Deflation Technique to Accelerate the Convergence of Iterative Solver for the Wave Scattering Problem

A. H. Sheikh, A. G. Shaikh, Hisamuddin, Naeem Faraz, and Asif Ali

***Abstract*—An iterative solution method for the discrete high wavenumber Helmholtz equation is presented. The basic idea for solution, already presented in [1], is to develop a preconditioner which is based on a Helmholtz operator with a complex-valued shift for a Krylov subspace iterative method. The preconditioner which can be seen as a strongly damped wave equation in Fourier space, can be approximately inverted by a multigrid method. Extensive deflation and spectral analysis, as Krylov subspace methods depends upon eigenvalues, highlights in this paper. Findings in analysis are validated by numerical results.**

***Index Terms*—Helmholtz equation, Multigrid Method, Preconditioning, Sparse linear systems, Deflation preconditioner.**

I. Introduction

WAVE scattering have many applications in physics, engineering and science. Examples include seismic imaging [1], [2], [3], [4], [5], radars, electromagnetism [6], bio medical imaging [7], (ultrasound), road-speed sensors etc. Wave scattering phenomena is mostly modeled by mathematicians in the form of the Helmholtz equation [8], [9] and [10]. Solving Helmholtz equation requires the use of iterative methods. The Helmholtz equation in two dimensional (2D) or three dimensional (3D), the convergence is typically characterized by indefiniteness of the eigenvalues of the corresponding coefficient matrix. With such a property, an iterative method either basic or advanced, encounters convergence problems. The method usually converges very slowly or diverges [11]. There are very few choices of numerical methods to compute solution of very large sparse systems for many reasons, including memory, sparsity, heterogeneity of medium and indefiniteness. Indefiniteness limits the choice narrowly. The sparse direct solver have been used in [12], [13], [14] and [15]. They are heavily constrained with memory and storage, hence are not practical for sufficiently large problems. The direct methods are not favorable for many obvious grounds, and they are too much time restrictions. They consume unaffordable memory for large problem which is under consideration. Discrete Helmholtz system, obtained by finite difference scheme, is approximated using Krylov subspace method. The preconditioner CSLP are tested with different shifts. Eigenvalue analysis of CSLP is given in accordance with solver performance. For small frequency, CSLP performs better whereas increasing frequency, CSLP becomes impractical in terms of memory and computational time. Deflation technique is used to address these types of issues.

The Helmholtz equation can be read as

$$-\Delta^2 u(x,y) - k^2(x,y)u(x,y) = f(x,y), \tag{1}$$

where $\Delta^2 = \frac{\partial^2}{(\partial x^2)} + \frac{\partial^2}{\partial y^2}$, and $u(x,y)$the unknown variable, defined on the unit square domain $\Omega = (0,1)\times(0,1)$, K wave number. The wave number k is related with wavelength λ as

$$k(x,y) = \frac{2\pi}{\lambda} = \frac{omega}{c(x,y} \tag{2}$$

where $\omega = 2\pi F$ is angular velocity, F the wave frequency, $\lambda = \frac{c(x,y)}{F}$ the wavelength and $c(x,y)$ is the speed of sound.

A. Model Problem

The Helmholtz problem considered this paper is non-homogeneous defined on the domain $\Omega = (x,y)\times(x,y)$ where $x, y \in (0,1)$. The wavenumber is constant, independent of geometry. The source function is given as

$$f(x,y) = \delta(x,y) = \delta(x - 1/2, y - 1/2). \tag{3}$$

With $x, y \in (0,1)$ where Dirac delta function is given as

$$\delta(x,y) = \begin{cases} +\infty & x = 0, y = 0 \\ 0 & x \neq 0, y \neq 0 \end{cases}. \tag{4}$$

