \beta E(Tx_{n-1}, x_n) \\ &= aE(x_{n-1}, x_n) + \beta E(x_n, x_n) \\ &\leq aE(x_{n-1}, x_n) \\ &= aE(Tx_{n-2}, Tx_{n-1}) \\ &\leq a^2 E(x_{n-2}, x_{n-1}) \\ &+ a\beta E(Tx_{n-2}, x_{n-1}) \\ &= a^2 E(x_{n-2}, x_{n-1}) + a\beta E(x_{n-1}, x_{n-1}) \\ &\leq a^2 E(x_{n-2}, x_{n-1}) \\ &\vdots \\ &\leq a^n E(x_0, x_1) \end{aligned}$$

For $n > m$

$$\begin{aligned} E(x_m, x_n) &\leq \delta[E(x_m, x_{m+1}) + E(x_{m+1}, x_n)] \\ &\leq \delta E(x_m, x_{m+1}) \\ &+ \delta\left[\delta\left[E(x_{m+1}, x_{m+2}) + E(x_{m+2}, x_n)\right]\right] \\ &\leq \delta E(x_m, x_{m+1}) + \delta^2 E(x_{m+1}, x_{m+2}) \\ &+ \delta^3 E(x_{m+2}, x_{m+3}) + \cdots \\ &+ \delta^n E(x_{n-1}, x_n) \\ &\leq (a^m \delta + a^{m+1}\delta^2 + a^{m+3}\delta^3 + \cdots \\ &+ a^{n-m}\delta^{n-1}) E(x_0, x_1) \\ &\leq a^m \delta(1 + a\delta + \cdots \\ &+ (a\delta)^{n-m-1}) E(x_0, x_1) \\ &\leq a^m \delta(1 + a\delta + \cdots) E(x_0, x_1) \\ &\leq \frac{a^m \delta}{1 - a\delta} E(x_0, x_1). \end{aligned}$$

As $m, n \to \infty$ implies that $\{E(x_m, x_n)\} \to 0$. Thus the sequence is Cauchy. Since X is complete then there is $x^* \in X$ such that

$$lim_{n\to\infty} x_n = x^*$$

. Now we show that $Tx^* = x^*$. Using (2) we have

$$E(Tx^*, Tx_n) \leq aE(x^*, x_n) + \beta E(Tx^*, x_n).$$

As $n \to \infty$, we obtain

$$E(Tx^*, x^*) \leq \beta E(Tx^*, x^*)$$

Since $\beta < 1$ then we have,

$$E(Tx^*, x^*) = 0.$$

Carrying out the same process, we can similarly have

$$E(x^*, Tx^*) = 0.$$

Therefore,

$$E(x^*, x^*) \leq \delta[E(x^*, Tx^*) + E(Tx^*, x^*)] = 0$$

This implies that $E(x^*, x^*) = 0$. Since $E(x^*, Tx^*) = 0$ then using the Lemma, we have

$$x^* = Tx^*.$$

Suppose there is another fixed point of T say y^* such that $y^* = Ty^*$ and $y^* \neq x^*$ then using (2) we have,

$$\begin{aligned} E(x^*, y^*) &= E(Tx^*, Ty^*) \\ &\leq aE(x^*, y^*) + \beta E(Tx^*, y^*) \\ &= aE(x^*, y^*) + \beta E(x^*, y^*) \\ &\leq (a + \beta) E(x^*, y^*) \end{aligned}$$

Since $a + \beta < 1$ then a contradiction , hence $x^* = y^*$.

Example II: Let $X = [0, 1]$ and define on $X \times X$ the map E given by $E(x, y) = |x^2 - y^2|$. Then, X is a complete E-uniform space. Suppose we define the map $T : X \to X$ by $T(x) = \frac{x}{4}$ and $a = \beta = \frac{1}{8}$. Then the conditions of Theorem III are satisfied. The unique fixed point of T is 0.

Remarks: Theorem III is an analogy result of ([10], Theorem II) if the maps reduce to a single map and the function coefficient is replaced with a scalar coefficient in the new setting.

IV. Conclusion

This research introduces the concepts of a new generalized uniform space which extends the notions of metric-type space and generalized uniform spaces. We prove some existence and uniqueness results for contraction and almost contraction mappings in this new setting. Examples are given to support our results.

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