This source functions is used to model the source centered at $(\frac{1}{2}, \frac{1}{2})$. The domain is bounded by the Sommerfeld radiation conditions [6] [1] [16], which are given as

$$\frac{\partial u}{\partial \eta} - \iota k u = 0. \tag{5}$$

This models the propagation of wave from center outwards direction.Discretization: two lines. The resultant linear system is written as

$$A_h u_h = f_h. \tag{6}$$

II. Helmholtz Solvers

Solving Helmholtz equation requires solution of resultant large sparse Linear System (6). For large, sparse matrix the Krylov subspaces are very popularchoice.The methods are developed on construction of iterants in the subspace.The space

$$K^j(A; r^0) = Span\{r^0, Ar^0, A^2r^0, \cdots\ldots A^{(j-1)}r^0\},$$

is called the Krylov subspace of dimension j, associated with A and r^0, and initial residual $r^0 := g - Au^0$ is related to the initial guess u^0. Among methods which are based

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A.H. Sheikh is with the Department of Mathematics & Statistics, QUEST Nawabshah, 67480 Pakistan e-mail: ah.sheikh@quest.edu.pk .

A. G. Shaikh is with department of Mathematics & Statistics, QUEST Nawabshah 67480, Pakistan e-mail: agshaikh@quest.edu.pk.

Hisamuddin is with department of Mathematics, Shah Abdul Latif University Khairpur, 66020 Pakistan e-mail: hisam.shaikh@salu.edu.pk.

Naeem Faraz is with Center of International Program, Donghua University Shanghai PRC e-mail: naeem@dhu.edu.cn .

Asif Ali is with department of Basic Science & Related Studies, Mehran University of Engineering & Technology, Jamshoro 76062, Pakistan e-mail: asif.shaikh@faculty.muet.edu.pk .

on construction the Krylov subspaces, a Conjugate Gradient (CG) [17],[18] [19] GMRES [20], CGS [21], Bi-CG [22], Bi-CGSTAB [23]and QMR are popular. For non-symmetric linear systems, Krylov subspace can be built from Arnoldi's process, which leads to GMRES(Saad and Schultz, 1986). GMRES is optimal method; it reduces the 2-norm of the residual at the every iteration. GMRES, however, require long recurrences, which is usually limited by the available memory. A remedy is by restarting, which some-times lead to slow convergence or stagnation [24].

A. Preconditioning

In an iterative method preconditioning is often vital component in enhancing the convergence of iterations, particularly when the system is large sized.. There are many iterative techniques for solving linear system. For large spare linear system, the convergence rate is always a concern for researchers; one can improve significantly the convergent rate by applying appropriate preconditioner. The convergence of iterative methods depends on eigenvalues of the coefficient matrix, it is often advantageous to use preconditioner that transfer the system to one with a better distribution of eigenvalues. The preconditioner is the key to successful iterative solver. In brief, to make linear system favorable for iterative solver, the coefficient matrix is scaled with a matrix Mcalled preconditioner. With choice of preconditionerMfor Linear System 6, where the inverse of M is relatively inexpensive to compute, and then the preconditioned system is $M^(-1)Au = M^(-1)f$ is supposed to be favorable for iterative solver. A few preconditioners have been tried for the Helmholtz equation, for details see [10] [2] [4].

B. Complex Shifted Laplace Preconditioner

The Complex Shifted Laplace Preconditoner(CSLP) is the discrete Helmholtz operator in addition with a complex shift $(a, \imath b)$. The CSLP is preconditioner based on operator, in contrast to decomposition type preconditioners, which are matrix-based. The CSLP obtained by(finite difference) discretization of the shifted Helmholtz operator i.e.

$$M(a, b) := -\Delta - (a - \imath b)k^2, \quad \text{where} \quad a, b \in \mathbb{R},$$

where a and b are real and imaginary numbers respectively. The first precedent in operator based preconditioner for the Helmholtz equation was simple Laplace operator Δ, used without any shifts. It works well, until mesh size is small. For large size of mesh, convergence starts to stagnates, and alot of unwanted eigenvalues appear, as shown in Fig:1, which shows that for large mesh size this shift is not good choice. Later different shifts were introduced, with real as well as imaginary parts, and found to be effective. The number of iterations taken by GMRES preconditioned CLSP $M(1, \pi/4)$ gorws with linear rate with wave number. This fact is illustrated by spectrum of preconditioned Helmholtz, as shown in Fig:2, where eigen values are getting more closer to orign. Some near-origin eigenvalues affect the convergence of solver. Deflation preconditioned, illustrated in next Sectin, is used to treat this drawback. A comparison of performance of CSLP with different shifts in given in Table I, where shift $(1, \pi/4)$ is the one which outperforms rest of choices of shifts for small as well large wave numbers.

TABLE I
COMPARISON OF GMRES NUMBER OF ITERATIONS BY CSLP WITH DIFFERENT SHIFTS

k	N	M(0,0)	M(0,1)	M(1,1)	M(1,pi/4)
10	16	09	11	10	09
20	32	20	21	20	18
30	48	40	35	33	29
40	64	71	53	44	37
50	80	110	75	57	47
60	96	154	98	67	56

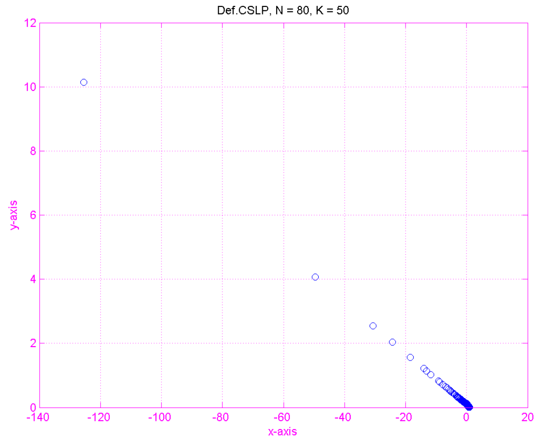

Fig. 1. Spectrum with $M(0, 0)$

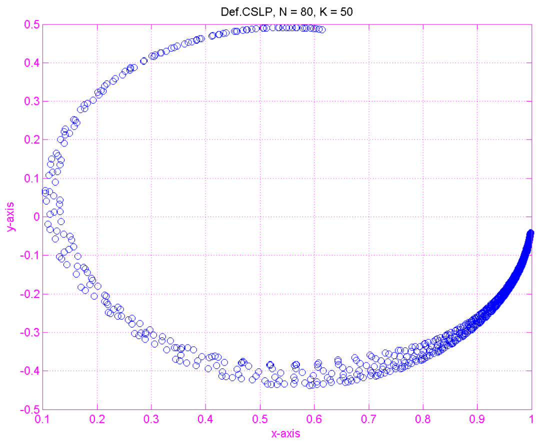

Fig. 2. Spectrum with $M(0, 1)$

III. DEFLATION TECHNIQUE

Convergence of the Krylov subspace method is typically adversely affected by small eigenvalues, as seen in Fig: 2. The small eigenvalues need to be special treatment. Deflation is special type of preconditioner. Deflation is a technique commonly used to get rid of certain part of the spectrum, and to force the "unfair" eigenvalues not to participate in the Krylov iterative method. In order to develop deflation preconditioner, we consider the linear system

$$A_h u_h = f_h. \tag{7}$$

For given a matrix $Z_h \in \mathbb{C}^{n \times r}$, the deflation precondioners are the projections of type

$$P_h = I_h - A_h Q_h, \tag{8}$$

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where $Q_h = Z_h A_{2h}^{-1} Z_h^T$ and $A_{2h} = Z_h^T A_h Z_h$. Choice of deflation vectors in matrix Z_h forms an interest area of research. Theoretically, eigenvectors gives ideal results, as they projects corresponding eigenvectors to zero. Since exact eigenvectors are impractical to compute, therefore many problem-specific possibilities have been explored. For the problem under consideration, few alternatives have been researched in [25], [9] and [26]. Getting motivation from property of resolving smaller error modes on coarser grids by multigrid, we choose multigrid coarsegrid operator as deflation matrix Z_h for our problem. This deflation preconditioner can be applied in combination with other preconditioners [27] and [28], and we have combined with CSLP as follows:

$$P_h M(a,b)_h^{-1} A_h u_h = P_h M(a,b)_h^{-1} f_h. \qquad (9)$$

Next, we plot the spectrum of operator given in Eq: 9 in one-dimension as well as two-dimension in Fig: 3 and Fig: 4. These spectral plots show clustered behavior of eigenvalues of deflated CSLP-preconditioned matrix in one dimension and two dimension respectively.

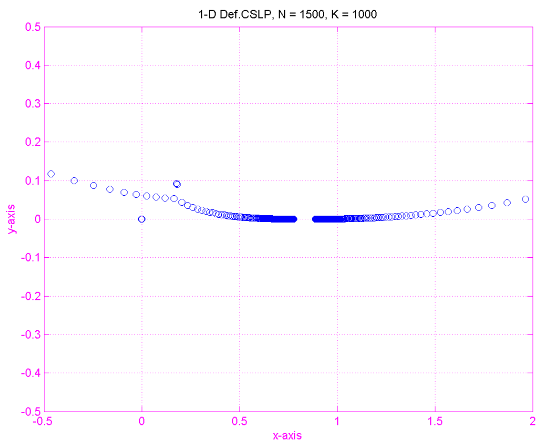

Fig. 3. One Dimensional Spectrum with CSLP and Def.

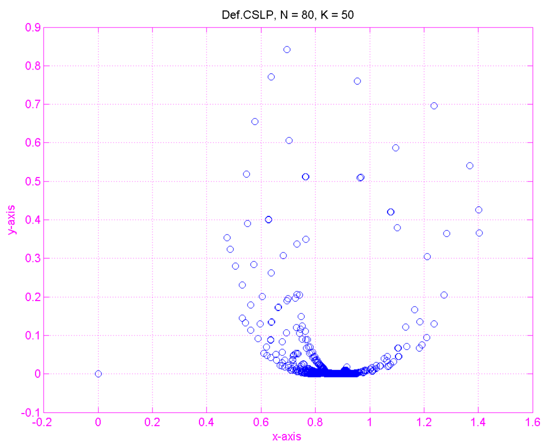

Fig. 4. Spectrum with $M(0,1)$

IV. Numerical Experiments

For all the experiments, u^0 (zero vector) is used as initial guess. The mesh size h is chosen such that for a wave number k, it satisfies relation $kh \leq 0.625$ (equivalent to 10 grid points per wave length). Iterations are stoppedn when the residul meets the tolerance

$$\|r_h\| \leq 10^{-5}.$$

A. Results

The first numerical result, using deflation, is presented in Table II, where CSLP is not used. This effort has been made to highlight affect of deflation preconditioner on its own. The readings show a substantial reduction in number of iterations and computational time. Subsequently, deflation is applied in combination with the CSLP, the first level preconditioner. A variety of shifts in CSLPj, in combination with defltaion, has been experimented and readings have been recorded and presented in Tables III, IV, V and VI for CSLP-shifts $(0,0)$, $(0,1)$, $(1,1)$ and $(1,\frac{\pi}{4})$ respectively. Such comparison is represented using consolidated bar plots given in Fig: 6, where bar representing iterations taken by solver preconditioned by CSLP and deflation is fairly smaller than the bar representing iterations taken by solver preconditioned by only CSLP. Comparison is performed with four different choices of shifts, comprehensible from figure. The two level preconditioned (CSLP and deflation perconditioner) solver is also tested for a very large wave number $k = 200$, and readings are presented in Table VII where inclusion of defltaion alongwith CSLP reduced the number of iterations significantly. Rate of reduction for shift $(1,\frac{\pi}{4})$ is 5 times. Lastly, the velocity potential for wave number ranging from $k = 5$ to $k = 30$ is plotted in Fig:5. Increasing wave number clearly highlights the need of more grid-points for large wave number.

TABLE II
Number of Iterations by GMRES and Deflated GMRES

k	N	Dim. of A	GMRES It.	Time	Def GMRES It.	Time
20	32	1089	74	00.79	13	00.11
40	64	4225	200	08.63	15	00.44
60	96	9409	405	51.95	17	01.10
80	128	16641	607	202.30	20	02.44
100	160	25921	782	362.79	24	04.36

TABLE III
Number of Iterations by CSLP and CSLP-Deflation with $M(0,0)$

k	N	Dim. A	CSLPIt	Time(S)	CSLP-Def It	Time(S)
20	32	1089	20	00.25	07	00.16
40	64	4225	71	02.86	10	00.70
60	96	9409	154	17.28	13	02.30
80	128	16641	257	66.02	16	05.26
100	160	25921	358	133.68	20	09.56

TABLE IV
Number of Iterations by CSLP and CSLP-Deflation with $M(0,1)$

k	N	Dim of A	CSLPIt	Time (S)	Def CSLP It	Time(S)
20	32	1089	21	00.27	07	00.16
40	64	4225	53	02.15	10	00.75
60	96	9409	98	10.56	13	02.32
80	128	16641	137	28.70	16	05.18
100	160	25921	175	56.93	20	09.73

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TABLE V
NUMBER OF ITERATIONS BY CSLP AND CSLP-DEFLATION WITH $M(1,1)$

k	N	Dim A	CSLP It	Time	Def. CSLP It	Time
20	32	1089	20	00.26	07	00.16
40	64	4225	44	01.77	10	00.73
60	96	9409	67	07.07	13	02.28
80	128	16641	88	17.87	16	05.37
100	160	25921	108	33.44	21	10.16

TABLE VI
NUMBER OF ITERATIONS BY CSLP AND CSLP-DEFLATION WITH $M(1,\pi/4)$

k	N	Dim A	CSLP It	Time	Def. CSLP It	Time
20	32	1089	18	00.24	07	00.16
40	64	4225	37	01.49	10	00.74
60	96	9409	56	05.84	13	02.27
80	128	16641	73	14.54	16	05.09
100	160	25921	88	26.88	20	09.64

TABLE VII
NUMBER OF ITERATIONS BY CSLP AND CSLP-DEFLATION, DIFFERENT SHIFTS $M(a,b)$

CSLP $M(a,b)$	Dim A	CSLP It	t(s)	D-CSLP It	t(s)
$M(0,0)$	103041	948	1681	48	107
$M(0,1)$	103041	349	542	49	108
$M(1,1)$	103041	208	267	49	86
$M(1,\pi/4)$	103041	169	248	49	83

V. CONCLUSION

In this paper, we discussed the ingredients of robust and efficient iterative solver for high wave number Helmholtz problems. Need of preconditioner is highlighted and a critical investigation of different preconditioners is presented. The CSLP preconditioner is applied and found to be very effective to enhance the convergence of Krylov subspace methods for small wave number problem. Increasing wave number stagnates convergence of CSLP preconditioned solver. The deflation is introduced and is used as a second-level in combination with CSLP, which not only pushes the small eigenvalues to origin (unwanted eigenvalues), also helps to achieve faster convergence fast. Specially when wave numbers are large, the deflation method takes less iterations as compared to the CSLP. It also reduces solve time for large wave number problem, which highlights the contribution of this paper.

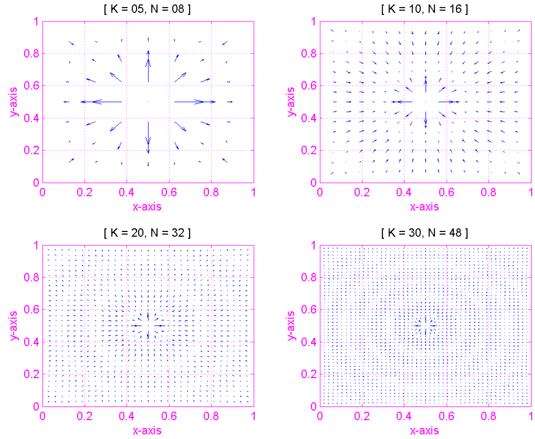

Fig. 5. Velocity Potential of App. Sol. Increasing wavenumber

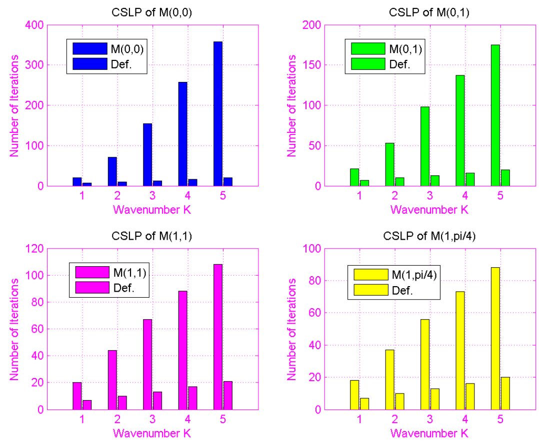

Fig. 6. Bar Iterative Comparison of CSLP and CSLP-Def.of $M(a,b)$

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